# A note on properties of condition numbers

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**Abstract**

The condition numbers described by Stewart, Todd and others for a matrix $A$ are given by the suprema of the norms of all the oblique projection operators respectively on the image space of $A^T$ and the null space of $A$. This paper shows that these two numbers coincide.

**Keywords:** Condition numbers, oblique projections.

## 1 Introduction

Let $A$ be an $m \times n$ matrix with rank $m$, and let $\mathcal{D}$ be the set of all positive definite $n \times n$ diagonal matrices. Define also a matrix $Z$ whose rows form a basis for the null space of $A$, $\mathcal{N}(A)$.

The following two numbers represent the suprema of the oblique projection operator norms respectively onto the range space of $A^T$, $\mathcal{R}(A^T)$, and onto the null space of $A$, $\mathcal{N}(A)$:

$$\bar{\chi}(A) = \sup\{\|A^T(ADA^T)^{-1}AD\| \mid D \in \mathcal{D}\} \tag{1}$$

$$\bar{\chi}(Z) = \sup\{\|Z^T(ZDZ^T)^{-1}ZD\| \mid D \in \mathcal{D}\} \tag{2}$$

Stewart [1] has shown that these numbers are always finite, and provide interesting condition numbers associated with the matrix $A$. These numbers were also studied by Todd [2] and by others (see references in [6]), and

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were intensively used in new results on complexity of linear programming algorithms. The number $\bar{\chi}(A)$ provides a stopping rule for interior point methods, thus avoiding the use of the bit size of the input data. This study was made by Vavasis and Ye in a series of papers [4, 5, 6].

In [3], Vavasis proves that $|\bar{\chi}(A) - \bar{\chi}(Z)| \leq 1$. In this paper we show that actually these two numbers coincide.

We now describe another way of computing the condition numbers, following Stewart and Vavasis and Ye. Let us define the following sets:

$$\begin{aligned} S &= \{s \in \mathcal{R}(A^T) \mid \|s\| = 1\}, &(3)\\ X &= \{x \in \mathcal{N}(AD) \mid D \in \mathcal{D}\}, &(4)\end{aligned}$$

and

$$\rho(A) = \inf\{\|x - s\| \mid x \in X,\ s \in S\}. \tag{5}$$

One can show that $\rho(A) = 1/\bar{\chi}(A)$(See, [1]).

The similar definitions for the matrix $Z$ give the following expressions. The equalities are easily shown:

$$\begin{aligned} S' &= \{s \in \mathcal{N}(ZD^{-1}) \mid D \in \mathcal{D}\} = \{s \in \mathcal{R}(DA^T) \mid D \in \mathcal{D}\}, &(6)\\ X' &= \{x \in \mathcal{R}(Z^T) \mid \|x\| = 1\} = \{x \in \mathcal{N}(A) \mid \|x\| = 1\}, &(7)\end{aligned}$$

and

$$\rho(Z) = \inf\{\|x - s\| \mid x \in X',\ s \in S'\}. \tag{8}$$

## 2 The main result

Before we prove the main result, let us derive the optimality conditions for the problem of minimizing $\|x - s\|$ subject to $s \in S$ and $x \in X$ for points in $X$ associated with a fixed $D \in \mathcal{D}$.

Let $D \in \mathcal{D}$ be given, and consider the problem

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\|x - s\|^2 \\ \text{subject to} & Zs = 0 \\ & ADx = 0 \\ & \|s\| = 1. \end{array} \tag{9}$$

The optimality conditions for this problem give for some $y \in I\!R^m$, $w \in I\!R^{n-m}$, $\alpha \in I\!R$,

$$\begin{aligned} \bar{x} - \bar{s} + DA^T y &= 0 \\ \bar{s} - \bar{x} + Z^T w - \alpha \bar{s} &= 0, \end{aligned}$$

and hence,

$$\bar{x} - \bar{s} \in \mathcal{R}(DA^T) \tag{10}$$
$$(1-\alpha)\bar{s} - \bar{x} \in \mathcal{N}(A). \tag{11}$$

**Theorem 2.1** *In the conditions above, $\rho(A) = \rho(Z)$.*

*Proof.* We shall prove that $\rho(A) \geq \rho(Z)$. This is sufficient because of the symmetry of the construction: if we start with the matrix $Z$, then the rows of $A$ provide a basis for $\mathcal{N}(Z)$ and we can conclude that $\rho(Z) \geq \rho(A)$.

Consider arbitrary vectors $x \in X$ and $s \in S$. By definition of $X$, there exists $D \in \mathcal{D}$ such that $ADx = 0$. We shall work with this fixed $D$ to show that $\|x - s\| \geq \rho(Z)$.

Let $\bar{x}$ and $\bar{s}$ be an optimal solution for the problem (9). Consider the following vectors:

$$x' = \frac{\bar{x} - (1-\alpha)\bar{s}}{\|\bar{x} - (1-\alpha)\bar{s}\|} \quad , \quad s' = \beta(\bar{x} - \bar{s}),$$

where $\beta \in I\!R$ will be specified in a moment.

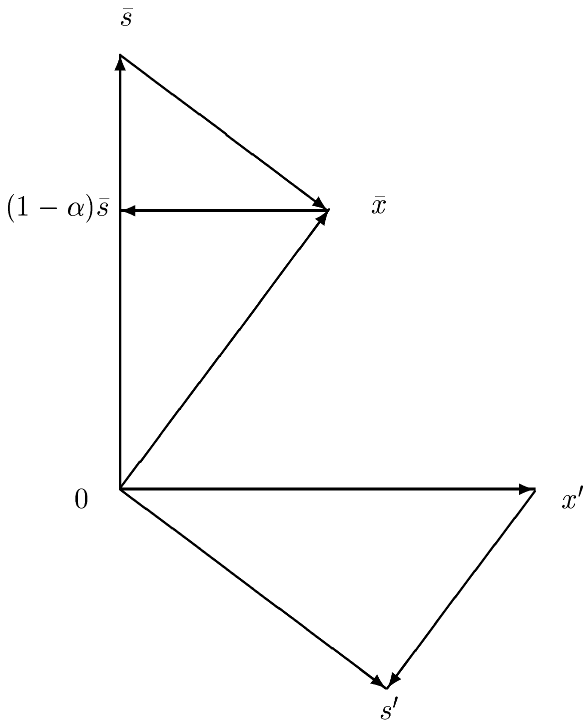
Then $x' \in X'$ due to (11) and the fact that $\|x'\| = 1$, and $s' \in S'$ for any $\beta \in I\!R$, due to (10).

All the action happens in the two-dimensional subspace defined by $\bar{x}$ and $\bar{s}$. We can use two-dimensional geometry, following the figure below. The triangle $(0, \bar{s}, \bar{x})$ is rectangle, because by (10) $\bar{x} - \bar{s} \perp \bar{x} \in \mathcal{N}(AD)$. Let $\theta$ be the angle between $\bar{x}$ and $\bar{s}$. Then $\|\bar{x} - \bar{s}\| = \sin\theta$.

We have that $x' \perp \bar{s}$, due to (11) and the fact that $\bar{s} \in \mathcal{R}(A^T)$. As we saw in the last paragraph, $s' \perp \bar{x}$, for any $\beta \in I\!R$. It follows that the signal of $\beta$ can be chosen in such a way that the angle between $x'$ and $s'$ is $\theta$, and its absolute value can be chosen such that $\|s'\| = \|\bar{x}\|$. Then the triangle $(0, x', s')$ is identical with the triangle $(0, \bar{s}, \bar{x})$, and it follows that

$$\|x - s\| \geq \|\bar{x} - \bar{s}\| = \|x' - s'\| \geq \rho(Z).$$

Since $x \in X$ and $s \in S$ were chosen arbitrarily, the inequality remains valid when taking the infimum, completing the proof. ∎

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