GENERATION OF DEGENERATE LINEAR PROGRAMMING PROBLEMS

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Abstract. In this paper we present a method for constructing linear programming problems with randomly generated data. Besides the number of variables and constraints, the dimensions of the primal and dual faces are given. We show that for problems in which the constraint matrix is carelessly constructed with random entries, with probability one only one between primal and dual degeneracy appears.

1. Introduction. A code for generating large sparse degenerate linear programming problems is very useful for testing algorithms. Generating problems with random data is easy, but controlling the amount of primal and dual degeneracy does not seem so trivial: in fact, we show in this paper that problems generated by naive methods will never possess both kinds of degeneracy. This paper describes a method for generating sparse problems with specified degrees of degeneracy, and displays most of its very short code in Matlab. The complete code is available in the web [5].

We shall study the linear programming problem

$$(P) \qquad \begin{array}{rl} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0, \end{array}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and A is an $m \times n$ matrix with full row rank $m < n$.

The dual problem associated with (P) is

$$(D) \qquad \begin{array}{rl} \text{maximize} & b^T y \\ \text{subject to} & A^T y + s = c \\ & s \geq 0, \end{array}$$

Formally, the problem to be solved is the following: generate a linear programming problem (P) with the following data (the data will be discussed ahead):
n: dimension of the space, $n > 1$.
m: number of constraints , $0 < m < n$.
$dens$: density of the sparse matrix A , $dens > 1/m$.
D_p: dimension of the primal optimal face, $0 \leq D_p \leq n - m - 1$.
D_d: dimension of the dual optimal face, $0 \leq D_d \leq m - 1$.

A linear programming problem could be easily generated in the following sequence: generate randomly the matrix A, choose optimal primal and dual solutions $x^*, s^* \geq 0$ satisfying the complementarity condition ${x^*}^T s^* = 0$ and finally define $b = Ax^*$ and $c = s^*$. The difficulty is in specifying the amount of degeneracy. The following fact, which will be proved below, illustrates this difficulty: if the density of A is sufficiently high, then the probability that $D_p > 0$ and $D_d > 0$ is null. This means that for whatever choice of x^* and s^*, only one kind of degeneracy (primal or dual) will be present.

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As far as we know, this means that primal *and* dual degenerate problems have never been generated randomly.

The paper is organized as follows: Sec. 2 summarizes some known facts about linear programming, Sec. 3 studies the optimal face dimensions. The actual construction is described in Sec. 4 and Sec. 5 shows how to modify it to specify an initial interior primal-dual solution.

The description of the method includes Matlab commands using routines for sparse matrices. A complete Matlab code is available at the author's home page [5].

2. Theoretical tools. Let us summarize some well-known facts about the primal and dual problems (P) and (D). These facts can be found in textbooks like Roos, Terlaky and Vial [3], Wright [4] or in our paper [2].

Optimality conditions. a primal-dual solution (x^*, y^*, s^*) with $x^*, s^* \in \mathbb{R}^n$, $y^* \in \mathbb{R}^m$ is optimal if and only if it satisfies the following optimality conditions:

$$\begin{aligned} x_i^* s_i^* &= 0, \quad i = 1, \ldots, n \\ Ax^* &= b \\ A^T y + s^* &= c \\ x^*, s^* &\geq 0. \end{aligned}$$

The primal and dual feasible sets are respectively $\mathcal{X} = \{x \in \mathbb{R}^n \mid Ax = b, x \geq 0\}$ and $\mathcal{D} = \{(y, s) \in \mathbb{R}^{m+n} \mid A^T y + s = c, s \geq 0\}$.

The set of feasible dual slacks is $\mathcal{S} = \{s \in \mathbb{R}_+^n \mid A^T y + s = c \text{ for some } y \in \mathbb{R}^m\}$. Note that the complete specification of a dual feasible solution (y, s) only needs one between y and s, because A has maximum rank. So we can talk freely of a "dual solution s" or a "dual solution y".

Let X^* be the set of optimal solutions for (P), and assume that $X :^* \neq \emptyset$. The optimal cost value will be $c^* = c^T x^*$, for an arbitrary $x^* \in X^*$. The we have:
(i) X^* is a face of the polyhedron X, with dimension denoted D_p. If $D_p > 0$ then the problem is *dual degenerate*, with a degree D_p of dual degeneracy.
(ii) $S^* = \{s^* \in S \mid {s^*}^T x^* \text{ for all } x^* \in X^*\}$ is the set of dual optimal solutions, a face of the polyhedron S with dimension denoted D_d. If $D_d > 0$, then the problem is *primal degenerate*, with degree D_d of primal degeneracy.
(iii) The set of indices $\{1, 2, \ldots, n\}$ can be partitioned into two sets B and N such that for any optimal primal-dual solution (x^*, s^*),

$$\begin{aligned} x_B^* \geq 0, \quad x_N^* = 0, \\ s_B^* = 0, \quad s_N^* \geq 0. \end{aligned}$$

(iv) Goldman-Tucker theorem [1]: there exists a strictly complementary optimal solution pair (x^*, s^*), i. e., an optimal solution pair such that $x_B^* > 0$ and $s_N^* > 0$.
(v) If the problem is (primal-dual) non-degenerate then the unique primal-dual optimal solution satisfies $|B| = m$, and is strictly complementary. The square sub-matrix A_B is non-singular.
(vi) If the problem admits an interior primal-dual feasible solution, i.e., a feasible pair (x, s) such that $x > 0$ and $s > 0$, then both X^* and S^* are non-empty and bounded.

We shall assume that $X^* \neq X$ and $S^* \neq S$, eliminating trivial cases. Reordering the variables if needed, we can assume that the optimal partition is given by $B = \{1, 2, \ldots, p\}$ and $N = \{p + 1, \ldots, n\}$, with $1 \leq p \leq n - 1$. If the problem is non-degenerate, then $p = m$. The optimal solutions x^* and s^* satisfying $x_B^* > 0$ and $s_N^* > 0$ define respectively the relative interiors of the primal and dual optimal faces.

3. Properties of degenerate problems. Consider the problem (P) and assume that the optimal faces are non-empty and bounded. Assume also that the optimal partition is $B = \{1, 2, \ldots, p\}$ and $N = \{p+1, \ldots, n\}$, with $1 \leq p \leq n-1$.

Define the matrix A_B composed by the columns A_i, $i = 1, 2, \ldots, p$ of A, and define $r = rank(A_B)$. The two lemmas below relate the optimal face dimensions with n, m, p and r.

LEMMA 3.1. *The primal optimal face satisfies $D_p = p - r \leq n - m - 1$.*

Proof. The primal optimal face is defined by $A_B x_B^* = b$, $x_N^* = 0$, where A_B is an $m \times p$ matrix. The solution set for $A_B x = b$ has dimension $D_p = p - rank(A_B) = p - r$, proving the first equality. The feasible set X has dimension $n - m$, because A has full rank m. Since the optimal face (which is contained in X) does not coincide with X, its dimension cannot be $n - m$, completing the proof. □

LEMMA 3.2. *The dual optimal face satisfies $D_d \leq \min\{n - p, m - r\}$.*

Proof. Let y^*, s^* be an optimal dual solution. The dual optimal face is the set of $s \in \mathbb{R}^n_+$ such that

$$A^T y + s = A^T y^* + s^*, \quad y \in \mathbb{R}^m.$$

Setting $z = y^* - y$, we obtain $s = s^* + A^T z$, or

$$A_B^T z = 0, \quad s_N = s_N^* + A_N^T z.$$

Let T be a matrix whose columns are a basis for the null space of A_B^T. Then z can be expressed as $z = Tw$, and $s_N = s_N^* + A_N^T T w$. Hence the dimension of the dual optimal face is

$$D_d = rank(A_N^T T).$$

We have: A_B^T is a $p \times m$ matrix with rank r. Its null space has dimension $m - r$, and hence T is $m \times (m - r)$. $A_N^T T$ is $(n - p) \times (m - r)$. Hence $D_d \leq \min\{n - p, m - r\}$, completing the proof. Note that if $A_N^T T$ has maximum rank, then this is true with equality. □

Randomly generated matrices. Consider a set $\{a_1, a_2, \ldots, a_q\}$ of $0 < q < m$ linearly independent vectors in $\mathbb{R}^m$, and let a_{q+1} be a randomly generated vector. Then with probability 1, the set $\{a_1, a_2, \ldots, a_{q+1}\}$ is linearly independent.

In fact, the subspace Q generated by $\{a_1, a_2, \ldots, a_q\}$ has measure zero in $\mathbb{R}^m$, and hence the probability that $a_{q+1} \in Q$ is null. Similarly, the probability that a_{q+1} is orthogonal to Q is also null. The next lemma summarizes the effect of this on our problem.

LEMMA 3.3. *If the matrix A is dense and randomly generated, then with probability one*
(i) Any sub-matrix of A has maximum rank.
(ii) The dual optimal face satisfies $D_d = \min\{n - p, m - r\}$.
(iii) If $p \leq m$ then $D_p = 0$ and $D_d = m - p$; if $p \geq m$ then $D_d = 0$ and $D_p = p - m$.

Proof. Assume that A is dense and randomly generated.
(i) The discussion above implies (i) immediately.
(ii) Let us examine the matrix $A_N^T T$ defined in the proof of Lemma 3.2. As we pointed out in that proof, we must prove that this matrix has maximum rank with probability one. This matrix is $(n - p) \times (m - r)$.
If $(n - p) \geq (m - r)$ then we must prove that its columns are linearly independent. By contradiction, assume that for some $w \neq 0$, $A_N^T T w = 0$. We know that $Tw \neq 0$

because by definition the columns of T are linearly independent, and hence Tw is orthogonal to all rows of A_N^T, which happens with probability zero.
If $(n-p) \leq (m-r)$ then we must prove that its rows are linearly independent. By contradiction, assume that for some $w \neq 0$, $w^T A_N^T T = 0$. Now $A_N w$ has random entries and is orthogonal to the columns of T, which happens with probability zero, completing the proof of (ii).
(iii) With probability one, we have the following: If $p \leq m$, then A_b has rank $r = p$, and by Lemmas 3.1 and 3.2, $D_p = 0$ and $D_d = \min\{n-p, m-p\} = m-p$. If $m \leq p$, then $r = m$ and by the same lemmas $D_p = p - m$ and $D_d = 0$, completing the proof. $\square$

So, if we do not make the matrix A_B intentionally rank deficient, there is no hope of generating problems which are both primal and dual degenerate. If the matrix A is sparse then the rank of A_B is easily controlled, as we shall see in a moment, but $A_N^T T$ may be rank deficient: in this case the primal optimal face will have the required dimension, but the dual optimal face dimension may be smaller than the desired value.

4. The construction. Now we show how to construct a degenerate problem for given n, m, D_p and D_d. Note that the Lemmas 3.1 and 3.2 require that

$$D_p \leq n - m - 1, \quad D_d \leq m - 1.$$

The construction will be done in the following sequence: first we choose r and p, then generate an interior optimal solution x^*, s^*, construct the matrix A and finally compute b and c. We shall describe the construction in words and in Matlab commands, which are preceded by the symbol $\bullet$. The construction is easily understood by watching the figures below. The figures use the Matlab **spy** function and show an example with $n = 320$ and $m = 160$. The final matrices are shown in Fig. 4.2 for two cases, with $D_p > D_d$ and with $D_p < D_d$.

- $r = m - D_d$;
- $p = r + D_p$;

With these values, Lemma 3.1 is trivially satisfied. To prove that Lemma 3.2 holds, we must show that $D_d \leq n - p$. In fact, since $D_p \leq n - m - 1$,

$$D_d = m - r = m - p + D_p \leq n - p - 1,$$

completing the proof.

Now construct the optimal solutions

$$x^* = \begin{bmatrix} x_B \\ 0 \end{bmatrix}, \quad s^* = \begin{bmatrix} 0 \\ s_N \end{bmatrix}, \quad x_B \in \mathbb{R}^p_{++},\ s_N \in \mathbb{R}^{n-p}_{++}.$$

The positive entries will have the format $\epsilon + \textbf{rand}$, where $\epsilon > 0$ is given (say, $\epsilon = 10^{-4}$) and **rand** is a random number in $[0, 1]$.

- $x^* = [\mathit{epsilon} + \textbf{rand}(p, 1);\ \textbf{zeros}(n-p, 1)]$;
- $s^* = [\textbf{zeros}(p, 1);\ \mathit{epsilon} + \textbf{rand}(n-p, 1)]$;

Construction of the matrix.

The matrix $A = [A_B\ A_N]$ will have the following characteristics, which can be observed in the figures:
— The rank is m, which is ensured by introducing a non-zero in each row.
— The rank of A_B is m, which is ensured by the format

$$A = \left[\begin{array}{c|c} A_{B_1} & \\ A_{B_2} & A_N \end{array} \right],$$

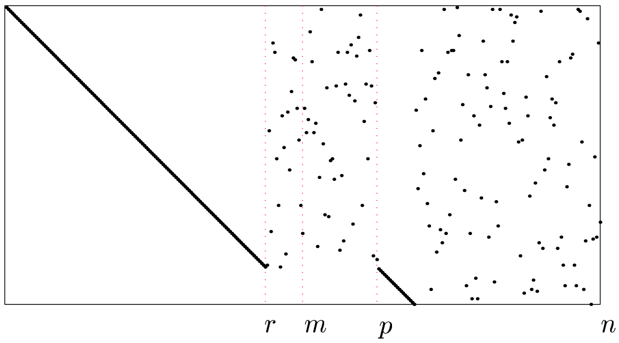

FIG. 4.1. *Matrix with one non-zero per column.*

where $A_{B_2} = 0$ and A_{B_1} is $r \times p$ with non-zeros on the main diagonal.
— Each column has at least on non-zero.

The construction is done in two steps:
First step: one non-zero per column. We construct a vector *row* which associates a row index to each column, and then a sparse matrix with non-zeros in these elements (see Fig 4.1):

$$row = [1, 2, \ldots, r \overset{r}{|} \text{ random in } 1 \ldots r \overset{p}{|} r+1, r+2, \ldots, m \overset{p+m-r}{|} \text{ random in } 1 \ldots n].$$

In Matlab, the construction is:

- **for** $i = 1 : r$, $row(i) = i$; **end;**
- **for** $i = r+1 : p$, $row(i) = \mathbf{floor}(1 + \mathbf{rand} * r)$; **end;**
- **for** $i = 1 : m - r$, $row(p+i) = r + i$; **end;**
- **for** $i = p + m - r : n$, $row(i) = \mathbf{floor}(1 + \mathbf{rand} * m)$; **end;**
- $t = -1 + 2 * \mathbf{rand}(1, n)$;
- $A = \mathbf{sparse(1{:}n, row, t)}$;

The Matlab commands here do the following: $\mathbf{rand}(1, n)$ generates a vector with random entries uniformly distributed in $[0, 1]$ (**rand** alone generates a number). Here we chose to generate the random entries of A uniformly distributed in $[-1, 1]$, but this can of course be changed to use for instance a normal distribution (in Matlab, **randn**(1,n)).

Now the matrix is like in Fig 4.1, with all the desired characteristics, except for the density. Now we spread random entries in the matrix to reach the desired density.
Second step: add random entries.
The total number of non-zeros to be added is $nz = dens \times (m-1)n$, because n non-zeros are already present. The total number of available positions, taking into account that the sub-matrix A_{B_2} is null, is $nt = r\,p + m(n-p)$. Hence the actual density of the matrix to be added is $d = nz/nt$. Now we can use the Matlab command **sprandn** to generate a sparse matrix and then add it to A:

- $R_{11} = \mathbf{sprandn}(r,\ p,\ d)$; % $r \times p$ sparse with density r.
- $R_{12} = \mathbf{sparse}(m - r,\ p)$; % This generates an empty matrix.
- $R_2 = \mathbf{sprandn}(m,\ n-p,\ d)$;
- $A = A + [\,[R_{11}\ ;\ R_{12}]\ R_2]$;

Finally, the construction is completed by computing b and c:

- $b = A\,x^*$;

- $c = s^*$;

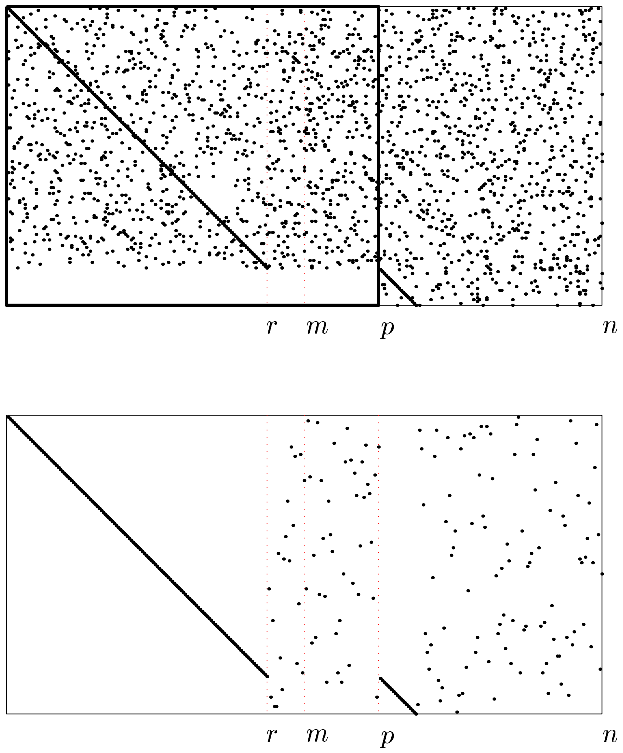

FIG. 4.2. *The final matrix, stressing the sub-matrix A_B, with $n = 320$, $m = 160$. Top: $D_p = 60$, $D_d = 30$. Bottom: $D_p = 40$, $D_d = 70$.*

5. Choosing initial interior solutions. The procedure above gives us strictly complementary optimal solutions (x^*, s^*). We usually need an interior initial solution for testing algorithms. Here we present a simple procedure for changing the data so that a given positive pair (x^0, s^0) is feasible. This pair may be randomly generated, or may use the vectors of ones, $x^0 = s^0 = [1\,1\,\ldots,1]^T$.

Let $(x^0, s^0) \in \mathbb{R}^{2n}_{++}$ be given and assume that

$$(x^0)^T s^* + (s^0)^T x^* = (x^0)^T s^0. \tag{5.1}$$

This condition can be easily ensured by multiplying x^* and s^* by a convenient constant before computing b and c. Here is the procedure:

— Change the last column of A to make x^0 primal feasible: define $A = [A_J\ A_n]$, where $J = \{1, 2, \ldots, n-1\}$, and set

$$A_n = (b - A_J x^0_J)/x^0_n.$$

In Matlab,

- $A(:, n) = (b - A(:, 1:n-1) * x^0(1:n-1))/x^0(n)$;

In fact, we now have $Ax^0 = A_J x_J^0 + A_n x_n = b$, and so x^0 is feasible. We also have $Ax^* = b$, because $x_n^* = 0$.

— Change the first row of A, to make s^0 dual feasible. Set

- $A(1, :) = (s^0 - c)'$;
- $b(1) = A(1, :) * x^*$;

Let us prove that s^0 and s^* are feasible. Denote the first row of A by a_1 and set $y^0 = (-1, 0, \ldots, 0)$. We have

$$A^T y^0 + s^0 = -a_1^T + s^0 = c,$$

by substitution of $a_1^T = s^0 - c$. So s^0 is feasible. To prove that s^* is feasible we must only check that $a_1 x^0 = b_1 = a_1 x^*$. Let us then prove that $a_1(x^0 - x^*) = 0$. Substituting $a_1 = (s^0 - c)^T = (s^0 - s^*)^T$, we get

$$(s^0 - s^*)^T (x^0 - x^*) = (x^0)^T s^0 - (x^0)^T s^* - (s^0)^T x^* + (x^*)^T s^* = 0,$$

because $(x^*)^T s^* = 0$ by construction and the remaining terms vanish due to (5.1), completing the proof.

This construction generates a dense column A_n, which may be undesirable. This can be avoided by the following procedure: instead of changing the elements $A(n, i)$, $i = 1, \ldots, m$, use the same procedure for changing elements $A(j, i)$, $i = 1, \ldots, m$ for j chosen (possibly at random) in $\{p + 1, \ldots, n\}$.

Note finally that these procedures increase the density of the matrix A. If this is undesirable, then the expected number of new non-zero elements (approximately equal to $m + n$) must be taken into consideration in the computation of the density d.

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