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Chebyshev pseudospectral method for wave equation with absorbing boundary conditions that does not use a first order hyperbolic system

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Abstract

The analysis and solution of wave equations with absorbing boundary conditions by using a related first order hyperbolic system has become increasingly popular in recent years. At variance with several methods which rely on this transformation, we propose an alternative method in which such hyperbolic system is not used. The method consists of approximation of spatial derivatives by the Chebyshev pseudospectral collocation method coupled with integration in time by the Runge-Kutta method. Stability limits on the timestep for arbitrary speed are calculated and verified numerically. Furthermore, theoretical properties of two methods by Jackiewicz and Renaut are derived, including, in particular, a result that corrects some conclusions of these authors. Numerical results that verify the theory and illustrate the effectiveness of the proposed approach are reported.

Keywords: Wave equation; Chebyshev pseudospectral methods; Eigenvalue stability; Pseudoeigenvalues

1. Introduction

We consider some numerical methods for the 1D wave equation

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, && 0 < x < 1,\ t > 0,\\ &u(x,0) = f(x), && 0 < x < 1,\\ &u_t(x,0) = 0, && 0 < x < 1 \end{aligned} \tag{1}$$

together with one-way wave equations as boundary conditions:

$$\begin{aligned} &u_t(0,t) - c u_x(0,t) = 0, && t > 0\\ &u_t(1,t) + c u_x(1,t) = 0, && t > 0 \end{aligned}. \tag{2}$$

Boundary conditions like these are introduced to avoid or absorb spurious reflections on the boundary, see, e.g., [3,5,9,11,12,15,19].

The pseudospectral collocation approach [1,6,8,10,11,13,16,18] has become an efficient way to generate several methods for solving time dependent partial differential equations (PDEs) due to its high precision and relatively lower computational cost compared with finite difference methods. These methods start by approximating spatial derivatives by the Chebyshev pseudospectral collocation (CPS) method, giving rise to a set of ordinary differential equations

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(ODEs) in which only the first time derivative appears. This set, referred to as semidiscrete system, is then integrated in time using a great variety of schemes.

As for the initial and boundary problem (1)–(2), a subclass of CPS methods that has been increasingly exploited in recent years is the one in which the analysis and solution of the problem is done by using a related first order hyperbolic system of PDEs [4,11,10,15]. Although these methods allow certain degree of flexibility for improvement of the accuracy because integration in time can be done by using an appropriate method for ODEs, some care is required when using them as the overall accuracy of computed solutions depends strongly on the chosen method and as the solution u is calculated from approximations to u_t. For this reason, methods that do not use such hyperbolic system of PDEs and that allow the calculation of u without using any estimate of u_t seem desirable. It is the object of the present paper to propose such a method. Essentially, we transform the PDE into a first order system of ODEs and then integrate in time by using the fourth-order Runge–Kutta (RK) method. Although the order of accuracy in time can be considered low when compared with the one of spatial derivatives, this is not a serious difficulty since highly accurate solutions can be constructed by taking the timestep fairly small while maintaining the storage requirements within reasonable bounds.

The paper is structured as follows. In Section 2 we reintroduce some methods of Jackiewicz and Renaut [11,10] and then describe the proposed method. Theoretical properties of all methods are described here, including, in particular, a result that corrects some conclusions in [11]. Stability of the proposed method as well as the stability limits on the timestep for arbitrary speed are included in Section 3. In Section 4 we give a numerical example that illustrates the effectiveness of the proposed approach. The paper ends with some concluding remarks in Section 5.

2. Methods and properties

As commented before, the methods we are interested in, use approximations to first order spatial derivatives calculated by the Chebyshev pseudospectral collocation method. From here on we use as collocation points the so-called Chebyshev–Gauss–Lobatto points, numbered from left to right on the interval [0,1], defined by

$$0 = x_0 < x_1 < \cdots < x_n = 1, \quad \text{with} \quad x_j = \frac{1}{2}[1 - \cos(j\pi/n)], \quad j = 0, \ldots, n. \tag{3}$$

The objective of this section is to review some methods by Jackiewicz and Renaut [10,11], as well as to describe the proposed method along with underlying properties. To this end, we start by describing some definitions and notation.

Throughout the paper the first order $(n+1) \times (n+1)$ Chebyshev differentiation matrix associated with these collocation points is denoted by D. The columns (resp., rows) of D are denoted by d_j (resp., l_j^T). Let D_1, D_2, and D_3 be matrices defined by

$$D_1 = [d_2, \ldots, d_n], \quad D_2 = [l_2, \ldots, l_n]^T, \qquad E = [d_1\, 0_{(n+1)\times(n-1)}\ -d_{n+1}], \tag{4}$$

where $0_{m\times n}$ denotes the $m \times n$ matrix having all entries equal to zero; when $m = n$ we simply write 0_m. Accordingly, I_{n+1} denotes the identity matrix of order $n+1$.

2.1. First order hyperbolic system-based methods

These methods rely on a conversion of the second order partial differential equation to a first order hyperbolic system of PDEs. With the notation $u_1(x,t) = u_t(x,t)$, and $u_2(x,t) = u_x(x,t)$, problem (1)–(2) is converted to the first order hyperbolic system

$$\begin{cases} u_{1,t} = c^2 u_{2,x}, & 0 < x < 1,\ t > 0, \\ u_{2,t} = u_{1,x}, & 0 < x < 1,\ t > 0, \end{cases} \tag{5}$$

subject to the initial and boundary conditions

$$\begin{aligned} &u_1(x,0) = 0, && 0 \le x \le 1,\\ &u_2(x,0) = f_x(x), && 0 \le x \le 1,\\ &u_1(0,t) - cu_2(0,t) = 0, && t > 0\\ &u_1(1,t) + cu_2(1,t) = 0, && t > 0 \end{aligned} \tag{6}$$

Approximation of spatial first order derivatives via the differentiation matrix D leads to several first order semidiscrete systems of ODEs which depend on the way the boundary conditions are incorporated into the model. To simplify the notation, here and throughout we use $u_1(x_j, t)$, $0 \le j \le n$, to denote both the exact value and the corresponding approximation arising from solving the related semidiscrete system; the same comment applies to $u_2(x_j, t)$ and $u(x_j, t)$. We shall concentrate on two methods by Jackiewicz and Renaut [11,10] which are briefly described as follows.

If the boundary conditions are incorporated by eliminating the variables $u_2(0, t)$ and $u_2(1, t)$ taking $u_2(0, t) = u_1(0, t)/c$ and $u_2(1, t) = -u_1(1, t)/c$, the semidiscrete system is

$$\begin{cases} U_{1,t} = A_1 U_1, \quad t > 0,\\ U_1(0) = [0_{1\times n+1}\ f_x(x_1), \ldots, f_x(x_{n-1})]^T \end{cases} \tag{7}$$

where

$$A_1 = \left[\begin{array}{c|c} c\,d_1 0_{(n+1)\times(n-1)} - cd_{n+1} & c^2 D(1:n+1, 2:n) \\ \hline D(2:n, 1:n+1) & 0_{(n-1)\times(n-1)} \end{array}\right],$$

and

$$U_1 = [u_1(x_0, t), \ldots, u_1(x_n, t), u_2(x_1, t), \ldots, u_2(x_{n-1}, t)]^T.$$

As in [11], MATLAB notation is being used. If, on the other hand, we choose $u_1(0, t) = cu_2(0, t)$ and $u_1(1, t) = -cu_1(1, t)$, the resulting semidiscrete system is

$$\begin{cases} U_{2,t} = A_2 U_2,\ t > 0,\\ U_2(0) = [0_{1\times(n-1)}\ f_x(x_0), \ldots, f_x(x_n)]^T \end{cases} \tag{8}$$

where

$$A_2 = \left[\begin{array}{c|c} 0_{(n-1)\times(n-1)} & c^2 D(2:n, 1:n+1) \\ \hline D(1:n+1, 2:n) & c\,d_1 0_{(n+1)\times(n-1)} - cd_{n+1} \end{array}\right],$$

and

$$U_2 = [u_1(x_1, t), \ldots, u_1(x_{n-1}, t), u_2(x_0, t), \ldots, u_2(x_n, t)]^T.$$

We now observe that because time integration of these systems does not give approximations to the solution u, but rather to u_1 and u_2, Jackiewicz and Renaut proposed the following two methods to calculate u from u_1

Method 1:

1. Step 1: Calculate approximations to u_1 by the fourth-order RK method.
2. Step 2: For $j = 0, \ldots, n$ and $v \ge 0$, calculate

$$u(x_j, t_{v+3}) = u(x_j, t_v) + \Delta t[\beta_0 u_1(x_j, t_v) + \beta_1 u_1(x_j, t_{v+1}) + \beta_2 u_1(x_j, t_{v+2}) + \beta_3 u_1(x_j, t_{v+3}),] \tag{9}$$

where $\beta_0 = -51/8$, $\beta_1 = 171/8$, $\beta_2 = -153/8$ and $\beta_3 = 57/8$.

The second method proposes integration in time by using a fourth-order diagonally implicit multistage integration method (DIMSIM). In these methods, approximations to u at time t_{v+1} are calculated by using approximations to u_1 at interior points of the interval $[t_v, t_{v+1}]$ (called internal stages). The method is as follows.

Method 2:

For $j = 0, \ldots, n$ and $v \geq 0$, calculate

$$u(x_j, t_{v+1}) = u(x_j, t_v) + \Delta t[\gamma_0 u_1(x_j, t_v + c_1\Delta t) + \gamma_1 u_1(x_j, t_v + c_2\Delta t) + \gamma_2 u_1(x_j, t_v + c_3\Delta t) + \beta_3 u_1(x_j, t_v + c_4\Delta t)] \tag{10}$$

where $\gamma_0 = \gamma_3 = 13/48$, $\gamma_1 = \gamma_2 = 11/48$, $c_1 = 1/8$, $c_2 = 3/8$, $c_3 = 5/8$, $c_4 = 7/8$.

The principal characteristic of Method 2 is that the internal stages $u_1(x_j, t_v + c_i\Delta t)$, $i = 1, \ldots, 4$, as well as the estimates $u(x_j, t)$ are all $\mathcal{O}(\Delta t^4)$ accurate; the reader is referred to [2,10] for details. As a consequence, Method 2 should outperform Method 1; explanation about this can be found in [10].

A comment is required concerning a conclusion in a paper by Jackiewicz and Renaut [11]. In that paper the authors analyze numerically the spectra of matrices A_1 and A_2 and conclude, among other things, that the spectrum of A_1 differs from the spectrum of A_2 (see Section 2 in [11]). The following proposition shows this conclusion is incorrect.[1]

Proposition 1. *System matrices A_1 and A_2 are similar.*

Proof. Define

$$S = \begin{bmatrix} 0_{(n-1)\times(n+1)} & c^2 I_{n-1} \\ I_{n+1} & 0_{(n+1)\times(n-1)} \end{bmatrix}.$$

It follows that S is non singular and that its inverse is

$$S^{-1} = \begin{bmatrix} 0_{(n+1)\times(n-1)} & I_{n+1} \\ \frac{1}{c^2} I_{n-1} & 0_{(n-1)\times(n+1)} \end{bmatrix}.$$

Using these matrices it is straightforward to see that

$$A_1 = S^{-1} A_2 S,$$

as claimed. □

Remark. It is worth noting that the statement of the above proposition continues to hold irrespective of the grid points being used, i.e., we might use, for instance, Legendre points, and the assertion will be still valid. Additional consequences of this proposition will be commented in Section 4.

The eigenvalues of these matrices, which come in complex conjugate pairs, are very sensitive to small perturbations such as rounding errors, except for some "outlier eigenvalues". This is illustrated in Fig. 1. Notice that for $n = 31$ the eigenvalues of A_1 match well those of A_2; this is not the case when $n = 63$. These results do not contradict Proposition 1, of course, they simply illustrate that matrices A_1 and A_2 are far from normal.

A useful measure of closeness to normality of $A \in \mathbb{R}^{m\times m}$ is the Jordan condition number of the corresponding eigenproblem. If A has a Jordan decomposition $A = V\Lambda V^{-1}$, the 2-norm Jordan condition number is

$$\kappa(V) = \|V\|_2 \|V^{-1}\|_2.$$

A a consequence, the smaller $\kappa(V)$, the closer A is to normality.

[1] After contacting Z. Jackiewicz, we now know that the analysis in [11] relies on a "system matrix A_1" that differs from that in Eq. (7) (Eq. (2.6) in [11]) and arises from one of the so many ways of incorporating the boundary conditions into the model. This clarifies what is claimed in [11] on the spectra of matrices A_1 and A_2 and related issues.

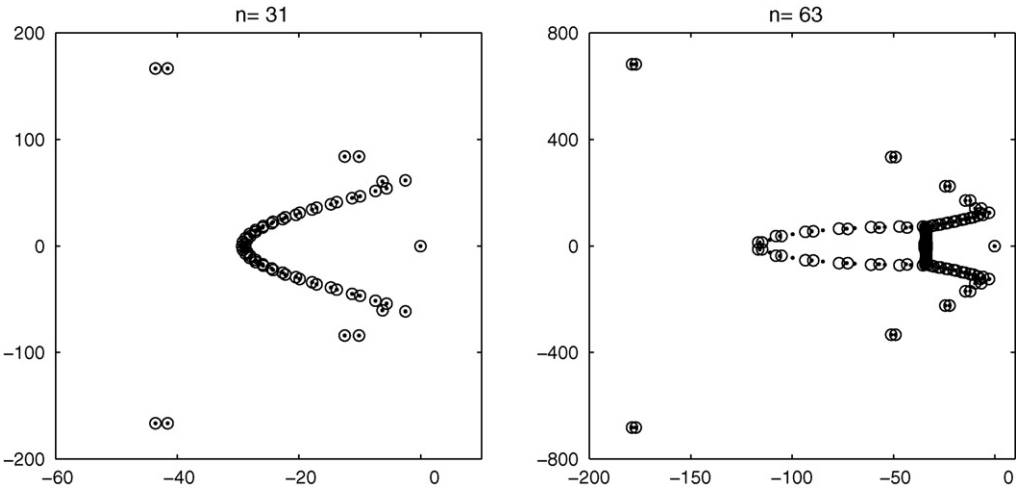

Fig. 1. Eigenvalues of system matrices A_1 (circles) and A_2 (dots).

As for the Jordan condition numbers of A_1 and A_2, for $c = 1$, $n = 31,\ 63$, we have

$$\kappa(V_1) \approx 6.5194e + 11, 8.5281e + 14, \quad \kappa(V_2) \approx 6.5191e + 11, 9.4652e + 14$$

which explains the behavior of eigenvalues displayed in Fig. 1.

An important consequence of Proposition 1 is that the Jordan condition numbers of the eigenproblems associated with system matrices A_1 and A_2 relate to each other by a constant factor that depends on the speed c. This is made precise in the following proposition.

Proposition 2. *Let $\kappa(V_1)$, $\kappa(V_2)$ be the Jordan condition numbers of the eigenvalue problems associated with A_1 and A_2, respectively. These numbers relate to each other by a constant factor not larger than $\Xi \geq 1$* defined by

$$\Xi = \begin{cases} c^2 & \text{if } c > 1, \\ 1 & \text{if } c = 1, \\ \dfrac{1}{c^2} & \text{if } c < 1. \end{cases}$$

Proof. Let $A_1 = V_1 \Lambda V_1^{-1}$ be a Jordan decomposition of A_1. By Proposition 1, the 2-norm Jordan condition number $\kappa(V_2)$ satisfies

$$\kappa(V_2) = \|SV_1\|_2 \|V_1^{-1} S^{-1}\|_2 \leq \|S\|_2 \|S^{-1}\|_2 \|V_1\|_2 \|V^{-1}\|_2 = \kappa(S)\kappa(V_1). \tag{11}$$

Now observe that if $c > 1$, because the columns of matrix S are orthogonal, we have $\|S\|_2 = c^2$, $\|S^{-1}\|_2 = 1$, and hence $\kappa(S) = c^2$. A similar reasoning leads to $\kappa(S) = 1/c^2$ if $c < 1$, and $\kappa(S) = 1$ if $c = 1$. Insertion of this value in (11) shows that $\kappa(V_2) \leq \Xi\kappa(V_1)$. If we reverse the roles of V_1 and V_2 in the above procedure we obtain $\kappa(V_1) \leq \Xi\kappa(V_2)$. What $\Xi \geq 1$ follows from the definition itself. This ends the proof. □

The conclusion that can be drawn from Proposition 1 and the above numerical results is that the eigenproblems associated with system matrices A_1 and A_2 are either almost equally conditioned when $c \neq 1$ or equally conditioned when $c = 1$, and that for any choice of the speed c, the corresponding condition numbers can be extremely huge when n is sufficiently large. From the practical point of view, this means that the stability properties of time integration methods for the semidiscrete systems (7) and (8) should not differ significantly.

2.2. Block companion approach

To describe the proposed method we need some notation. Let $\bar{u} = [u(x_0, t), \ldots, u(x_n, t)]^T$. Since the inner product of jth row of D and $\bar{u}$ approximates $u_x(x_{j-1}, t)$, $1 \le j \le n+1$, the boundary conditions (2) imply that

$$l_1^T \bar{u} \approx u_x(x_0, t) = \frac{u_t(x_0, t)}{c}, \quad l_{n+1}^T \bar{u} \approx u_x(1, t) = -\frac{u_t(1, t)}{c}. \tag{12}$$

In view that

$$\bar{u}_{tt} = c^2 \bar{u}_{xx} \approx c^2 D^2 \bar{u},$$

using (12) we obtain

$$D^2\bar{u} = d_1 l_1^T \bar{u} + (d_2 l_2^T \bar{u} + \cdots + d_n l_n^T \bar{u}) + d_{n+1} l_{n+1}^T \bar{u} \approx [d_1/c \quad 0_{(n+1)\times(n-1)} \quad -d_{n+1}/c]\bar{u}_t + D_1 D_2 \bar{u}.$$

Neglecting the approximation error we obtain the set of second order system of ODEs

$$\bar{u}_{tt} = cE\bar{u}_t + c^2 D_1 D_2 \bar{u}. \tag{13}$$

If we introduce

$$U_3 = [u_t(x_0, t), \ldots, u_t(x_n, t), u(x_0, t), \ldots, u(x_n, t)]^T,$$

a first order system in companion block form associated with (13) is

$$\begin{cases} U_{3,t} = A_3 U_3, \\ U_3(0) = [0_{1\times(n+1)},\ f(x_0), \ldots, f(x_n)]^T, \end{cases} \tag{14}$$

where

$$A_3 = \left[\begin{array}{c|c} cE & c^2 D_1 D_2 \\ \hline I_{n+1} & 0_{n+1} \end{array}\right] \tag{15}$$

with D_1, D_2 and E being defined in (4). Observe that A_3 is now of order $2n+2$, as opposed to matrices A_1 and A_2 which are of order $2n$. In spite of this, an important characteristic of the semidiscrete system (14) is that the related system matrix contains essentially the same information as A_1 and A_2. This is the subject of the following proposition.

Proposition 3. *Except for a zero eigenvalue of multiplicity two*, $\lambda(A_3)$, *the spectrum of matrix* A_3, *equals* $\lambda(A_1)$, *the spectrum de* A_1.

Proof. We first recall a well known result from numerical linear algebra concerning eigenvalues. Let $A \in \mathbb{R}^{m\times q}$ and $B \in \mathbb{R}^{q\times m}$, $m \ge q$. Then except for $(m-q)$ zero eigenvalues, we have that $\lambda(AB) = \lambda(BA)$.

Now observe that A_3 can be rewritten as

$$A_3 = \left[\begin{array}{c|c} cE & c^2 D_1 D_2 \\ \hline I_{n+1} & 0_{n+1} \end{array}\right]_{(2n+2)\times(2n+2)} = \left[\begin{array}{c|c} cE & c^2 D_1 \\ \hline I_{n+1} & 0_{(n+1)\times(n-1)} \end{array}\right] \left[\begin{array}{c|c} I_{n+1} & \\ \hline & D_2 \end{array}\right].$$

Application of the above property to the right hand side of the previous equality gives that, except for two zero eigenvalues, we have

$$\lambda(A_3) = \lambda\left(\left[\begin{array}{c|c} I_{n+1} & \\ \hline & D_2 \end{array}\right] \left[\begin{array}{c|c} cE & c^2 D_1 \\ \hline I_{n+1} & 0_{(n+1)\times(n-1)} \end{array}\right]\right) = \lambda(A_1),$$

which proves the assertion of the proposition. □

Another point to be discussed here is the dependence of the spectral radius of the system matrices on constant c. To highlight the dependence of the system matrix A_3 on n and c, we shall use the notation $A_3(n, c)$ instead of A_3. Our goal is to relate the spectrum of $A_3(n, c)$ for arbitrary $c > 0$ to the spectrum of $A_3(n, 1)$.

Proposition 4. *Let* $\rho(A_3(n,c))$ *denote the spectral radius of the system matrix* $A_3(n,c)$ *defined in* (15). *Then* $\lambda(A_3(n,c)) = c\lambda(A_3(n,1))$, *and thus*

$$\rho(A_3(n,c)) = c\,\rho(A_3(n,1)), \tag{16}$$

and the same property continues to hold if A_3 is replaced with A_1 or A_2.

Proof. Let $A_4 = ZA_3(n,c)Z^{-1}$ where Z is defined by

$$Z = \left[\begin{array}{c|c} \frac{1}{c}I_{n+1} & 0 \\ \hline 0 & I_{n+1} \end{array}\right].$$

A simple calculation reveals that $A_4 = cA_3(n,1)$. This proves that $\lambda(A_3(n,c)) = c\lambda(A_3(n,1))$, which in turn proves (16); the last part of the statement of the proposition is a consequence of Proposition 3. □

The method proposed in this work relies on the observation that matrix A_4 is better scaled than A_3, in which case the eigenvalue problem of the former is better conditioned than the eigenvalue problem of the latter [7]. Essentially, we change the system coordinates U_3 in (14) to $U_4 = ZU_3$, which leads to the semidiscrete system

$$\begin{cases} U_{4,t} = A_4U_4, \\ U_4(0) = [0_{1\times(n+1)},\ f(x_0),\ldots,f(x_n)]^T, \end{cases} \tag{17}$$

and propose to integrate in time by the fourth-order Runge–Kutta method. The method, which we denote by CPS-RK, can be described as follows.

Method 3 (CPS-RK): For $v \geq 0$, calculate

$$U_4(t_{v+1}) = U_4(t_v) + \frac{\Delta t}{6}(F_1 + 2F_2 + 2F_3 + F_4)$$

where

$$F_1 = A_4U_4(t_v), \quad F_2 = A_4\left(U_4(t_v) + \frac{\Delta t}{2}F_1\right)$$
$$F_3 = A_4\left(U_4(t_v) + \frac{\Delta t}{2}F_2\right), \quad F_4 = A_4\left(U_4(t_v) + \Delta tF_3\right).$$

The principal feature of using the semidiscrete system (17) is that now $u(x_j,t)$ can be calculated directly from initial data, as opposed to the approaches of Jackiewicz and Renaut which estimate u from approximations to u_1 and u_2. As far as accuracy is concerned, CPS-RK will certainly outperform Method 1 but nothing can be affirmed with respect to its performance compared with Method 2 because both methods are fourth-order accurate. Even so, it is worth emphasizing that CPS-RK has the capability to construct highly accurate approximations by simply taking the stepsize Δt fairly small.

3. Stability of the proposed method

In this section we discuss the choice of the maximum stepsize that assures stable integration for a given spatial grid. A discussion about this involving matrices A_1 and A_2 is carried out by Jackiewicz and Renaut in [11]. They analyze the spectra and ϵ-pseudospectra of these matrices and conclude that the maximum stepsize that assures stability of his RK-based method can be chosen by requiring the spectrum of the system matrix scaled by Δt to lie within the stability region of the method. Limits on allowable stepsizes for stability of both formulations for speed $c = 1$ are also reported (Table 1 in [11]). Ignoring the fact that most part of these reported values are incorrect because they contradict the statement of Proposition 1, what must be observed here is that the conclusion of these authors concerning the appropriateness of eigenvalue analysis for stability is actually correct. This is supported by the results displayed in Fig. 2. With regard to the stability limits on the stepsizes for the proposed method, they can be found by coupling the line of analysis of Jackiewicz and Renaut with Proposition 4. They are displayed in Table 1 for several values of n. Observe that in the sense of "eigenvalue stability", these values apply to all matrices A_i, $i = 1,2,3,4$, since by Proposition 4, all these matrices share the same spectral radius.

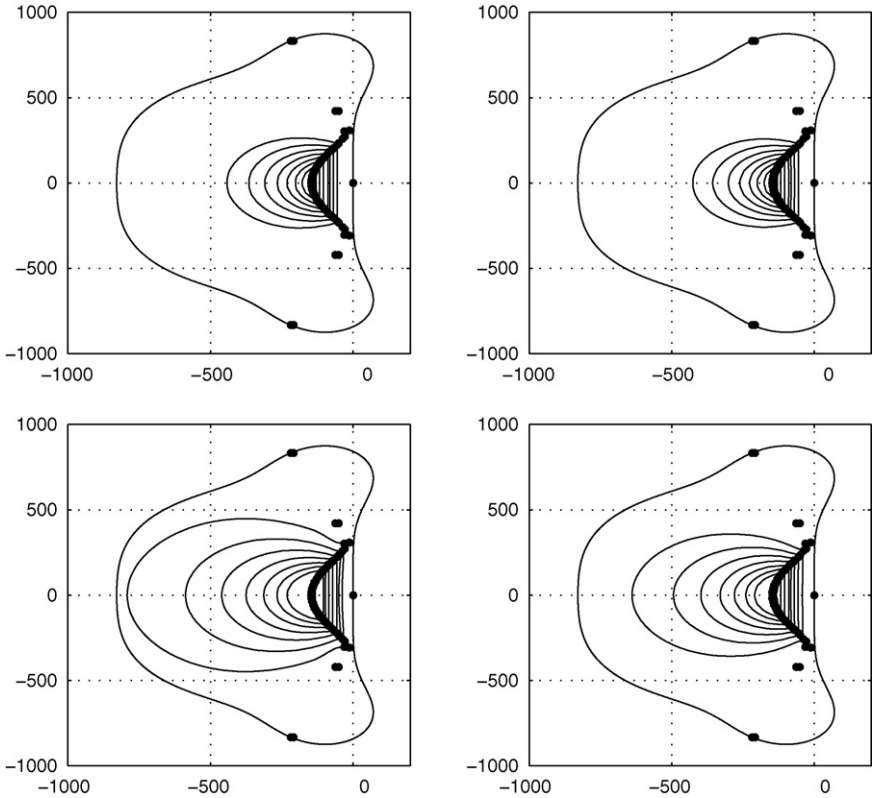

Fig. 2. Region of absolute stability of fourth-order Runge–Kutta method scaled by $1/\Delta t_{\max}$ and spectra and ϵ-pseudospectra of system matrices A_1 (top left) A_2 (top right), A_3 (bottom left), and A_4 (bottom right), for $\epsilon = 10^{-9}, 10^{-8}, \ldots, 10^{-3}$ (from inner to outer), $n = 31$, $c = 5$, and $\Delta t_{\max} = 0.01680$.

Since these stability limits were shown to hold in practice in [11] when solving the semidiscrete system involving matrix A_2, it remains to verify whether the same occurs in conjunction with the system matrix A_4 of the proposed method. As demonstrated in [14] and illustrated in [17,18], the right way to carry out such verification is via pseudoeigenvalues. In words, this requires choosing the timestep in such a way that the pseudospectra of A_4 lie within a distance $\mathcal{O}(\epsilon) + \Delta t$ of the scaled region of stability as $\epsilon \to 0$ and $\Delta t \to 0$ [14,18]. The spectra and ϵ-pseudospectra of all matrices A_i for $n = 31$ and the scaled region of stability of RK method are displayed in Fig. 2.

Several conclusions can be drawn from Fig. 2. First, it shows that the eigenvalues of matrices A_1 and A_2 are almost equally sensitive to perturbation, as predicted in theory (see Proposition 2), and that among the eigenvalue problems associated with matrices A_i ($i = 1 : 4$), the most sensitive to perturbation is the one associated with matrix A_3. Moreover, in view that the ϵ-pseudospectra of A_4 are well within the scaled region of stability, the most important conclusion is that the pseudoeigenvalue stability criterion is indeed satisfied, thus verifying that the $\Delta t_{\max}$ calculated using the standard eigenvalue analysis is appropriate in this case.

Table 1
Maximum stable stepsize for arbitrary speed c, for matrices A_i, $i = 1, 2, 3, 4$ of several orders, according to eigenvalue stability analysis

$n+1$	$c\Delta t_{\max}$ for A_i, $i = 1 : 4$
4	0.61413877092330
8	0.23643588530295
16	0.06906986008198
32	0.01680029485722
64	0.00410202619991
128	0.00101143197421

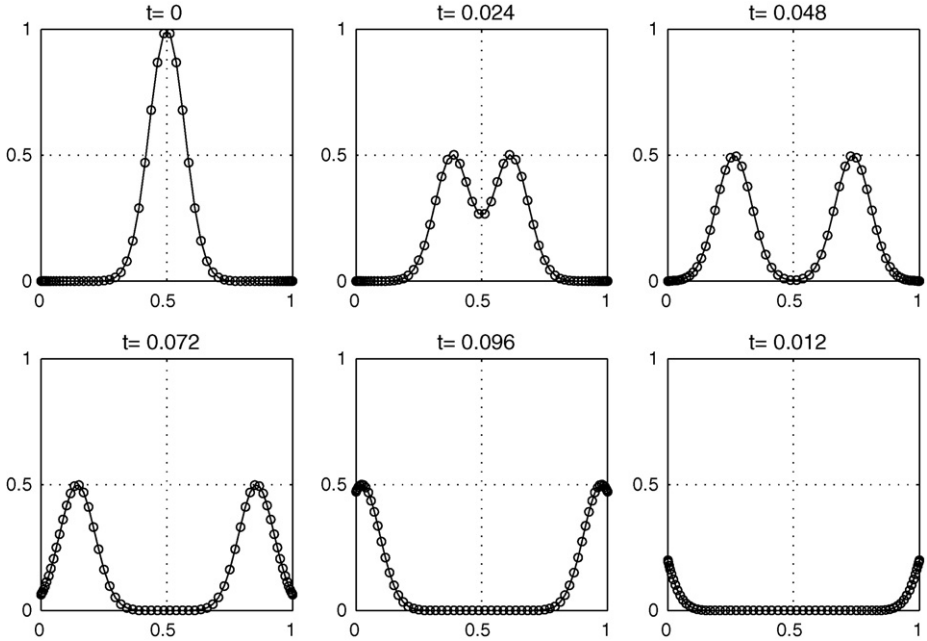

Fig. 3. Exact and CPS-RK-based approximate solution for 1D wave equation.

4. Numerical example

We give a numerical example to illustrate the effectiveness of the proposed method. The example addresses the solution of a 1D wave equation with speed $c = 5$ and initial condition defined by

$$u(x, 0) = e^{-100(x-0.5)^2}$$

The problem was solved on the interval $[0, 0.12]$, $n + 1 = 64$, and $\Delta t = 0.004/c = 0.0008$ (see Table 1). This test problem is taken from [11] in which the solution is computed on the interval [0,0.6] with $c = 1$ and $\Delta t = 0.004$. For comparison, we also solve the problem using Method 1 and Method 2. For Method 2 we provide exact internal stages $u_1(x_j, t_v + c_i\Delta t)$ (see (10), and we do so to illustrate that this method should not outperform CPS-RK. All numerical

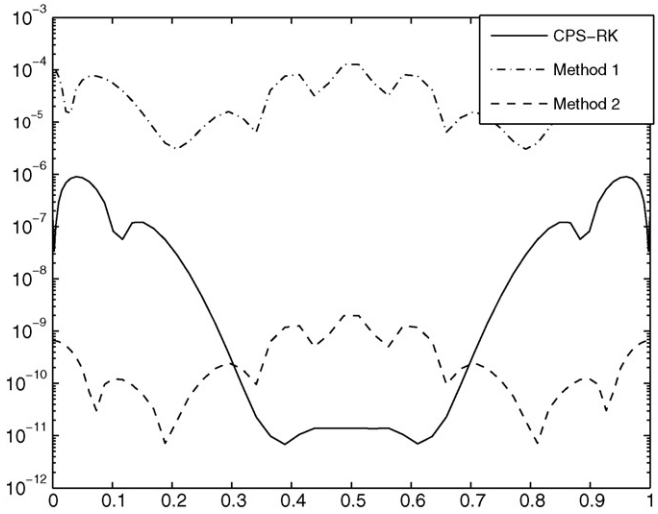

Fig. 4. Error in absolute value of approximate solutions at time level $t = 0.12$.

computation were carried out in MATLAB. In our implementation we use a slightly modified version of function *cheb.m* by Trefethen [18] (see also, Trefethen and Embree [17], p. 410).

The numerical results of PCS-RK method are presented in Fig. 3, in which the exact solution is plotted by solid line and approximate solutions by small circles.

To assess the performance of the methods, the error of the approximate solutions was calculated. This is simple to do since for 1D problems the exact solution is known in closed form. Fig. 4 presents the error in absolute value of numerical solutions calculated by the tested methods. The numerical results are clear: even that Method 2 is shown in an advantageous light because exact internal stages were provided, it does not outperform CPS-RK, as expected.

5. Concluding remarks

We have presented a method for numerical simulation of 1D wave propagation with absorbing boundary conditions that relies on the Runge–Kutta method and that does not use a first order hyperbolic system. Doing so, we maintained the simplicity and accuracy of the Runge–Kutta method and avoided the difficulty of recovering the solution u from low order approximations to derivatives u_t, which is typical of methods like Method 1 and Method 2. Stability of the proposed method was also discussed and the stability limits on the stepsize for arbitrary speed were calculated and verified in practice.

The method we have presented can be extended to 2D wave equations without difficulties. Indeed, a block companion approach for 2D problems appears naturally following the same line of reasoning as the one used in Section 3. This extension was not included here because the error of the approximate solutions for 2D problems as well as the assessment of how much reflection is avoided at the artificial numerical boundaries is not as simple to estimate as for the 1D case. This is the subject of future work.

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References

[1] F.S.V. Bazán, M.C. Cunha, Solving damped wave equations with the method of lines and quadratic matrix eigenvalues, Far East J. Appl. Math. 17 (1) (2004) 105–120.
[2] J.C. Butcher, Z. Jackiewicz, Implementation of diagonally implicit multistage integration methods for ordinary differential equations, SIAM J. Numer. Anal. 31 (6) (1997) 2119–2141.
[3] R.W. Clayton, B.Q. Engquist, Absorbing boundary conditions for acoustic and elastic wave equations, Bull. Seismic Soc. Am. 67 (6) (1977) 1529–1540.
[4] T.A. Driscoll, L.N. Trefethen, pseudospectra for the wave equation with a absorbing boundary, J. Comput. Appl. Math. 69 (1996) 125–142.
[5] B. Engquist, A. Majda, Absorbing boundary conditions for the numerical simulation of waves, Math. Comput. 31 (1977) 629–651.
[6] B. Fornberg, A Practical Guide to Pseudospectral Methods, Cambridge University Press, Cambridge, 1996.
[7] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd ed., Johns Hopkins University Press, Baltimore, 1996.
[8] D. Gotlieb, S.A. Orzag, Numerical Analysis of Spectral Methods: Theory and Practice, SIAM, Philadelphia, PA, 1977.
[9] L. Halpern, Absorbing boundary conditions for the discretization schemes of the one-dimensional wave equation, Math. Comput. 38 (158) (1982) 415–429.
[10] Z. Jackiewicz, R.A. Renaut, Diagonally implicit multistage integration for pseudospectral solutions of the wave equations, Appl. Numer. Math. 34 (2000) 219–229.
[11] Z. Jackiewicz, R.A. Renaut, A note on stability of pseudospectral methods for wave propagation, J. Comp. Appl. Math. 143 (2002) 127–139.
[12] R. Kosloff, D. Kosloff, Absorbing boundaries for wave propagation problems, J. Comput. Phys. 63 (1986) 363–376.
[13] D. Kosloff, H. Tal Ezer, Modified Chebyshev pseudospectral method with $O(N^{-1})$ time step restriction, J. Comput. Phys. 104 (1993) 457–469.
[14] S. Reddy, N. Trefethen, Stability analysis of the method of lines, Numer. Math. 62 (1992) 237–267.
[15] R.A. Renaut, Stability of Chebyshev pseudospectral solution of the wave equation with absorbing boundaries, J. Comput. Appl. Math. 87 (1997) 243–259.
[16] R.A. Renaut, J. Fröhlich, A pseudospectral Chebyshev method for 2-D wave equation with domain stretching and absorbing boundary conditions, J. Comput. Phys. 124 (1996) 324–336.
[17] L.N. Trefethen, M. Embree, Spectra and Pseudospectra—The Behavior of Nonnormal Matrices and Operators, Princeton University Press, Princeton, 2005.
[18] L.N. Trefethen, Spectral Methods in Matlab, SIAM, Philadelphia, PA, 2000.
[19] K. Veselić, On linear vibrational systems with one dimensional damping, Applicable Analysis 29 (1988) 1–18.