GALOIS CORRESPONDENCES FOR PARTIAL GALOIS AZUMAYA EXTENSIONS

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Abstract. Let α be a partial action, having globalization, of a finite group G over a unital ring R. Let R^α denote the subring of the α-invariant elements of R and $C_R(R^\alpha)$ the centralizer of R^α in R. In this paper we show that there are one-to-one correspondences among sets of suitable separable subalgebras of R, R^α and $C_R(R^\alpha)$. In particular, we extend to the setting of partial group actions similar results due to F. DeMeyer [4], and R. Alfaro and G. Szeto [2].

1. Introduction

DeMeyer [4], Kanzaki [10] and Harada [8] investigated central Galois algebras (Galois algebras A over k such that k is the center of A) and Alfaro and Szeto [1] generalized this class of algebras to the class of Galois Azumaya extensions (Galois extensions of an Azumaya algebra) as well as characterized such extensions in terms of properties of the corresponding skew group ring [1, Theorem 1] (see also [2, Theorem 1]). In [2] Alfaro and Szeto pushed ahead their study started in [1] and presented two nice one-to-one correspondence theorems for Galois Azumaya extensions [2, Theorems 2 and 3], the last one being a generalization of a similar result due to DeMeyer [4, Lemma 2].

In this paper we will show that the Alfaro-Szeto's results in [2] in fact hold in the more general setting of the partial group actions having globalization.

Throughout, rings and algebras are always associative and unital. For any ring R, any non-empty subset X of R and any subring Y of R we will denote by $C_Y(X)$ the centralizer of X in Y. If $X = Y = R$ then $C_Y(X)$ is the center of R and we will denote it simply by $C(R)$.

Following [6], a *partial action* α of a group G on a ring R is a pair

$$\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}),$$

where for each $g \in G$, D_g is an ideal of R and $\alpha_g : D_{g^{-1}} \to D_g$ is an isomorphism of (non-necessarily unital) rings, satisfying the following conditions:

(i) $D_1 = R$ and α_1 is the identity automorphism of R;

(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$;

(iii) $\alpha_g \circ \alpha_h(r) = \alpha_{gh}(r)$, for every $r \in D_{h^{-1}} \cap D_{(gh)^{-1}}$.

Notice that if $D_g = R$ for every $g \in G$, then α is a global action of the group G on R, by automorphisms of R.

For our purposes we will assume hereafter that every ideal D_g is unital, with its identity element denoted by 1_g (in particular, each 1_g is a central idempotent of R). By [6, Theorem 4.5], this condition is equivalent to say that α has a *globalization* (or an *enveloping action*), which means that there exist a ring T and a global action of G on T, by automorphisms β_g ($g \in G$), such that R can be considered an ideal of T and the following conditions hold:

(i) $T = \sum_{g\in G} \beta_g(R)$;

(ii) $D_g = R \cap \beta_g(R)$, for all $g \in G$;

(iii) $\alpha_g = \beta_g|_{D_{g^{-1}}}$.

In particular, under these conditions, we have

$$1_g = 1_R\beta_g(1_R), \quad \alpha_g(r1_{g^{-1}}) = \beta_g(r)1_R \quad \text{and} \quad \alpha_g(1_h1_{g^{-1}}) = 1_g1_{gh}$$

for every $g, h \in G$ and $r \in R$.

Following [7] the subring of *invariants* of R under α is defined as

$$R^\alpha = \{r \in R : \alpha_g(r1_{g^{-1}}) = r1_g\},$$

and a finite set $\{x_i, y_i\}_{i=1}^m$ of elements of R is called a *partial Galois coordinate system* of R over R^α if $\sum_{i=1}^m x_i\alpha_g(y_i1_{g^{-1}}) = \delta_{1,g}1_R$, for every $g \in G$.

Given any non-empty subset X of R, we say that X is *α-invariant* (or *G-invariant*, if α is global) if $\alpha_g(X1_{g^{-1}}) \subseteq X$, for every $g \in G$. In particular, the centralizer $C_R(X)$ of a non-empty α-invariant subset X of R is also α-invariant.

Given any α-invariant subring X of R, the restriction of α to X is also a partial action of G on X given by $\alpha|_X = (\{X1_g\}_{g\in G}, \{\alpha_g|_{X1_{g^{-1}}}\}_{g\in G})$. If in addition such a subring X contains a partial Galois coordinate system over X^α, then the restriction $\alpha|_X$ is necessarily faithful, and we say that X is an *α-partial Galois extension of X^α* whenever G is assumed to be finite.

A ring extension $S \supseteq R$ is called *separable* (see [9]) if the multiplication map $m_S : S \otimes_R S \to S$ is a splitting epimorphism of S-bimodules or, equivalently, if there exists an element $x \in C_{S\otimes_R S}(S)$ such that $m_S(x) = 1_S$. Such an element x is an idempotent in $S \otimes_R S$ and it is called a *separability idempotent* of S over R. If $R \subseteq C(S)$ (resp., $R = C(S)$) we also say that S is a separable (resp., an *Azumaya*) R-algebra. A ring R is called *Azumaya* if it is an Azumaya $C(R)$-algebra. Furthermore, provided the existence of a partial action α of a finite group G on a ring R, we say that R is an *α-partial Galois Azumaya extension* (of R^α) if R is an α-partial Galois extension of R^α, R^α is an Azumaya ring and $C(R^\alpha) = C(R)^\alpha$.

Our main purpose in these notes is to prove the following theorems.

Theorem 1.1. *Let α be a partial action having globalization of a finite group G on a ring R. Suppose that R is an α-partial Galois Azumaya extension of R^α. Then there exists a one-to-one correspondence between the set of the separable $C(R^\alpha)$-subalgebras X of R^α and the set of the separable $C(R^\alpha)$-subalgebras Y of R which are α-partial Galois Azumaya extensions of Y^α containing $C_R(R^\alpha)$, given by $X \overset{\mu}{\mapsto} C_R(X)$ with inverse $Y \overset{\nu}{\mapsto} C_{R^\alpha}(Y)$.*

Theorem 1.2. *Let α be a partial action having globalization of a finite group G on a ring R. Suppose that R is an α-partial Galois Azumaya extension of R^α. Then there exists a one-to-one correspondence between the set of the separable $C(R^\alpha)$-subalgebras X of R containing R^α and the set of the separable $C(R^\alpha)$-subalgebras Y of $C_R(R^\alpha)$, given by $X \overset{\mu}{\mapsto} C_X(R^\alpha)$ with inverse $Y \overset{\nu}{\mapsto} R^\alpha Y$.*

Theorem 1.3. *Let α be a partial action having globalization of a finite group G on a ring R. Suppose that R is an Azumaya ring and $C(R)$ is an α-partial Galois extension of $C(R)^\alpha$. Then there exists a one-to-one correspondence between the set of the separable $C(R)^\alpha$-subalgebras X of R containing R^α and the set of the separable $C(R)^\alpha$-subalgebras Y of $C(R)$, given by $X \overset{\mu}{\mapsto} C(X)$ with inverse $Y \overset{\nu}{\mapsto} R^\alpha Y$.*

We set their proofs in the section 3. In the proof of Theorem 1.1 we proceed by similar arguments as those used in the proof of [2, Theorem 2]. For the proof of Theorem 1.2 we use the corresponding global Alfaro-Szeto's result [2, Theorem 3] and we give an explicit way to go, step by step, from the partial case to the global one and conversely. The proof of Theorem 1.3 is done in the same way, but in this case we simplify the procedures used before translating them in terms of one-to-one correspondences between the partial and the global cases.

We will present in the next section some lemmas which are the necessary preparation to prove the above theorems. In particular, Lemmas 2.6 and 2.7 below are generalizations to the partial case of [2, Lemmas 1 and 2] respectively.

2. Prerequisites

From now on, G will denote a finite group and $\alpha = (\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$ a partial action of G on a given ring R, with globalization (T, β). As usually, we will also denote the subring of invariants of T under β by T^G.

Since $T = \sum_{g\in G} \beta_g(R)$, putting $G = \{g_1 = 1, g_2, \dots, g_n\}$, we have that $1_T = e_1 \oplus e_2 \oplus \cdots \oplus e_n$, where $e_1 = 1_R$ and $e_i = (1_T - 1_R)\cdots(1_T - \beta_{g_{i-1}}(1_R))\beta_{g_i}(1_R)$, for every $2 \le i \le n$, (see [7]). Also in [7] the authors introduced a (left and right) T^G-linear and multiplicative map $\psi : T \to T$ given by

$$\psi(x) = \sum_{i=1}^{n} \beta_{g_i}(x)e_i = \sum_{1\le l\le n} \sum_{i_1<\cdots<i_l} \beta_{g_{i_1}}(1_R)\cdots\beta_{g_{i_{l-1}}}(1_R)\beta_{g_l}(x),$$

for every $x \in T$. It was showed in [3, Proposition 2.5] that the restriction $\psi|_{R^\alpha}$ gives a ring isomorphism from R^α onto T^G, whose inverse is given by the map $x \mapsto x1_R$, for every $x \in T^G$. We will see in the first lemma bellow a generalization of this result.

Lemma 2.1. *The following statements hold.*

(i) *Let X (resp., X') be a non-empty subset of R (resp., T) and Y (resp., Y') a subring of R (resp., T) such that $X'1_R = X$ (resp., $Y'1_R = Y$). Then, $C_Y(X) = C_{Y'}(X')1_R$.*

(ii) *Let X be a non-empty subset of R and Y a subring of R. Then, the map*

$$\psi|_{C_Y(X)} : C_Y(X) \to C_{\psi(Y)}(\psi(X))$$

is a ring isomorphism with inverse induced by $t \mapsto t1_R$, for all $t \in T$.

(iii) *Let X be an α-invariant subring of R and Y a G-invariant subring of T such that $Y1_R = X$, then the map*

$$\psi|_{X^\alpha} : X^\alpha \to Y^G$$

is a ring isomorphism with inverse induced by $t \mapsto t1_R$, for all $t \in T$.

(iv) *Let X and Y be subrings of R such that Y is α-invariant and $C(Y^\alpha) \subseteq X$. Then the map*

$$\theta : X \otimes_{C(Y^\alpha)} X \to \psi(X) \otimes_{C(\psi(Y^\alpha))} \psi(X)$$

given by $x \otimes x' \mapsto \psi(x) \otimes \psi(x')$ is a ring isomorphism with inverse induced by $t \mapsto t1_R$, for all $t \in T$.

(v) *Let X and Y be subrings of R such that Y is α-invariant and $C(Y)^\alpha \subseteq X$. Then the map*

$$\gamma : X \otimes_{C(Y)^\alpha} X \to \psi(X) \otimes_{\psi(C(Y)^\alpha)} \psi(X)$$

given by $x \otimes x' \mapsto \psi(x) \otimes \psi(x')$ is a ring isomorphism with inverse induced by $t \mapsto t1_R$, for all $t \in T$.

(vi) *Consider X and Y as given in (iv). Then the map*

$$\theta|_{C_A(X)} : C_A(X) \to C_B(\psi(X))$$

is a ring isomorphism, where $A = X \otimes_{C(Y^\alpha)} X$ and $B = \theta(A)$.

(vii) *Consider X and Y as given in (v). Then the map*

$$\gamma|_{C_U(X)} : C_U(X) \to C_V(\psi(X))$$

is a ring isomorphism, where $U = X \otimes_{C(Y)^\alpha} X$ and $V = \gamma(U)$.

(viii) *Consider X and Y as given in (iv). Then X is $C(Y^\alpha)$-separable if and only if $\psi(X)$ is $C(\psi(Y^\alpha))$-separable.*

(ix) *Consider X and Y as given in (v). Then X is $C(Y)^\alpha$-separable if and only if $\psi(X)$ is $\psi(C(Y)^\alpha)$-separable.*

Proof. (i) It is straightforward.

(ii) Clearly $\psi|_{C_Y(X)}$ is a well-defined ring isomorphism from $C_Y(X)$ onto $C_{\psi(Y)}(\psi(X))$, with inverse induced by the multiplication by 1_R.

(iii) The inclusion $\psi(X^\alpha) \subseteq Y^G$ can be proved by a similar way as in the proof of [3, Proposition 2.5]. Moreover, $Y^G 1_R \subseteq X^\alpha$. Indeed, $\alpha_g(y1_R1_{g^{-1}}) = \beta_g(y1_R)1_R = y\beta_g(1_R)1_R = (y1_R)1_g$ for every $y \in Y^G$ and $g \in G$. Since ψ is (right and left) T^G-linear, we have $\psi(y1_R) = y\psi(1_R) = y1_T = y$ for any $y \in Y^G$, and $\psi(x)1_R = x$ for all $x \in X^\alpha$.

(iv) Note that X is a $C(Y^\alpha)$-module and consequently $\psi(X)$ is a $\psi(C(Y^\alpha))$-module. Besides this, from (ii) we have $\psi(C(Y^\alpha)) = C(\psi(Y^\alpha))$ and therefore $\psi(X)$ is an $C(\psi(Y^\alpha))$-module.

Now putting $Q = C(Y^\alpha)$ and $P = C(\psi(Y^\alpha))$ it is clear that the maps

$$\begin{array}{ccc} X \times X & \to & \psi(X) \otimes_P \psi(X) \\ (x, x') & \mapsto & \psi(x) \otimes \psi(x') \end{array} \quad \text{and} \quad \begin{array}{ccc} \psi(X) \times \psi(X) & \to & X \otimes_Q X \\ (\psi(x), \psi(x')) & \mapsto & x \otimes x' \end{array}$$

are bi-additive, multiplicative and balanced over P and Q, respectively.

Thus, we have the respective induced ring homomorphisms:

$$\begin{array}{cccc} \theta: & X \otimes_Q X & \to & \psi(X) \otimes_P \psi(X) \\ & x \otimes x' & \mapsto & \psi(x) \otimes \psi(x') \end{array} \quad \text{and} \quad \begin{array}{cccc} \theta': & \psi(X) \otimes_P \psi(X) & \to & X \otimes_Q X \\ & \psi(x) \otimes \psi(x') & \mapsto & x \otimes x' \end{array}$$

which satisfy $\theta \circ \theta' = I_{\psi(X)\otimes_P\psi(X)}$ and $\theta' \circ \theta = I_{X\otimes_Q X}$.

(v) The proof is similar to that of (iv).

(vi) It is an immediate consequence of (iv).

(vii) It is an immediate consequence of (v).

(viii) Notice first that by (ii) $C(Y^\alpha) \subseteq C(X)$ if and only if $C(\psi(Y^\alpha)) \subseteq C(\psi(X))$. The rest of the proof follows from (vi) and the definition of separability.

(ix) The proof is similar to that of (viii) and it follows from (ii), (vii) and the definition of separability. □

Lemma 2.1 has some immediate consequences, which we put in the next two corollaries.

Corollary 2.2. *The following statements hold:*

(i) $R^\alpha = T^G 1_R$.
(ii) $C(R) = C(T)1_R$.
(iii) $C(R^\alpha) = C(T^G)1_R$.
(iv) $C(R)^\alpha = C(T)^G 1_R$.

Proof. (i) It follows from Lemma 2.1(iii).

(ii) It follows from Lemma 2.1(i).

(iii) It follows from (i) and Lemma 2.1(i).

(iv) It follows from (ii) and Lemma 2.1(iii). □

Corollary 2.3. *The following statements are equivalent:*

(i) *T is a Galois Azumaya extension of T^G.*
(ii) *R is an α-partial Galois Azumaya extension of R^α.*

Proof. By [7, Theorem 3.3], T is a Galois extension of T^G if and only if R is an α-partial Galois extension of R^α. By Lemma 2.1(viii), T^G is $C(T^G)$-separable if and only if R^α is $C(R^\alpha)$-separable. And from (iii)-(iv) of Corollary 2.2, we have $C(T^G) = C(T)^G$ if and only if $C(R^\alpha) = C(R)^\alpha$. The proof is complete. □

Lemma 2.4. *Let X' be an α-invariant subring of R. Then, the following statements hold:*

(i) $\alpha' = (\{X'_g = X'1_g\}_{g\in G}, \{\alpha'_g = \alpha_g|_{X'_{g^{-1}}}\}_{g\in G})$ *is a partial action of G on X'.*

(ii) *$(Y' = \sum_{g\in G}\beta_g(X'), \beta')$, with $\beta' : G \to Aut(Y')$ given by $\beta'_g = \beta_g|_{Y'}$, is a globalization of (X', α').*

If in particular $X' = C_R(X)$ for some non-empty subset X of R^α, then $C_T(\psi(X)) = Y'$.

Proof. (i) Under the assumptions on X' it is immediate that each X'_g is an ideal of X' and each $\alpha'_g : X'_{g^{-1}} \to X'_g$ is an isomorphism of unital rings. Now, since

$$\alpha'_g(X'_{g^{-1}} \cap X'_h) = \alpha_g(X'1_{g^{-1}}1_h) = \alpha_g(X'1_{g^{-1}})\alpha_g(1_h1_{g^{-1}}) = (X'1_g)1_g1_{gh} = X'1_g1_{gh} = X'_g \cap X'_{gh},$$

for any $g, h \in G$, and

$$(\alpha'_g\alpha'_h)(x) = \alpha_g(\alpha_h(x)) = \alpha_{gh}(x) = \alpha'_{gh}(x),$$

for any $x \in X'_{h^{-1}} \cap X'_{(gh)^{-1}}$, the required follows.

(ii) Since $\beta_g(X')X' = \beta_g(X')1_RX' = \alpha_g(X'1_{g^{-1}})X' \subseteq X'$ and $X'\beta_g(X') \subseteq X'$ as well, for every $g \in G$, it follows that X' is an ideal of Y'. In particular $X' = Y'1_R$ and hence $X' \cap \beta_g(X') = Y'1_R \cap \beta_g(Y')\beta_g(1_R) = Y'1_R\beta_g(1_R) = X'_g$. Finally, $\alpha'_g(x) = \alpha_g|_{X'_{g^{-1}}}(x) = \beta_g|_{X'_{g^{-1}}}(x) = \beta'_g(x)$ for every $g \in G$ and $x \in X'_{g^{-1}}$.

For the last statement notice first that $X' = C_R(X) = C_T(\psi(X))1_R$ by Lemma 2.1(i). Moreover, because $X \subseteq R^\alpha$ we have $\psi(X) \subseteq T^G$ and from this it easily follows that $C_T(\psi(X))$ is G-invariant. Thus, $\beta_g(X') = \beta_g(C_T(\psi(X))1_R) = \beta_g(C_T(\psi(X)))\beta_g(1_R) = C_T(\psi(X))\beta_g(1_R) \subseteq C_T(\psi(X))$, for any $g \in G$, and consequently $Y' \subseteq C_T(\psi(X))$.

Indeed, Y' is an ideal of $C_T(\psi(X))$. To see this, take $t \in C_T(\psi(X))$, $g \in G$ and set $t' = \beta_{g^{-1}}(t)$. Then, for any $r \in X'$ and any $x \in X$ we have $(t'r)x = t'xr = \beta_{g^{-1}}(t)\psi(x)r = \beta_{g^{-1}}(t\psi(x))r = \beta_{g^{-1}}(\psi(x)t)r = \psi(x)\beta_{g^{-1}}(t)r = \psi(x)t'r = x(t'r)$. Therefore, $t'r \in X'$ and $tY' = \sum_{g\in G} t\beta_g(X') = \sum_{g\in G}\beta_g(t'X') \subseteq \sum_{g\in G}\beta_g(X') = Y'$. We also get $Y't \subseteq Y'$ by similar arguments.

Finally, $1_T = \psi(1_R) = \sum\limits_{1\le l\le n}\ \sum\limits_{i_1<\cdots<i_l} \beta_{g_{i_1}}(1_R)\cdots\beta_{g_{i_l}}(1_R) \in Y'$ and the result follows. □

Remark 2.5. *For the globalization (Y', β') of (X', α'), both constructed in Lemma 2.4, one can consider, by restriction to Y', the same map ψ also from Y' into Y'. In particular, all the statements of Lemma 2.1 and Corollaries 2.2 and 2.3 remain valid when we replace (R, α) (resp., (T, β)) by (X', α') (resp., (Y', β')).*

Following [6], the *partial skew group ring* $R \star_\alpha G$ is defined as the direct sum

$$\bigoplus_{g\in G} D_g\delta_g,$$

where the $\delta'_g s$ are symbols, with the usual sum and the multiplication defined by the rule

$$(r\delta_g)(s\delta_h) = r\alpha_g(s1_{g^{-1}})\delta_{gh}$$

for all $g, h \in G$, $r \in D_g$ and $s \in D_h$. Since every D_g is unital by assumption, then $R \star_\alpha G$ is associative (see [6, Proposition 2.5 and Theorem 3.1]) and unital, with the identity element given by $1_R\delta_1$.

Lemma 2.6. *Suppose that R is an α-partial Galois Azumaya extension of R^α. Let X be a separable $C(R^\alpha)$-subalgebra of R^α, $X' = C_R(X)$ and $\alpha' = (\{X'_g = X'1_g\}_{g\in G}, \{\alpha'_g = \alpha_g|_{X'_{g^{-1}}}\}_{g\in G})$ the partial action of G on X' constructed in Lemma 2.4. Then the following statements hold.*

(i) *X' is an α'-partial Galois Azumaya extension of $X'^{\alpha'}$.*
(ii) *X' is a $C(R^\alpha)$-separable algebra.*
(iii) *$R *_\alpha G$ is an Azumaya ring and $C(R *_\alpha G) = C(R)^\alpha = C(R^\alpha)$.*

Proof. To prove this lemma we need some preparation. First of all, by Corollary 2.3 we have that T is a Galois Azumaya extension of T^G and so, it follows that $C(T *_\beta G) = C(T^G) = C(T)^G$ by [2, Theorem 1]. Corollary 2.2(i) implies that $\psi(X) \subseteq \psi(R^\alpha) = T^G$ and by Lemma 2.1(viii) we have that $\psi(X)$ is a separable $C(T^G)$-subalgebra of T^G. Therefore, it follows from [2, Lemma 1] that

(1) $C_T(\psi(X))$ is a Galois Azumaya extension of $C_T(\psi(X))^G$ and
(2) $C_T(\psi(X))$ is a separable $C(T^G)$-subalgebra of T.

From (1), [4, Theorem 1] and [5, Theorem II.3.4] we have, in particular, that $C_T(\psi(X))$ is a finitely generated projective $C(C_T(\psi(X))^G)$-module. Hence, from (2) and [11, Proposition III.2.4(c)] it follows that

(3) $C(C_T(\psi(X))^G)$ is a separable $C(T^G)$-subalgebra of T.

Moreover, it follows from Lemma 2.4 that

(4) $(C_T(\psi(X)), \beta')$, with β' given by $\beta'_g = \beta_g|_{C_T(\psi(X))}$, is a globalization of (X', α').

Now, we are able to conclude this proof.

(i) It follows from (1), (4) and Corollary 2.3.

(ii) It follows from (i) that X' is $C(X'^{\alpha'})$-separable. From (3), (4), Remark 2.5 and the statements (i)-(iii) and (viii) of Lemma 2.1 we have that $C(X'^{\alpha'})$ is $C(R^\alpha)$-separable. So the claim follows by [11, Proposition III.2.4(b)].

(iii) By assumption we have that: (a) R is an α-partial Galois extension over R^α; (b) R^α is an Azumaya algebra and (c) $C(R^\alpha) = C(R)^\alpha$. By arguments similar to those used in the proof of [7, Theorem 4.1], it follows from (a) that R is a finitely generated projective right R^α-module and $R *_\alpha G \simeq End(R_{R^\alpha})$ as $C(R)^\alpha$-algebras. It follows from (b), (c) and [5, Theorem II.3.4] that R is a finitely generated projective $C(R)^\alpha$-module. Thus, $End_{C(R)^\alpha}(R)$ is an Azumaya $C(R)^\alpha$-algebra by [5, Proposition II.4.1]. Since $End(R_{R^\alpha}) = C_{End_{C(R)^\alpha}(R)}(R^\alpha)$, the result follows by [5, Theorem II.4.3]. □

Lemma 2.7. *Suppose that R is an α-partial Galois Azumaya extension of R^α and let W be a separable $C(R^\alpha)$-subalgebra of R which is α-invariant. Assume that $C_R(R^\alpha) \subseteq W$ and W is an α^W-partial Galois Azumaya extension of W^{α^W}, with $\alpha^W = (\{W_g = W1_g\}_{g\in G}, \{\alpha^W_g = \alpha|_{W_{g^{-1}}}\}_{g\in G})$. Then, there exists a separable $C(R^\alpha)$-subalgebra V of R^α such that $W = C_R(V)$ and $V = C_{R^\alpha}(W)$.*

Proof. By Lemma 2.4 we have that $(W' = \sum_{g\in G} \beta_g(W), \beta')$, with β' given by $\beta'_g = \beta_g|_{W'}$, $g \in G$, is a globalization of (W, α^W). Note that W' is a subring of T. In the sequel we will show that T and W' satisfy all the conditions listed in [2, Lemma 2]. We will proceed by steps.

Step 1: $W' \supseteq C_T(T^G)$.
We start by observing that $T = \bigoplus_{1\le j\le n} \beta_{g_j}(R)e_j$ because this direct sum is clearly an ideal of T that contains 1_T. Now, take $y \in C_T(T^G)$ and recall that $T^G = \psi(R^\alpha)$. Then, $y = \sum_{1\le j\le n} \beta_{g_j}(r_j)e_j$, with $r_j \in R$, and for any $r \in R^\alpha$ we have $y\psi(r) = \psi(r)y$, which easily implies that $r_j\beta_{g_j^{-1}}(e_j)r = rr_j\beta_{g_j^{-1}}(e_j)$, for every $1 \le j \le n$. Therefore $r_j\beta_{g_j^{-1}}(e_j) \in C_R(R^\alpha)$ for all $1 \le j \le n$, and so $y \in \sum_{1\le j\le n} \beta_{g_j}(C_R(R^\alpha)) \subseteq \sum_{1\le j\le n} \beta_{g_j}(W) = W'$.

Step 2: T and W' are Galois Azumaya extensions of T^G and W'^G, respectively.
This is clear by Corollary 2.3 and Remark 2.5.

Step 3: $C(T \star_\beta G) = C(T^G)$.
This is an immediate consequence of Step 2 and Lemma 2.6 (or [2, Theorem 1]).

Step 4: W' is a separable $C(T^G)$-algebra.
It follows from Remark 2.5, Corollary 2.2(iii) and Lemma 2.1(viii) that $\psi(W)$ is a separable $C(T^G)$-algebra. Hence, there exist elements $x_i, y_i \in \psi(W) \subseteq W'$, $1 \le i \le m$, such that $\sum_i x_iy_i = 1_{\psi(W)} = 1_{W'}$ and $\sum_i \psi(w)x_i \otimes y_i = \sum_i x_i \otimes y_i\psi(w)$, for every $w \in W$. From this second equality we have that

$$\bigoplus_{1\le j\le n} \left(\sum_i \beta_{g_j}(w)e_jx_i \otimes y_i \right) = \bigoplus_{1\le j\le n} \left(\sum_i x_i \otimes y_i\beta_{g_j}(w)e_j \right)$$

which implies

$$\sum_i \beta_{g_j}(w)e_jx_i \otimes y_i = \sum_i x_i \otimes y_i\beta_{g_j}(w)e_j$$

for every $1 \leq j \leq n$. On the other hand, $W' = \bigoplus_{1\leq i\leq m} \beta_{g_j}(W)e_j$ (by the same arguments used in the proof of Step 1) and, consequently, $\sum_i w'x_i \otimes y_i = \sum_i x_i \otimes y_iw'$, for every $w' \in W'$, which proves the required.

Now, it follows from [2, Lemma 2] that there exists a separable $C(T^G)$-subalgebra V' of T^G such that $V' = C_{T^G}(W')$ and $W' = C_T(V')$. Put $V = V'1_R$ and notice that $W = W'1_R$. Then $\psi(V) = V'$ and so V is a separable $C(R^\alpha)$-subalgebra of R^α by Corollary 2.2(iii) and Lemma 2.1(viii). Moreover, $V = C_{T^G}(W')1_R = C_{R^\alpha}(W)$ and $W = W'1_R = C_T(V')1_R = C_R(V)$ by Lemma 2.1(i). The proof is complete. □

3. The proofs

Proof of Theorem 1.1:

Let X be a separable $C(R^\alpha)$-subalgebra of R^α. It is clear that $C_R(X)$ contains $C_R(R^\alpha)$ and it follows from Lemma 2.6 that $C_R(X)$ is an α-partial Galois Azumaya extension and a separable $C(R^\alpha)$-subalgebra of R. Thus, μ is well-defined. From Lemma 2.7, it follows that μ is surjective and ν is well-defined.

To show that μ is one-to-one, it is enough to prove that $(\nu\mu)(X) = C_{R^\alpha}(C_R(X)) = X$, for any separable $C(R^\alpha)$-subalgebra X of R^α. By Lemma 2.6 $R *_\alpha G$ is an Azumaya algebra with $C(R *_\alpha G) = C(R^\alpha)$. Thus, it follows from [5, Theorem II 4.3] that $X = C_{R*_\alpha G}(C_{R*_\alpha G}(X))$. Noting that $C_R(X)$ is α-invariant, it is easy to check that $C_{R*_\alpha G}(X) = C_R(X) *_\alpha G$. Obviously $X \subseteq C_{R^\alpha}(C_R(X))$ and $C_{R^\alpha}(C_R(X)) \subseteq C_{R*_\alpha G}(C_R(X) *_\alpha G) = C_{R*_\alpha G}(C_{R*_\alpha G}(X)) = X$. The proof is complete. □

Proof of Theorem 1.2:

By Corollary 2.3 and Lemma 2.6 (or [2, Theorem 1]) we have that T is a Galois Azumaya extension of T^G and $C(T *_\beta G) = C(T^G) = C(T)^G$.

Let X be a separable $C(R^\alpha)$-subalgebra of R containing R^α. It follows from Corollary 2.2 and Lemma 2.1(viii) that $T^G = \psi(R^\alpha) \subseteq \psi(X)$ and $\psi(X)$ is a separable $C(T^G)$-subalgebra of T. By [2, Theorem 3], $C_{\psi(X)}(T^G)$ is a separable $C(T^G)$-subalgebra of $C_T(T^G)$. By Lemma 2.1(i) we have that $C_X(R^\alpha) = C_{\psi(X)}(T^G)1_R$ and again by Lemma 2.1(viii) it follows that $C_X(R^\alpha)$ is a separable $C(R^\alpha)$-subalgebra of $C_R(R^\alpha)$. So, μ is well-defined.

Conversely, let Y be a separable $C(R^\alpha)$-subalgebra of $C_R(R^\alpha)$. Then, it follows from (ii) and (viii) of Lemma 2.1 that $\psi(Y)$ is a separable $C(T^G)$-subalgebra of $C_{\psi(R)}(T^G) \subseteq C_T(T^G)$. Thus, $T^G\psi(Y)$ is a separable $C(T^G)$-subalgebra of T that contains T^G by [2, Theorem 3]. Again by (ii), (iii) and (viii) of Lemma 2.1 we get that $R^\alpha Y = T^G\psi(Y)1_R$ is a separable $C(R^\alpha)$-subalgebra of R containing R^α. Hence, ν is well-defined.

Finally, we observe that

$$C_X(R^\alpha)R^\alpha = C_{\psi(X)}(T^G)T^G1_R \stackrel{(*)}{=} \psi(X)1_R = X$$

and

$$C_{R^\alpha Y}(R^\alpha) = C_{\psi(R^\alpha Y)}(T^G)1_R = C_{T^G\psi(Y)}(T^G)1_R \stackrel{(*)}{=} \psi(Y)1_R = Y,$$

where the equalities $(*)$ follow from [2, Theorem 3] and the others are ensured by Lemma 2.1. □

Proof of Theorem 1.3:

Note that R is a separable $C(R)^\alpha$-algebra, by assumption and [7, Theorem 4.2]. Thus, by Lemma 2.1(ix) and Corollary 2.2(iv) we have that $\psi(R)$ is a separable $C(T)^G$-algebra, and by the same argumentation used in the step 4 of the proof of Lemma 2.7 we have that T is a separable $C(T)^G$-algebra. Consequently, T is Azumaya by [5, Proposition II.3.8].

Following [3, Proposition 2.4], $(C(T), \beta_{|_{C(T)}})$ is a globalization of $(C(R), \alpha_{|_{C(R)}})$ and so, by assumption and [7, Theorem 3.3], we get that G acts faithfully on $C(R)$ and $C(T)$ is a Galois extension of $C(T)^G$. Therefore, from [2, Corollary 1] we have the one-to-one correspondence between the set A' of the separable $C(T)^G$-subalgebras of T containing T^G and the set B' of the separable $C(T)^G$-subalgebras of $C(T)$, given by $U \overset{\mu'}{\mapsto} C(U)$ with inverse $V \overset{\nu'}{\mapsto} T^G V$.

Let A denote the set of the separable $C(R^\alpha)$-subalgebras of R containing R^α and B the set of the separable $C(R^\alpha)$-subalgebras of $C(R)$. As a consequence of Lemma 2.1, the map ψ induces a one-to-one correspondence, denoted by $\psi_A^{A'}$ (resp., $\psi_B^{B'}$), between the sets A (resp., B) and A' (resp., B'), with inverse given by the multiplication by 1_R.

The required one-to-one correspondence between the sets A and B is given by the composition $\mu = 1_R \mu' \psi_A^{A'}$, with inverse $\nu = 1_R \nu' \psi_B^{B'}$. □

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