On the concept of n-diversity and the Banach spaces $C(K^n)$

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The concept of n-diversity and Banach spaces $C(K^n)$

A joint work with professor Piotr Koszmider from the Polish Academy of Sciences

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L. Candido, P. Koszmider

On complemented copies of $c_0(\omega_1)$ in $C(K^n)$ spaces, arxiv.org/abs/1501.01785.

Outline of the talk

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If X is inf. dim. Banach space, there is a sequence $(x_n^*)_{n\in\mathbb{N}} \subseteq X^*$ with $\|x_n^*\| = 1$ such that $x_n^* \overset{w^*}{\to} 0$.

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Theorem (E. M. Galego & J. Hagler 2012)

If $c_0(\omega_1) \hookrightarrow C(K)$ and there is a sequence $(x_\alpha^*)_{\alpha<\omega_1} \subseteq C(K)^*$ with $\|x_\alpha^*\| = 1$ such that such that $(x_\alpha^*(x))_{\alpha<\omega_1} \in c_0(\omega_1)$ for each $x \in C(K)$, then $c_0(\omega_1) \overset{c}{\hookrightarrow} C(K \times K)$.

Exposition

Theorem (S. Todorcevic 2006)

(MM) For every Banach space X of density ω_1 there is $(x^*_\alpha)_{\alpha<\omega_1} \subseteq X^*$ with $\|x^*_\alpha\| = 1$ such that such that $(x^*_\alpha(x))_{\alpha<\omega_1} \in c_0(\omega_1)$ for each $x \in X$.

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Theorem (E. M. Galego & J. Hagler 2012)

(MM) Let K be a compact Hausdorff space such that $C(K)$ has density ω_1. Then,

$$c_0(\omega_1) \hookrightarrow C(K) \Longrightarrow c_0(\omega_1) \overset{c}{\hookrightarrow} C(K \times K).$$

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Problem (E. M. Galego & J. Hagler 2012)

Can MM be removed from the previous theorem?

First approach

Theorem (P. Koszmider & P. Zieliński 2011)

(♣) There is a weakly Lindelöf $C(K)$ space of density ω_1 such that $K^{(\omega_1)} = \emptyset$, $c_0(\omega_1) \hookrightarrow C(K)$, but $c_0(\omega_1) \stackrel{c}{\not\hookrightarrow} C(K)$.

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Theorem (P. Koszmider & P. Zieliński 2011)

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Theorem (P.Koszmider & P. Zieliński 2011)

(PID) If $C(K)$ is a weakly Lindelöf space such that $K^{(\omega_1)} = \emptyset$ and $c_0(\omega_1) \hookrightarrow C(K)$, then $c_0(\omega_1) \overset{c}{\hookrightarrow} C(K)$

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Theorem (L.C. & P. Koszmider)

Suppose that $K = \omega_1 \cup \{\omega_1\}$ is the one point compactification of a locally compact, Hausdorff space ω_1 which carries a a bigger topology than the order topology. Then $c_0(\omega_1) \overset{c}{\hookrightarrow} C(K \times K)$

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On some results of A. Dow, H. Junilla and J. Pelant

A. Dow, H. Junilla, J. Pelant, Chain condidition and weak topologies, Topology Appl. 156 (2009), 1327–1344.

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1. A topological space is p.c.c. if every point-finite family of open subsets of the space is countable
2. X is weakly p.c.c. if every point finite family of weakly open sets in X is countable.
3. X is half-p.c.c. if every point finite family of half spaces ($\{x : \varphi(x) > a\}$ for some $\varphi \in X^*$ and $a \in \mathbb{R}$) in X is countable.

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3. X is half-p.c.c. if every point finite family of half spaces ($\{x : \varphi(x) > a\}$ for some $\varphi \in X^*$ and $a \in \mathbb{R}$) in X is countable.
4. X is half-p.c.c. iff every $T : X \to c_0(\omega_1)$ has separable range

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Theorem (A. Dow, H. Junilla, J. Pelant 2009)

($\diamondsuit$) There exists a compact Hausdorff space K such that K admits a finite-to-one continuous mapping onto the ordinal space $[0, \omega_1]$ and $C(K)$ is weakly pcc.

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Proposition (L.C & P. Koszmider)

If $C(K)$ is weakly pcc then $C(K^n)$ is weakly pcc for all $n \in \mathbb{N}$.

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If $C(K)$ is weakly pcc then $C(K^n)$ is weakly pcc for all $n \in \mathbb{N}$.

Theorem (Implicitely in Dow, Junilla, Pelant, 2009)

($\diamondsuit$)There is a scattered compact K which maps onto $[0, \omega_1]$ such that $c_0(\omega_1) \hookrightarrow C(K)$ but for all $n \in \mathbb{N}$, $c_0(\omega_1) \overset{c}{\not\hookrightarrow} C(K^n)$.

On some results of A. Dow, H. Junilla and J. Pelant

Theorem (L.C & P. Koszmider)

(♣) There is a scattered compact K such that $C(K)$ is half-p.c.c. but is not weakly p.c.c.

On some results of A. Dow, H. Junilla and J. Pelant

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$\Delta_n = \{(x_1, \ldots, x_n) \in K^n : x_i = x_j \text{ for some } i = j\}$.

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Theorem (Arhangelskii & Tkachuk 86)

$C(K)$ is pointwise p.c.c. iff for every $n \in \mathbb{N}$ every uncountable set in $K^n \setminus \Delta_n$ has an accumulation point in $K^n \setminus \Delta_n$.

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Theorem (A. Dow, H. Junilla, J. Pelant 2009)

If K scattered then $C(K)$ is weakly p.c.c. iff $C(K)$ is pointwise p.c.c.

Our main results

Definition

Let K be a compact space, $m \in \mathbb{N}$ and let $F_1, \ldots, F_k$ a partition of $\{1, \ldots, m\}$. A point $(x_1, \ldots, x_m) \in K^m$ is said to be $(F_1, \ldots, F_k)$-*diverse* if $\{x_j : j \in F_i\} \cap \{x_j : j \notin F_i\} = \emptyset$ for all $1 \leq i \leq k$.

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Definition (n-diversity)

Let K be a Hausdorff compact and $n \in \mathbb{N}$. We say that K is n-*diverse* if for any given $m \in \mathbb{N}$ and for any partition $F_1, \ldots, F_k$ of $\{1, \ldots, m\}$ with $k \leq n$, any sequence $\{(x_1^\alpha, \ldots, x_m^\alpha)\}_{\alpha<\omega_1} \subseteq K^m$ of $(F_1, \ldots, F_k)$-diverse points admits a cluster point which is $(F_1, \ldots, F_k)$-diverse.

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Theorem

If K is a scattered compact space then $C(K)$ is weakly pcc iff K is n-diverse for each $n \in \mathbb{N}$.

Our main results

Theorem (L.C. & P. Koszmider)

If a compact scattered Hausdorff K is $(n+1)$-diverse for some $n \in \mathbb{N}$, then $C(K^n)$ is half-pcc.

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Theorem (L.C. & P. Koszmider)

Let K be compact totally disconnected space and $\infty \in K$. If there exists a continous surjective map $\phi : K \setminus \{\infty\} \to [0, \omega_1)$ such that $|\phi^{-1}[\{\alpha\}]| \leq n$ for all $\alpha < \omega_1$ and some $n \in \mathbb{N}$, where $[0, \omega_1)$ is endowed with the order topology, then $c_0(\omega_1) \overset{c}{\hookrightarrow} C(K^{n+1})$. In particular $C(K^{n+1})$ is not half-pcc.

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Let K be a scattered compact Hausdorff space and $n \in \mathbb{N}$. Each of the following conditions implies the next.

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1. K is $(n+1)$-diverse,
2. $C(K^n)$ is half-pcc,
3. $c_0(\omega_1) \overset{c}{\not\hookrightarrow} C(K^n)$,
4. There is no point $\infty \in K$ such that $K \setminus \{\infty\}$ can be mapped onto $[0, \omega_1)$ by an $(n-1)$-to-1 continuous map.

Our main results

Theorem (L.C. & P. Koszmider)

(♣) For each $n \in \mathbb{N}$ there is a scattered compact Hausdorff space K_n such that $C(K_n)$ is weakly Lindelöf, K_n is $(n+1)$-diverse and there is a point $\infty \in K_n$ such that $K_n \setminus \{\infty\}$ can be mapped onto $[0, \omega_1)$ by an n-to-1 continuous map.

Our main results

Theorem (L.C. & P. Koszmider)

It is consistent that there are compact Hausdorff spaces K_n for all $1 \leq n < \omega$ such that $c_0(\omega_1) \hookrightarrow C(K_n)$ and $c_0(\omega_1) \stackrel{c}{\hookrightarrow} C(K_n^m)$ if and only if $n < m < \omega$.

An open question

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There exist in ZFC a compact Hausdorff 2-diverse space K such that $C(K)$ is not weakly p.c.c.?

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Theorem

Suppose that K is compact scattered space which contains a point ∞ such that $K \setminus \{\infty\}$ maps injectively and continuously onto a subset of R. If K is 2-diverse, then $C(K)$ is weakly p.c.c.

Thank you for your attention!