# Discrete group actions preserving a proper metric. Amenability and property (T)

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*von Neumann* (1929) : Given a group $G$ acting on a set $X$, when is there an invariant mean ?

Let $G$ be a group acting on a set $X$. An **invariant mean** is a map $\mu$ from the collection of subsets of $X$ to $[0, 1]$ such that

**(i)** $\mu(A \cup B) = \mu(A) + \mu(B)$ when $A \cap B = \emptyset$ ;

**(ii)** $\mu(X) = 1$ ;

**(iii)** $\mu(gA) = \mu(A)$ for all $g \in G$ and $A \subset X$.

If such a mean exists, we say that **the action is amenable**.

*Hausdorff* (1914) : There is no $SO(3)$-invariant mean on $X = SO(3)/SO(2)$.

*Tarski* (1929) : There exists a $G$-invariant mean iff the action is not paradoxical.

*von Neumann* : Every action of an amenable group is amenable. If a **free** action is amenable, then the group is amenable.

Let $G \curvearrowright X$. We have the equivalence (*Greenleaf* (1969), *Eymard* (1972)) :

- there exists an invariant mean ;
- there exists an invariant state on $\ell^\infty(X)$ ;
- the trivial representation of $G$ is weakly contained in the Koopman representation $\lambda_X$ of $G$ on $\ell^2(X)$ ;
- for every $\varepsilon > 0$ and every finite subset $F \subset G$, there exists a finite subset $E$ of $X$ such that

$$|E \Delta sE| < \varepsilon |E|, \quad \forall s \in F.$$

Assume that $G$ acts by left translations on $X = G/H$, where $H$ is a subgroup of $G$. Then the above conditions are equivalent to :

- every affine continuous action of $G$ on a compact convex subset of a separated locally convex topological vector space having an $H$-fixed point has also a $G$-fixed point.

Warning : this is not the amenability *in the sense of Zimmer* which can be defined by the existence of a map $m : x \mapsto m_x$ from $X$ into the set of states on $\ell^\infty(G)$ such that $m_{gx}(f) = m_x(gf)$ for $x \in X$, $g \in G$ and $f \in \ell^\infty(G)$.

When $H$ is a subgroup of $G$ and $G$ acts on $G/H$ by translations, this latter notion is equivalent to the amenability of $H$, whereas, when $H$ is a normal subgroup of $G$, the amenability of $G \curvearrowright G/H$ *in the sense of von Neumann* is equivalent to the amenability of the group quotient $G/H$.

In the sequel, amenability will always mean "in the sense of von Neumann". When $G \curvearrowright G/H$ is amenable, one also says that $H$ is **co-amenable** in $G$.

**Q1** (*von Neumann* (1929), *Greenleaf* (1969)) : If $G$ acts faithfully, transitively and amenably on $X$, does this imply that $G$ is amenable ?

**Q2** (*Eymard* (1972)) : Let $G$ act transitively and amenably on $X$, let $G_1$ be a subgroup of $G$. Then $G_1 \curvearrowright X$ is amenable, but is the action of $G_1$ on each orbit $G_1 x_0$ amenable ?

**Q3** : Is the amenability of a transitive action of $G$ on $X$ equivalent to the injectivity of $\lambda_X(G)''$, where $\lambda_X$ is the Koopman representation ?

Answers to **Q2** and **Q3** are positive when $X = G/H$ and $H$ is a normal subgroup of $G$, since the amenability of $G \curvearrowright G/H$ is then equivalent to the amenability of the group $G/H$.

Answers to all three questions are negative in general.

**Q1** (*von Neumann* (1929), *Greenleaf* (1969)) : If $G$ acts faithfully, transitively and amenably on $X$, does this imply that $G$ is amenable ?

Denote by $\mathcal{A}$ the class of countable groups that admit a faithful, transitive, amenable action.

*van Douwen* (1990) : **finitely generated free groups are in** $\mathcal{A}$. There are even examples with almost free actions, that is, every non trivial element has only a finite number of fixed points.

*Glasner-Monod* (2006) and *Grigorchuk-Nekrashevych* (2007) have provided other constructions of faithful, transitive, amenable actions of free groups.

*Glasner-Monod* : **the class** $\mathcal{A}$ **is stable under free products. Every countable group embeds in a group in** $\mathcal{A}$. More examples obtained by *S. Moon* (2010-2011) and *Fima* (2012).
Obstruction : groups with Kazhdan property (T) are not in $\mathcal{A}$.

**Q2** (*Eymard* (1972)) : Let $G$ act transitively and amenably on $X$, let $G_1$ be a subgroup of $G$ and $x_0 \in X$. Is the action of $G_1$ on $G_1 x_0$ amenable ?

Counterexamples given by *Monod-Popa* and *Pestov* (2003).

*Monod-Popa* : Let $Q$ be a discrete group,
$H = \oplus_{n \geq 0} Q$, $G_1 = \oplus_{n \in \mathbb{Z}} Q$, $G = G_1 \rtimes \mathbb{Z} = Q \wr \mathbb{Z}$.

$\rightsquigarrow$ $G \curvearrowright X = G/H$ **is amenable** (whatever $Q$, but $G_1 \curvearrowright G_1/H$ is amenable only if $Q$ is amenable) :

*Claim :* there exists of a $G$-invariant mean on $\ell^\infty(G/H)$.

- Enough to show the existence of a $G_1$-invariant mean since the group $G/G_1$ is amenable.
- Set $m_k = \delta_{t^{-k}H} \in \ell^\infty(G/H)^*_+$ where $t = 1 \in \mathbb{Z} < G$. This mean is invariant by the subgroup $t^{-k}Ht^k$. Since $G_1 = \cup_k t^{-k}Ht^k$, every limit point of the sequence $(m_k)$ gives a $G_1$-invariant mean.

In this example, $H$ is "*very non-normal*" in $G$, when $Q$ is non trivial.

- The **commensurator of** $H$ **in** $G$ is the set of $g \in G$ such that

$$[H : H \cap gHg^{-1}] < +\infty \quad \text{and} \quad [gHg^{-1} : H \cap gHg^{-1}] < +\infty$$

  It is a subgroup $\mathcal{C}\mathrm{om}_{\mathbf{G}}(\mathbf{H})$, which contains the normalizer $\mathcal{N}_G(H)$.

- *Observation* : $g \in \mathcal{C}\mathrm{om}_G(H)$ iff the $H$-orbits of $gH$ and $g^{-1}H$ in $G/H$ are finite.

In the previous example of Monod-Popa

$$H = \oplus_{n \geq 0} Q, \quad G_1 = \oplus_{n \in \mathbb{Z}} Q, \quad G = G_1 \rtimes \mathbb{Z}$$

we have

$$\mathbf{Com_G(H) = G_1 \lneqq G}$$

**Q3** : Is the amenability of a transitive action of $G$ on $X$ equivalent to the injectivity of $\lambda_X(G)''$, where $\lambda_X$ is the Koopman representation ?

In the example :

$$H = \oplus_{n\geq 0} Q, \quad G_1 = \oplus_{n\in\mathbb{Z}} Q, \quad G = G_1 \rtimes \mathbb{Z}$$

$G \curvearrowright G/H$ is always amenable but :

*the commutant $\lambda_{G/H}(G)'$ of $\lambda_{G/H}(G)''$ is isomorphic to $\mathcal{L}(Q)^{\otimes\infty}$, where $\mathcal{L}(Q)$ is the group von Neumann algebra of $Q$. It is injective only when $Q$ is an amenable group.*

$\rightsquigarrow$ So, amenability of $G \curvearrowright G/H \not\Rightarrow$ injectivity of $\lambda_{G/H}(G)''$.

$\rightsquigarrow$ The injectivity of $\lambda_{G/H}(G)'' \not\Rightarrow$ amenability of $G \curvearrowright G/H$ (see later).

Let $H$ be a subgroup of $G$. A notion weaker than normality is almost normality.

We say that $H$ is **almost normal** in $G$ if its commensurator $\mathcal{C}\mathrm{om}_G(H)$ is equal to $G$, that is, for all $g \in G$ the $H$-orbit of $gH$ in $G/H$ is finite. One also says that $(G, H)$ is a **Hecke pair** and write $H \underset{\sim}{\triangleleft} G$.

## Digression on the existence of $G$-invariant proper metrics.

Let $G \curvearrowright X$ be given. We say that a metric $d$ on $X$ is **proper**, or **locally finite** if the balls have a finite number of elements.

- For $G \curvearrowright G$ by left translations, there is a $G$-invariant proper metric when $G$ is countable.

- Let $G = \mathbb{Q} \rtimes \mathbb{Q}_*^+$, $H = \mathbb{Q}_*^+$. On $X = G/H \sim \mathbb{Q}$, there does not exist a proper $G$-invariant metric.

- Let $X$ be the set of vertices of a connected locally finite graph $\Gamma = (X, E)$ (i.e. each vertex has a finite degree) and let $G$ be a subgroup of the automorphism group of $\Gamma$. Then the geodesic metric on $X$ is proper and $G$-invariant.

Denote by $\text{Map}(X)$ the set of maps from $X$ to $X$ endowed with the topology of pointwise convergence and by $\text{Bij}(X)$ its subset of bijections. $\text{Bij}(X)$ is a topological group acting continuously on $X$, not locally compact if $X$ is infinite.

A-D (2012) : Let $G$ be a group acting on a countable set $X$. Let $\rho$ be the corresponding homomorphism from $G$ into $\mathrm{Bij}(X)$ and denote by $G'$ the closure of $\rho(G)$ in $\mathrm{Map}(X)$. The following conditions are equivalent :

**(i)** there exists a $G$-invariant locally finite metric $d$ on $X$ ;

**(ii)** the orbits of all the stabilizers of the $G$-action are finite ;

**(iii)** $G'$ is a subgroup of $\mathrm{Bij}(X)$ acting properly on the discrete space $X$.

In this case the group $G'$ is locally compact and totally disconnected.

For a transitive action $G \curvearrowright G/H$, we get the equivalence of the following conditions :

**(i)** there exists a $G$-invariant locally finite metric $d$ on $G/H$ ;

**(ii)** $H$ is almost normal in $G$ ;

**(iii)** the closure $G'$ of the image of $G$ in $\mathrm{Map}(G/H)$ is a subgroup of $\mathrm{Bij}(G/H)$ which acts properly on the discrete space $G/H$.

$\leadsto$ $(G, H)$ is a Hecke pair iff $G$ acts by isometries on a locally finite metric space and $H$ is the stabilizer of some point.

Let $H'$ be the closure of $H$ in $G'$. Then $G'$ is a is locally compact and totally disconnected group and $H'$ is a compact open subgroup of $G'$. The pair $(G', H')$ is called the *Schlichting* completion of $(G, H)$. (*Schlichting* (1980))

**Examples of almost normal subgroups :**

- Trivial examples : $H < G$ with $H$ normal subgroup, or finite subgroup, or finite index subgroup.
- $H = SL_n(\mathbb{Z}) < G = SL_n(\mathbb{Z}[1/p])$. Then $H' = SL_n(\mathbb{Z}_p)$, $G' = SL(n, \mathbb{Q}_p)$, $p$ prime number.
- $H = SL_n(\mathbb{Z}) < G = SL_n(\mathbb{Q})$. Then $H' = SL_n(\mathcal{R})$ and $G' = SL_n(\mathcal{A}_f)$ where $\mathcal{A}_f$ is the ring of finite adèles and $\mathcal{R}$ the subring of integers.
- $H = \mathbb{Z} \rtimes \{1\} < G = \mathbb{Q} \rtimes \mathbb{Q}_+^*$. Then $H' = \mathcal{R} \rtimes \{1\}$ and $G' = \mathcal{A}_f \rtimes \mathbb{Q}_+^*$.
- $H = \langle x \rangle < BS(m,n) = \langle t, x : t^{-1}x^m t = x^n \rangle$.
- $SL_n(\mathbb{Z})$, $n \geq 3$ only has finite, or finite index, almost normal subgroups (*Margulis* (1979) *Venkataramana* (1987)).

*Tzanev* (2000) : Let $H$ be an almost normal subgroup of $G$. The action of $G$ on $G/H$ is amenable iff the group $G'$ of Schlichting is amenable.

A-D (2012) : Let $G \curvearrowright X$ be an amenable transitive action by isometries on a locally finite metric space and let $G_1$ be a subgroup of $G$. The action of $G_1$ on each $G_1$-orbit is amenable.

In particular, the answer of Eymard's question

**Q2** : Let $G$ act amenably on $X = G/H$, and let $G_1$ be a subgroup of $G$ containing $H$. Is the action of $G_1$ on $G_1/H$ amenable ?

is positive when $H$ is almost normal.

**Q'1** : If $G$ acts faithfully, transitively and amenably by isometries on a locally finite metric space $X$, does this imply that $G$ is amenable ?

We are looking for an example of a group $G$ acting faithfully, transitively and by isometries on a locally finite metric space $X$ such that $G'$, the closure of $G$ in $\mathrm{Map}(X)$, is an amenable group, but $G$ is not amenable, and we will take for $H$ the stabilizer of any point.

The simplest examples of spaces $X$ carrying a locally finite metric are the sets of vertices of locally finite connected graphs $\Gamma = (X, E)$ with the geodesic length. Necessary and sufficient conditions for a closed subgroup $G'$ of the group $\mathrm{Aut}(\Gamma)$ of automorphisms of $\Gamma$ to be amenable have been studied by several authors.

Let $\Gamma = (X, E)$ be a connected graph. A *ray* (or half-line) is a sequence $[x_0, x_1, \dots]$ of successively adjacent vertices without repetitions. Two rays $R_1$ and $R_2$ are said to be in the same **end** if there is a ray $R_3$ which contains infinitely many vertices in $R_1$ and in $R_2$. In particular, when $\Gamma$ is a tree, two rays are in the same end if and only if their intersection is a ray.

*Nebbia* (1988), *Woess* (1989), *Soardi-Woess* (1990) : Let $\Gamma = (X, E)$ be a locally finite graph and let $G'$ be a **closed** subgroup of $\mathrm{Aut}(\Gamma)$.

- **(i)** If $G'$ is amenable then $G'$ fixes a finite subset of $X$, or an end of $\Gamma$, or a pair of ends of $\Gamma$.
- **(ii)** Assume that $\Gamma$ is a tree. Then $G'$ is amenable iff it fixes a vertex, an edge, an end, or a pair of ends.
- **(iii)** Assume that $\Gamma$ has infinitely many ends and that $G'$ acts transitively on $X$. Then $G'$ is amenable iff if fixes an end.

We would like to exhibit a non amenable group $G$ of automorphisms of a locally finite graph, acting transitively on the graph, whose closure is amenable. Does there exist such a group $G$, containing a free group ?

*Nebbia* : a closed group of automorphisms of a locally finite tree is amenable if and only if it does not contain a discrete free subgroup.

*Pays-Valette* (1991) : Let $\Gamma = (X, E)$ be a locally finite tree and let $G$ be a subgroup of $\mathrm{Aut}(\Gamma)$. The following properties are equivalent :

**(i)** the closure $G'$ of $G$ is amenable ;

**(ii)** $G$ does not contain a free group discrete in $\mathrm{Aut}(\Gamma)$ ;

**(iii)** $G$ does not contain a free group acting freely on $X$.

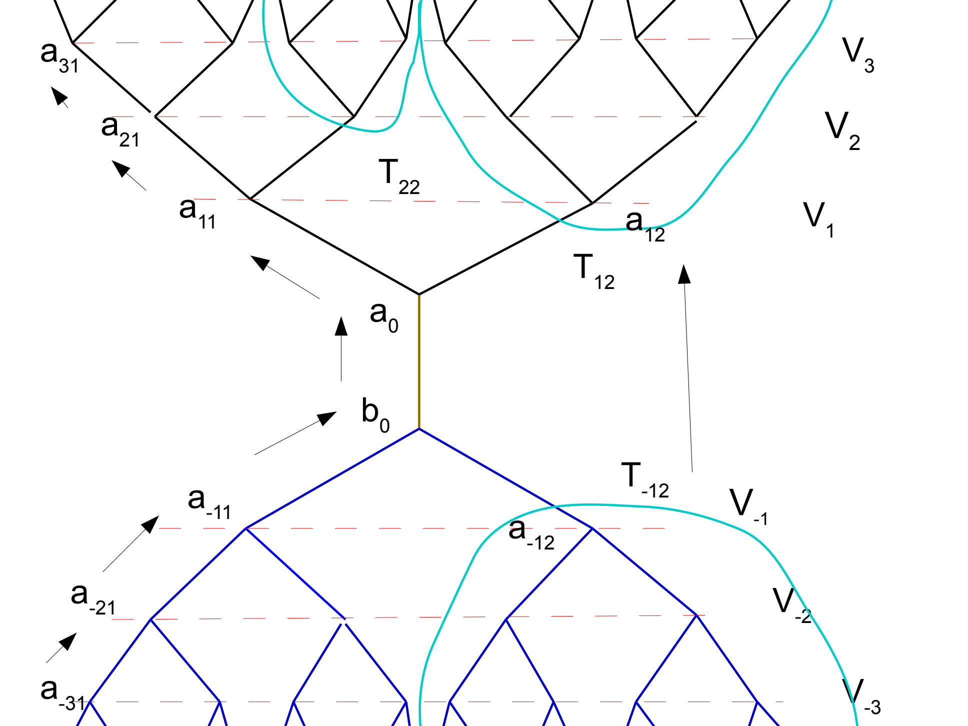
Let $\mathcal{C}$ be a class of group. We say that a group $G$ is **residually** $\mathcal{C}$ if for every $g \neq e$ in $G$, there exists a normal subgroup $N$ of $G$ such that $g \notin N$ and $G/N \in \mathcal{C}$.

Denote by $\mathcal{A}_{Iso}$ the class of countable groups that admit a faithful, transitive and amenable action by isometries on a locally finite metric space $X$.

A-D (2013) (after a discussion with *N. Monod*) : Let $p$ be a prime number. Any residually finite $p$-group $P$ can be embedded into a countable group $G$ that belongs to the class $\mathcal{A}_{Iso}$.

More precisely, we may construct $G$ as a subgroup of the automorphism group of the regular tree $T_p$ of degree $p+1$, generated by $P$ and an infinite cyclic element $\varphi$, in such a way that $G$ acts transitively on $T_p$ and its closure $G'$ is amenable.

We use the fact that a residually finite $p$-group is isomorphic to a subgroup of the automorphism group of a spherically homogeneous regular rooted tree of index $p$ (the root has degree $p$ and the other vertices have degree $p+1$).



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$a_{-11}$
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$V_{-3}$

Non-amenable residually finite $p$-groups are abundant :

► for every prime number $p$ and every integer $k \geq 2$, the free group $\mathbb{F}_k$ is a residually finite $p$-group ;

► for $n \geq 3$, the congruence subgroup $\Gamma_n(k) = \ker\theta : SL_n(\mathbb{Z}) \to \mathrm{SL}_n(\mathbb{Z}/k\mathbb{Z})$ is residually $p$-finite when $p$ divides $k$.

There are some obstructions for a group $G$ to belong to $\mathcal{A}_{Iso}$. For instance, if $G \in \mathcal{A}_{Iso}$, every subgroup of $G$ with the Kazhdan property is residually finite.

On the contrary, *Glasner-Monod* proved that every countable group embeds in a group in $\mathcal{A}$. So

$$\{amenable\ groups\} \subsetneqq \mathcal{A}_{Iso} \subsetneqq \mathcal{A}.$$

Do the non abelian free groups belong to $\mathcal{A}_{Iso}$ ?

**Q3** : Is the amenability of a $G \curvearrowright G/H$ equivalent to the injectivity of $\lambda_{G/H}(G)''$, where $\lambda_{G/H}$ is the quasi-regular representation of $G$ ?

Let $\xi \in \ell^2(G/H)$ and $f_g = \mathbf{1}_{HgH}$ where $g \in \mathcal{C}\mathrm{om}_G(H)$. Then

$$(R(f_g)\xi)(\dot{y}) = \sum_{k \in \langle G/H \rangle} \xi(\dot{k}) f_g(k^{-1}y).$$

is a bounded operator in $\lambda_{G/H}(G)'$.

*Mackey* (1951), *Kleppner* (1961), *Binder* (1993) : the von Neumann algebra $\lambda_{G/H}(G)'$ is generated by the operators $R(f_g)$, where $g$ runs into $\mathcal{C}\mathrm{om}_G(H)$.

In particular, $\lambda_{G/H}$ is irreducible iff $\mathcal{C}\mathrm{om}_G(H) = H$.

A-D (2012) : Let $H$ be an almost normal subgroup of $G$. Then $G \curvearrowright G/H$ is amenable iff there exists a net $(\varphi_i)$ of $H$-bi-invariant positive type functions on $G$, which converges to 1 pointwise, and is such that $\varphi_i$ is supported in a finite union of double $H$-cosets for every $i$.

Let $\varphi$ be such a function. Then

$$\Phi : \mathbf{1}_{HgH} \mapsto \varphi(g)\mathbf{1}_{HgH}$$

extends to a normal finite rank, completely positive map from $\lambda_{G/H}(G)'$ into itself. It follows that

Let $H < G$ such that $H$ is co-amenable in its commensurator $\mathcal{C}\mathrm{om}_G(H)$. Then $\lambda_{G/H}(G)''$ is an injective von Neumann algebra.

**Remark :** *Even when $H$ is almost normal in $G$, the injectivity of $\lambda_{G/H}(G)'$ does not imply that $H$ is co-amenable in $G$.*
See for example $H = SL_n(\mathbb{Z}) \underset{\sim}{\lhd} G = SL_n(\mathbb{Q})$ : then $\lambda_{G/H}(G)'$ is abelian.

About co-rigidity. This notion was considered by several authors : *Popa*, A-D (1986), *Tzanev* (2000), *Larsen-Palma* (2014).

Let $H$ be a subgroup of $G$. We say that $H$ is **co-rigid** in $G$ if there exists a finite subset $F$ of $G$ and $\varepsilon > 0$ such that if $\pi$ is a unitary representation of $G$ on a Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ such that $\pi(h)\xi = \xi$ for every $h \in H$ and $\|\pi(g)\xi - \xi\| \leq \varepsilon$ for $g \in F$, then $\mathcal{H}$ contains a non-zero $G$-invariant vector.

This is equivalent to the following property :

every sequence $(\varphi_n)_n$ of $H$-bi-invariant positive definite functions on $G$ that converges to 1 pointwise also converges to 1 uniformly on $G$.

- If $G$ has the Kazhdan property (T), every subgroup of $G$ is co-rigid.
- If $H$ is a normal subgroup of $G$, then $H$ is co-rigid iff the group $G/H$ has the Kazhdan property (T).

Let $H$ be a subgroup of $G$. We say that $H$ is **co-rigid** in $G$ if there exists a finite subset $F$ of $G$ and $\varepsilon > 0$ such that if $\pi$ is a unitary representation of $G$ on a Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ such that $\pi(h)\xi = \xi$ for every $h \in H$ and $\|\pi(g)\xi - \xi\| \leq \varepsilon$ for $g \in F$, then $\mathcal{H}$ contains a non-zero $G$-invariant vector.

*Kazhdan* (1967), *Margulis* (1982), *Cornulier* (2005) Let $X$ be a subset of $G$. We say that $(G, X)$ has **relative Property (T)** if for every $\varepsilon > 0$ there exist a finite subset $F \subset G$ and $\delta > 0$ such that whenever $\pi$ is a unitary representation of $G$ on a Hilbert space $\mathcal{H}$ with a unit vector $\xi \in \mathcal{H}$ such that $\max_{g \in F} \|\pi(g)\xi - \xi\| \leq \delta$ then $\mathcal{H}$ contains a $X$-invariant vector $\eta$ with $\|\xi - \eta\| \leq \varepsilon$.

► If $X$ is finite or if $X$ is a subgroup of $G$ with Property (T), then $(G, X)$ has the relative property (T).

Let $(G, X)$ with relative Property (T) and let $H$ be a subgroup of $G$. We assume that there exists an integer $n$ such that $G = (HX)(HX) \cdots HX$ $n$-times. Then $H$ is co-rigid in $G$.

Example : Let $G = \mathbb{Q} \rtimes \mathbb{Q}^*$ and $H = \mathbb{Q}^*$. Then $G$ acts faithfully on $G/H$ and $H$ is co-rigid in $G$.

Indeed take $X = \{(1, 1), (-1, 1)\}$. Then $G = HXHX$.

- In this example the group $G$ is amenable.
- Every subgroup of finite index is co-rigid.
- In case $H$ is an almost normal co-amenable subgroup of a group $G$, $H$ is co-rigid in $G$ if and only if it has a finite index in $G$.

Let $N$ be a group with Property (T) and $H$ a countable subgroup of $\mathrm{Aut}(N)$. Then $H$ is co-rigid in $N \rtimes H$.

Example : $N = \mathrm{SL}_n(\mathbb{Z}) \ltimes \mathrm{M}_{n,m}(\mathbb{Z})$, $H =$ any subgroup of $\mathrm{GL}_m(\mathbb{Z})$ acting by $g(s, x) = (s, xg^{-1})$.

Denote by $\mathcal{T}$ (resp. $\mathcal{T}_{alnor}$) the class of countable groups that have a co-rigid subgroup (resp. almost normal co-rigid subgroup) $H$ such that $G \curvearrowright G/H$ is faithful. Then

$$\{\textit{Kazhdan groups}\} \subset \mathcal{T}_{alnor} \subsetneq \mathcal{T}.$$

**Q :** Is the first inclusion strict ?

$$\{Kazhdan\ groups\} \subset \mathcal{T}_{alnor} \subsetneq \mathcal{T}.$$

**Q :** Is the first inclusion strict ?

We have to look for a group $G$ acting transitively and faithfully by isometries on a locally finite metric space $X$, such that $G$ has not the property (T) but its closure $G'$ in Map($X$) has the property (T).

▶ We cannot take $X$ to be a tree.

▶ If $X$ is the set of vertices of a connected locally finite graph $\Gamma$, then $\Gamma$ must be an expander, i.e. $\inf \{|\partial U|/|U| : U \subset X,\ \text{finite}\} > 0$. (Soardi-Woess)