Approximate Ramsey properties of finite dimensional normed spaces.

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Notation: $[n] := \{1, \cdots, n\}$. Recall the well-know Ramsey Theorem: Given integers d, m and r there is an integer n such that for every coloring

$$c : [n]^d := \{s \subseteq [n] : \#s = d\} \to [r] \tag{1}$$

there is

$$s \in [n]^m \tag{2}$$

such that

$$c \upharpoonright [s]^d \text{ is constant.} \tag{3}$$

This can be rephrased as follows: Let $\mathbf{A} = (A, <_A)$ and $\mathbf{B} = (B, <_B)$ be two finite linearly ordered sets and let $r \in \mathbb{N}$. Then there exists $\mathbf{C} = (C, <_C)$ such that for every coloring

$$c : \binom{\mathbf{C}}{\mathbf{A}} := \{A' \subseteq C \,:\, (A', <_C) \text{ and } (A, <_A) \text{ are order-isomorphic}\} \to [r] \tag{4}$$

there is

$$\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}} \tag{5}$$

such that

$$c \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.} \tag{6}$$

So, given a family $\mathcal{K}$ of structures of the *same sort*, we say that $\mathcal{K}$ has the *Ramsey Property* when for every $\mathbf{A}, \mathbf{B} \in \mathcal{K}$ and $r \in \mathbb{N}$ there exists $\mathbf{C} \in \mathcal{K}$ such that for every coloring

$$c : \binom{\mathbf{C}}{\mathbf{A}} := \{A' \subseteq C \,:\, \mathbf{A}' \cong \mathbf{A}\} \to [r] \tag{7}$$

there is

$$\mathbf{B}' \in \binom{\mathbf{C}}{\mathbf{B}} \tag{8}$$

such that

$$c \restriction \binom{\mathbf{B}'}{\mathbf{A}} \text{ is constant.} \tag{9}$$

We will abbreviate this by

$$\mathbf{C} \to (\mathbf{B})^{\mathbf{A}}_r \tag{10}$$

Examples

Example

The class of all finite ordered *Graphs* has the Ramsey property (Nesetril and Rodl).

Example

The class of finite-dimensional vector spaces over a finite field has the Ramsey property (Graham, Leeb and Rothschild)

Example

The class of all finite ordered metric spaces has the Ramsey property (Nesetril).

Example

The class of naturally ordered finite boolean algebras is Ramsey (Graham and Rothschild, *Dual Ramsey Theorem*)

Definition

Let $1 \leq p \leq \infty$, $n \in \mathbb{N}$. *The p-norm* $\|\cdot\|_p$ *on* $\mathbb{R}^n$ *is defined for* $(a_i)_{i<n}$ *by*

$$\|(a_i)_{i<n}\|_p := \Big(\sum_{i<n} |a_i|^p\Big)^{\frac{1}{p}} \text{ for } p < \infty \tag{11}$$

$$\|(a_i)_{i<n}\|_\infty := \max_{i<n} |a_i|. \tag{12}$$

Definition

Given two Banach spaces X *and* Y, *by a (linear isometric) embedding from* X *into* Y *we mean a linear operator* $T : X \to Y$ *such that*

$$\|T(x)\|_Y = \|x\|_X \text{ for all } x \in X. \tag{13}$$

Definition

Let

$$\mathrm{Emb}(X, Y) \tag{14}$$

be the collection of all embeddings from X into Y, and let

$$\binom{Y}{X} := \{Z \subseteq Y \,:\, Z \text{ is isometric to } X\}. \tag{15}$$

Note that $\mathrm{Emb}(X, Y)$ is a metric space with the norm distance

$$d(T, U) := \|T - U\| := \sup_{x \in S_X} \|T(x) - U(x)\|. \tag{16}$$

When Y is finite dimensional $\binom{Y}{X}$ is also a metric space when considering the Hausdorff distance between the unit balls of copies X' and X'' of X in Y.

The approximate Ramsey property would be: For every $1 \leq p \leq \infty$ every integers d, m and r there is n such that for every coloring

$$c : \binom{\ell_p^n}{\ell_p^d} \to [r] \tag{17}$$

there exist

$$X \in \binom{\ell_p^n}{\ell_p^m} \text{ and } 1 \leq i \leq r \tag{18}$$

such that

$$\binom{X}{\ell_p^d} \subseteq (c^{-1}(i))_\varepsilon. \tag{19}$$

In fact we have a more demanding notion.

Definition

Given $1 \leq p \leq \infty$, integers d, m and r and $\varepsilon > 0$, let $\mathbf{n_p}(d, m, r, \varepsilon)$ be the minimal integer n (if exists) such that for every coloring

$$c : \mathrm{Emb}(\ell_p^d, \ell_p^n) \to [r] \tag{20}$$

there exist

$$\gamma \in \mathrm{Emb}(\ell_p^m, \ell_p^n) \text{ and } 1 \leq i \leq r \tag{21}$$

such that

$$\gamma \circ \mathrm{Emb}(\ell_p^d, \ell_p^m) \subseteq (c^{-1}(i))_\varepsilon. \tag{22}$$

This property implies the first one about $\binom{\ell_p^n}{\ell_p^d}$.

Another reformulation: Let $\bar{u}^n = (u_i)_{i<n}$ be the unit basis of $\mathbb{R}^n$. Given $1 \le p \le \infty$ and $m \le n$ let

$$\mathcal{I}_{m,n}^p := \{A \in M_{n,m} : A \text{ is the matrix in the unit bases of } \mathbb{R}^d \text{ and } \mathbb{R}^n \text{ of an isometric embedding}\}.$$

Then $\mathbf{n_p}(d, m, r, \varepsilon)$ is the minimal integer n (if exists) such that for every coloring

$$c : \mathcal{I}_{d,m}^d \to [r] \tag{23}$$

there exist

$$A \in \mathcal{I}_{m,n}^p \text{ and } 1 \le i \le r \tag{24}$$

such that

$$A \cdot \mathcal{I}_{d,m}^p \subseteq (c^{-1}(i))_\varepsilon. \tag{25}$$

Proposition

$A \in \mathcal{I}^{\infty}_{m,n}$ if and only if for every column vector c of A one has that $\|c\|_\infty = 1$ and for every row vector r of A one has that $\|r\|_1 \le 1$.

Proposition

$A \in \mathcal{I}^{2}_{m,n}$ if and only if the sequence $(c_i)_{i<m}$ of columns of A is orthonormal.

Proposition

Given $1 \le p < \infty$, $p \ne 2$, $A \in \mathcal{I}^{p}_{m,n}$ if and only if for every column vector c of A one has that $\|c\|_p = 1$ and every two column vectors have disjoint support.

Theorem

$\mathbf{n_p}(d, m, r, \varepsilon)$ *exists.*

The intention is to relate our result with the Borsuk-Ulam Theorem. Recall that one of the several equivalent versions (*Lusternik-Schnirelmann* Theorem) of the *Borsuk-Ulam* theorem states that if the unit sphere $\mathbb{S}^n$ of ℓ_2^{n+1} is covered by $n+1$ many open sets, then one of them contains a point x and its antipodal $-x$.

Definition

Let (X,d) be a metric space, $\varepsilon > 0$. We say that an open covering $\mathcal{U}$ of X is ε-fat when for every $U \in \mathcal{U}$ there is V_U open such that $(V_U)_\varepsilon \subseteq U$ and $\{V_U\}_{U\in\mathcal{U}}$ is still a covering of X.

It is not difficult to see that if X is compact, then every open covering is ε-fat for some $\varepsilon > 0$.

Now the previous Theorem on embeddings can be restated as follows

Theorem

For every $1 \leq p \leq \infty$, every integers d, m and r and every ε there is some $\mathbf{n}_p(d, m, r, \varepsilon)$ such that for every ε-fat open covering $\mathcal{U}$ of $\mathcal{I}^p_{d,n}$ with cardinality at most r there exists some $A \in \mathcal{I}^p_{m,n}$ such that

$$A \cdot \mathcal{I}^p_{d,m} \subseteq U \text{ for some } U \in \mathcal{U}. \tag{26}$$

For example, Borsuk-Ulam Theorem is the statement

$$\mathbf{n}_2(1, 1, r, \varepsilon) = r \text{ for all } \varepsilon > 0, \tag{27}$$

because $\mathcal{I}^2_{1,n}$ consists on 1-column-matrices (v) of vectors v of the sphere of ℓ^n_2, and $\mathcal{I}^2_{1,1} = \{(1), (-1)\}$, so $(v) \cdot \mathcal{I}^2_{1,1} = \{(-v), (-v)\}$.

(1) Case $p = \infty$ Bartosova, Lopez-Abad, Mbombo (2014), case $p \neq \infty$ Ferenczi, Lopez-Abad, Mbombo and Todorcevic (2014).

(1) The result for embeddings and $d = 1$ was proved by Odell, Schlumprecht and Rosenthal (1993), and by Matoušek and Rödl (1995) independently.

(2) The case $p = 2$ (i.e. the Hilbert case) is an indirect consequence of the fact that the Unitary group of ℓ_2 is *extremely amenable*, proved by Gromov and Milman (1983).

(3) The result is true for real or complex Banach spaces.

(4) There are several extensions to the context of *operator spaces* (Lupino and Lopez-Abad (2014)).

Definition

Recall that a topological group G is extremely amenable (EA in short) when every (continuous) flow on a compact set K has a fixed point, that is, there is some $p \in K$ such that $g.p = p$ for every $g \in G$.

The terminology is consistent with one of the characterizations of *amenable* groups: Every action by affine mappings on a compact convex set has a fixed point.

1. The unitary group $\mathcal{U}(\ell^2)$, equipped with strong operator topology (Gromov-Milman, 1984).
2. $\mathrm{Aut}(\mathbb{Q}, \leq)$ the group of all order-preserving bijections of the rationals (Pestov, 1998).
3. In general automorphism groups of certain Fraissé structures (Kechris-Pestov-Todorcevic). Namely Fraissé class with structural Ramsey property.
4. $\mathrm{Iso}(\mathbb{U})$ where $\mathbb{U}$ is the universal Urysohn space. (Pestov, 2002)
5. The group $\mathrm{Iso}(L_p(X, \mu))$ (for every $1 \leq p < \infty$) where (X, μ) is a standard Borel measure space with a non-atomic measure (X, μ), equipped with the strong operator topology. (Giordano and Pestov 2007)

Until now, there are basically two techniques to prove the extreme amenability of a group:

(1) Proving the (approximate) ramsey property. This way was initiated by Kechris, Pestov and Todorcevic, (2005).
(2) Proving a concentration of measure phenomenon on G, initiated by Gromov and Milman, (1983).

The "discrete version" of (1) is the following:

Theorem (Kechris, Pestov and Todorcevic, (2005))

Suppose that $\mathcal{M}$ is a countable ultra homogeneous ordered structure. Then the automorphism group of $\mathcal{M}$ is EA if and only if

$$\mathrm{Age}(\mathcal{M}) := \textit{Finitely generated substructures of } \mathcal{M} \tag{28}$$

has the Ramsey property.

Recall that a structure $\mathcal{M}$ is called ultra homogeneous when every isomorphism between two finitely generated substructures of $\mathcal{M}$ extends to an automorphism of $\mathcal{M}$.

When dealing with *metric* structures, the right notion is the *approximate Ramsey property for embeddings* (the one we presented for ℓ_p's) and the right structures are the separable *approximate ultra homogeneous ones* (every isometric isomorphism between finitely generated substructures ε-extends to an isometric automorphism of $\mathcal{M}$). This work was initiated recently by, among others, Ben Yaacov, Melleray and Tsankov.

Example

$\mathrm{Aut}(\mathbb{Q},<)$ is EA (classical Ramsey property).

Example

$\mathrm{Iso}(\mathbb{U})$ is EA (Ramsey property of finite ordered metric spaces).

Example

$\mathrm{Iso}(\mathbb{F}^{<\infty})$ (ordered) is EA for every finite field $\mathbb{F}$ (Ramsey property of finite dimensional vector spaces over $\mathbb{F}$).

Concentration of Measure

There is a strength of EA, called the *Lévy* property.

Definition

An mm space is a metric space with a measure on it. Given such mm space (X, d, μ), *and* $\varepsilon > 0$,

$$\alpha_X(\varepsilon) := \inf\{\mu(A_\varepsilon) \,:\, \mu(A) \geq \frac{1}{2}\}. \tag{29}$$

A sequence $(X_n)_n$ *of mm-spaces is called Lévy when*

$$\alpha_{X_n}(\varepsilon) \to_n 1 \text{ for every } \varepsilon > 0. \tag{30}$$

Definition

A metrizable group G is called Lévy when there exists a sequence $(G_n)_n$ of compact subgroups of G such that the union is dense in G, a right-invariant compatible distance d on G such that $(G_n, d, \mu_n)_n$ is Lévy (μ_n being the Haar probability measure on G_n).

Theorem (Gromov and Milman, 1983)

Every Lévy Group is EA.

Gurarij space

Definition

The Gurarij space $\mathbb{G}$ is the unique (up to isometry) separable space with the following property: Given finite-dimensional normed spaces $X \subseteq Y$, given $\varepsilon > 0$, and given an isometric linear embedding $\gamma : X \to \mathbb{G}$ there exists an injective linear operator $\psi : Y \to \mathbb{G}$ extending γ and satisfying that

$$(1-\varepsilon)\|y\| \leq \|\psi(y)\| \leq (1+\varepsilon)\|y\|. \tag{31}$$

Theorem (Bartosova, Lopez-Abad and Mbombo, 2014)

The automorphism group of $\mathbb{G}$ is extremely amenable.

Poulsen Simplex

Definition

A compact and convex set K is called a (Choquet) simplex if for every $x \in K$ there is a unique probability measure supported on $\mathrm{Ext}(K)$ such that the barycenter of μ is x, that is

$$x = \int p d\mu(p) \tag{32}$$

The Poulsen simplex $\mathbb{P}$ is the unique (up to affine homeomorphism) metrizable simplex such that

$$\mathrm{Ext}(\mathbb{P}) \textit{ is dense in } \mathbb{P}. \tag{33}$$

Theorem (Bartosova, Lopez-Abad and Mbombo, 2014)

The universal minimal flow of the Poulsen simplex is the Poulsen simplex itself.

$L_p[0,1]$

Theorem (Ferenczi, Lopez-Abad, Mbombo and Todorcevic 2014)

For $0 < p < \infty$, $L_p[0,1]$ is approximate ultrahomogeneous for copies of ℓ_p^n's.

Together with the approximate Ramsey property of ℓ_p^n's we obtain:

Corollary (Giordano and Pestov, 2007)

For $1 \leq p < \infty$ the automorphism group of $L_p[0,1]$ is extremely amenable.

Embeddings of ℓ_p^d into ℓ_p^n are determined by the image $(x_i)_{i<d}$ of the unit basis $(u_i)_{i<d}$. Each sequence $\mathbf{x}=(x_i)_{i<d}$ in $\mathbb{R}^n$ is determined by its support: Given $\mathbf{a}=(a_i)_{i<d}$, let

$$\operatorname{supp}_{\mathbf{a}}\mathbf{x} := \{\xi < n \,:\, (x_i(\xi))_{i<n} = \mathbf{a}\}. \tag{34}$$

Observe that $(x_i(\xi))_{i<d} \in B_{(\ell_p^d)^*} = B_{\ell_q^d}$. After discretizing, we may assume that the sequences $\mathbf{x}=(x_i)_{i<d}$ to consider takes value in a finite ε-dense set $\Delta \subseteq B_{\ell_q^d}$. On the other hand, given $F : n \to \Delta$, we can define $\mathbf{x}_F = (x_i)_{i<d}$,

$$x_i := \sum_{\mathbf{a}\in\Delta} a_i \frac{1}{(\#F^{-1}(\mathbf{a}))^{\frac{1}{p}}} \mathbb{1}_{F^{-1}(\mathbf{a})}. \tag{35}$$

In this way, for $\mathbf{a}\in\Delta$,

$$\operatorname{supp}_{\mathbf{a}(\#F^{-1}(a))^{-\frac{1}{p}}} \mathbf{x}_F = F^{-1}(\mathbf{a}). \tag{36}$$

It is not difficult to understand when the sequence $\mathbf{x}_F$ defines an isometric embedding. It is natural to study then the set ${}^nS$ of mappings $n \to \Delta$ and the (approximate) Ramsey properties associated to them.
We consider the mm-space $X_n = ({}^nS, d_{\mathrm{H}}, \mu_{\mathrm{C}})$, where d_{H} is the *Hamming* distance on S^n and μ_{C} is the corresponding normalized *counting* measure.
It is well-known that $(X_n)_n$ is a normal Lévy sequence (i.e. it has the concentration phenomenon).

Definition

Given two finite sets S, T, we say that $F \in {}^TS$ is an ε-equipartition when

$$\max_{s,t \in S} \frac{\#F^{-1}(s)}{\#F^{-1}(t)} \le 1 + \varepsilon. \tag{37}$$

Let $\mathrm{Equi}_\varepsilon(T, S)$ be the collection of ε-equipartitions.

Theorem

$(\mathrm{Equi}_\varepsilon(T, S), d_{\mathrm{H}}, \mu_{\mathrm{C}})_n$ is an asymptotic normal Lévy sequence.

Recall that $A = \{a_i\}_{i<k}$ such that $\sum_{i<k} |a_i|^p = 1$ ($\max_i |a_i| = 1$), given n, and given $F : n \to A$ *onto*, we define the vector $v_F \in S_{\ell_p^n}$ by

$$v_F^A := \sum_{i<k} a_i \frac{1}{(\#F^{-1}(a_i))^{\frac{1}{p}}} \mathbb{1}_{F^{-1}(a_i)} \tag{38}$$

Proposition

For $p \neq \infty$ the mapping $v^A : \mathrm{Equi}_\varepsilon(n, A) \to \ell_p^n$ is uniformly continuous, with modulus of continuity independent of n. this is not true for arbitrary partitions or for $p = \infty$.

So, for $p \neq \infty$, we can use the following approximate Ramsey result.

Proposition

For every integers d, m and r and every $\varepsilon, \delta > 0$ there is some n such that for every coloring $c : \mathrm{Equi}_{\bar{\varepsilon}}(n, d) \to [r]$ there exists $F \in \mathrm{Equi}_\varepsilon(n, m)$ and $\bar{r} \in [r]$ such that

$$F \circ \mathrm{Equi}_\varepsilon(m, d) \subseteq (c^{-1}(\bar{r}))_\delta \tag{39}$$

$p \neq \infty$

The case $p = 2$ is a little simpler because

Proposition

$\mathbf{n_2}(d, m, r, \varepsilon) = \mathbf{n_2}(m, m, r, \varepsilon)$.

The case $p \neq 2$ reduces to the case $p = 1$, because it is a classical result of Ribe that the *Mazur map*

$$M_{p,q}((a_i)_i) := (\operatorname{sgn}(a_i)|a_i|^{\frac{p}{q}})_i \tag{40}$$

is an uniform homemorphism between the spheres of ℓ_p and ℓ_q $(p, q < \infty)$, with modulus of continuity

$$\omega_{p,q}(t) \leq \frac{p}{q} t \text{ if } p > q$$
$$\omega_{p,q}(t) \leq c t^{\frac{p}{q}} \text{ if } p < q.$$

It is also used that if $\gamma \in \mathrm{Emb}(\ell_p^d, \ell_p^n)$, $p \neq 2, \infty$, then $(\gamma(u_i))_{i<d}$ is a pairwise disjointly supported sequence in ℓ_p^n. So,

Proposition

$\mathbf{n_p}(d, m, r, \varepsilon) = \mathbf{n_q}(d, m, r, \omega_{p,q}(\varepsilon))$ for every $p, q \neq 2, \infty$, and where $w_{p,q}$ is the modulus of continuity of $M_{p,q}$

This approach does not work for $p = \infty$ because $v_F^{\mathbf{a}}$ is not uniformly continuous independent of n. Instead, we use the Dual Ramsey Theorem of Graham and Rothschild (1971),

Let $\mathcal{E}_n^d$ be the set of all partitions of $[n]$ into d-many pieces. Given a partition $\mathcal{Q} \in \mathcal{E}_n^m$, and $d \leq m$, let $\langle \mathcal{Q} \rangle^d$ be set of all partitions $\mathcal{P} \in \mathcal{E}_n^d$ *coarser* than $\mathcal{Q}$.

Theorem (Graham and Rothschild)

For every d, r and r there exists n such that for every coloring

$$c : \mathcal{E}_n^d \to [r] \tag{41}$$

there exists

$$\mathcal{Q} \in \mathcal{E}_n^m \tag{42}$$

such that

$$c \upharpoonright \langle \mathcal{Q} \rangle^d \text{ is constant.} \tag{43}$$

Let $\mathcal{EQ}_n^d$ be the set of all partitions of n whose pieces have cardinality $n|d$.

Problem

Given $d|m$ and r does there exists $m|n$ such that for every coloring

$$c : \mathcal{EQ}_n^d \to [r] \tag{44}$$

there exists

$$\mathcal{Q} \in \mathcal{EQ}_n^m \tag{45}$$

such that

$$c \upharpoonright \langle \mathcal{Q} \rangle^d \cap \mathcal{EQ}_n^d \text{ is constant?} \tag{46}$$