Supramenability of a group and tracial states on partial crossed products

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Definiton

Let G be a group acting on a set X. A non-empty subset A of X is said to be paradoxical if there exist disjoint subsets B and C of A, finite partitions $\{B_i\}_{i=1}^{n}$ and $\{C_j\}_{j=1}^{m}$ of B and C and elements $s_1, \ldots, s_n, t_1, \ldots, t_m \in G$ such that $A = \sqcup_{i=1}^{n} s_i B_i = \sqcup_{j=1}^{m} t_j C_j$.

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Suppose a group G contains a free semigroup SF_2 generated by two elements a and b. Then $SF_2 \subset G$ is paradoxical with respect to the action of the group on itself.

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Theorem (Tarski '29)

Let G be a group acting on a set X. A subset A of X is non-paradoxical if and only if there is a finitely additive, invariant measure $\mu : \mathcal{P}(X) \to [0, +\infty]$ such that $\mu(A) = 1$.

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Proposition (Rosenblatt '74)

Groups of sub-exponential growth are supramenable.

Example (Lamplighter group)

The group

$$\left(\bigoplus_{\mathbb{Z}} \frac{\mathbb{Z}}{2\mathbb{Z}} \right) \rtimes \mathbb{Z}$$

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This example shows that the class of supramenable groups is not closed under semi-direct products. It is unknown if the direct product of two supramenable groups is still supramenable.

Definiton (Exel '94 + McClanahan '95)

Let X be a topological space and $\{D_g\}_{g \in G}$ be a family of open subsets of X. A partial action of a group G on X is a map

$$\theta : G \to \mathrm{pHomeo}(X)$$
$$g \mapsto \theta_g : D_{g^{-1}} \to D_g$$

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such that:
1) $\theta_e = \mathrm{Id}_X$*;*
2) For all $g, h \in G$ *and* $x \in D_{g^{-1}}$*, if* $\theta_g(x) \in D_{h^{-1}}$*, then* $x \in D_{(hg)^{-1}}$ *and* $\theta_h \circ \theta_g(x) = \theta_{hg}(x)$*.*

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Example

Let $\theta : G \to \mathrm{Homeo}(X)$ be a (global) action of a group G on a topological space X. Given $D \subset X$ an open set, define, for all $g \in G$, $D_g := D \cap \theta_g(D)$. Then one can check that the restrictions of the maps θ_g to the sets D_g give rise to a partial action of G on D.

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be a partial action of a group G on a topological space X. We say a measure ν on X is invariant if for all $E \in \mathcal{B}(X)$ and $g \in G$, we have that

$$\nu(\theta_g(E \cap D_{g^{-1}})) = \nu(E \cap D_{g^{-1}}).$$

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It is well known that a group is amenable if and only if whenever it acts on a compact Hausdorff space, then the space admits an invariant probability measure. For supramenable groups we have the following:

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Proof.

($\Leftarrow$) If G is a non-supramenable group, then it has a subset A which is paradoxical with respect to the action of the group on itself. Let $j : G \to \beta G$ be the imbedding of G on its beta-compactification. Consider the partial action obtained by restricting the canonical action of G on βG to $\overline{j(A)}$. Then this partial action does not admit an invariant probability measure. $\square$

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Definiton (Exel '94 + McClanahan '95)

Let A be a C^*-algebra and $\{I_g\}_{g\in G}$ be a family of ideals of A. A partial action of a group G on A is a map

$$\begin{aligned}\theta : G &\to \mathrm{pIso}(A)\\ g &\mapsto \theta_g : I_{g^{-1}} \to I_g\end{aligned}$$

such that:
1) $\theta_e = Id_A$;
2) For all $g, h \in G$, $x \in I_{g^{-1}}$, *if* $\theta_g(x) \in I_{h^{-1}}$, *then* $x \in I_{(hg)^{-1}}$ *and* $\theta_h \circ \theta_g(x) = \theta_{hg}(x)$.

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Given a partial action θ of a group G on a C*-algebra A, one associates to it another C*-algebra, called partial crossed product and denoted by $A \rtimes_\theta G$. It contains the C*-algebra A, and the data of the partial action. Its construction is a generalization of the usual crossed product.

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Theorem

Let θ be a partial action of a supramenable group G on a unital C^*-algebra A which has a tracial state. Then $A \rtimes_\theta G$ has a tracial state.

It is well known that if τ is a positive functional defined on an ideal of a C*-algebra, then it has a unique extension, with same norm, to the whole C*-algebra. It is a straightforward computation to check that if τ is a trace, then the extension will also be a trace.

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Lemma

Let I be an ideal of a C^*-algebra A and τ a trace on I. Then there exists a unique extension of τ to a trace τ' on A satisfying $\|\tau\| = \|\tau'\|$.

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Let A be a unital C^*-algebra which has a tracial state and τ be an extreme point of $T(A)$, the set of tracial states of A. Then, for every ideal I of A, $\|\tau|_I\|$ is either 0 or 1.

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Proof.

We would like to have some "invariant" tracial state σ on A. Invariant in the sense that $\sigma(\theta_g(a)) = \sigma(a)$ for all $g \in G$, $a \in I_{g^{-1}}$. If such a tracial state exists, then, by using the canonical conditional expectation $\Phi : A \rtimes_\theta G \to A$, we get that $\sigma \circ \Phi$ is a tracial state on $A \rtimes_\theta G$.

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In order to produce σ, we start with some tracial state τ which is an extreme point of $T(A)$. For each $g \in G$, $\tau \circ \theta_{g^{-1}}$ is a trace on I_g. Use the lemma to extend it to a trace τ_g defined on all of A.

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For each $a \in A$, let $\hat{a} \in \ell^\infty(G)$ be defined by $\hat{a}(g) := \tau_g(a)$, $g \in G$.
Let $Y := \|g \in G : \|\tau|_{I_g}\| = 1\|$. Use Tarski's Theorem to get an invariant, finitely additive measure on G such that $\mu(Y) = 1$.
Then $\sigma(a) := \int_G \hat{a} d\mu$ is the desired invariant tracial state. □

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We say a partial action

$$\theta : G \to \mathrm{pHomeo}(X)$$
$$g \mapsto \theta_g : D_{g^{-1}} \to D_g$$

on a compact Hausdorff space X, such that each D_g is clopen, is amenable if there exists a net $(m_i)_{i\in I}$ of continuous maps $m_i : X \to \mathrm{Prob}(G)$ such that:
(1) For every $x \in X$ and $i \in I$, $\mathrm{supp}(m_i^x) \subset \{g \in G : x \in D_g\}$;
(2) For every $g \in G$, $\sup_{x\in D_{g^{-1}}} \|g.m_i^x - m_i^{g.x}\|_1 \to 0$.

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Proposition

A partial action on a compact Hausdorff space, with clopen domains, is amenable if and only if the groupoid of the partial action is amenable, if and only if the associated Fell bundle has the approximation property.

Proposition (Kellerhals-Monod-Rørdam '13 + Exel-Laca-Quigg '02 + Giordano-Sierakowski '14)

Let G be a countable, amenable and non-supramenable group. Then G admits a free, minimal, amenable and purely infinite partial action on the Cantor set K, with compact-open domains. The associated partial crossed product $C(K) \rtimes G$ of any such partial action is a simple, purely infinite and nuclear C^*-algebra.

Thank you!