WHAT IS...
Quantum Chaos?

Ze'ev Rudnick

A referee of one of my grant proposals complained recently that the text did not explain "what is quantum chaos"; the desire for an answer to that question was the sole reason he had agreed to review the proposal. I was bemused, since that particular proposal had nothing to do with the subject. However, the incident did make me amenable to suggestions that I try and provide such an explanation.

Quantum chaos began as an attempt to find chaos, in the sense of extreme sensitivity to changes in initial conditions, in quantum mechanical systems. That attempt failed, since it was eventually realized that such sensitivity does not exist. However, along the way it was found that chaos (or the lack of it) is reflected in quantum systems in other ways. I will describe one such phenomenon, concerning the statistics of the energy spectrum, and the mathematical challenges it poses.

We will consider a simple model system, that of a billiard particle (discussed in this column by Yakov Sinai in the April 2004 *Notices*). The description of the system in the language of *classical mechanics* is of a point particle moving without friction in a billiard table—a bounded planar enclosure where the particle reflects from the boundary so that the angle of incidence equals the angle of reflection. The total energy E is conserved during the motion. The energy can attain a *continuum* of values: the faster the particle travels, the greater the energy.

A *quantum mechanical* description of this system at a given instant of time includes the wave function of the particle $\psi(x,t)$, which vanishes at the boundary of the billiard. The probability density of finding the particle in position x at time t is $|\psi(x,t)|^2$, once we have normalized the total integral over the billiard to be unity. The time evolution is described by Schrödinger's equation $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2\mu}\Delta\psi$, where $\hbar$ is Planck's constant, μ is the mass of the particle, and $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$ is the Laplacian. A solution of the equation whose amplitude does not change in time, that is a *stationary* solution, is of the form $\psi_n(x,t) = e^{-itE_n/\hbar}\varphi_n(x)$, where φ_n satisfies the eigenvalue equation $-\frac{\hbar^2}{2\mu}\Delta\varphi_n = E_n\varphi_n$. The numbers E_n are the quantal energy levels of the system. Unlike the case of classical mechanics, where energy is a continuous variable, here the energy levels are a *discrete* set. It is convenient in our case to work with the rescaled levels $\lambda_n := 2\mu E_n/\hbar^2$. A simple example is the rectangular billiard with sides a and b, in which case the levels are $\lambda_{m,n} = \pi^2\left\{(\frac{m}{a})^2 + (\frac{n}{b})^2\right\}$, where m, n range over all positive integers (this is one of the very rare cases where one can explicitly write down the levels!).

How can we relate the two different descriptions of our billiard system? How is the classical mechanics description reflected in the quantum description when Planck's constant $\hbar$ is small (or equivalently in the case at hand, when $\lambda \to \infty$) relative to the characteristic actions of the system? Are there universal laws to be found in the energy spectrum? What are the statistical properties of highly excited eigenfunctions? These are some of the questions that quantum chaos tries to answer. In this article I will focus on the statistics of the spectrum. There has been substantial progress on other aspects, see the recent reviews by Marklof and Zelditch in [3].

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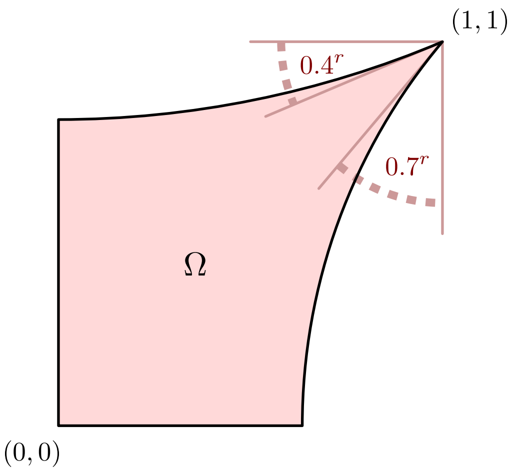

Figure 1. One of the regions proven by Sinai to be classically chaotic is this region Ω constructed from line segments and circular arcs.

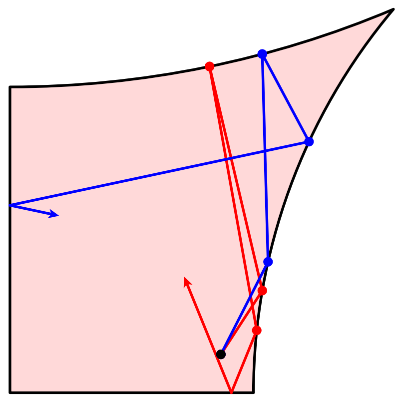

Figure 2. This figure gives some idea of how classical ergodicity arises in Ω.

Traditionally, analysis of the spectrum recovers information such as the *total area* of the billiard, from the asymptotics of the counting function $N(\lambda) = \#\{\lambda_n \le \lambda\}$: As $\lambda \to \infty$, $N(\lambda) \sim \frac{area}{4\pi}\lambda$ (Weyl's law). Quantum chaos provides completely different information: The claim is that we should be able to recover the coarse nature of the *dynamics* of the classical system, such as whether they are very regular ("integrable") or "chaotic". The term *integrable* can mean a variety of things, the least of which is that, in two degrees of freedom, there is another conserved quantity besides energy, and ideally that the equations of motion can be explicitly solved by quadratures. Examples are the rectangular billiard, where the magnitudes of the momenta along the rectangle's axes are conserved, or billiards in an ellipse, where the product of angular momenta about the two foci is conserved, and each billiard trajectory repeatedly touches a conic confocal with the ellipse. The term *chaotic* indicates an exponential sensitivity to changes of initial condition, as well as *ergodicity* of the motion. One example is Sinai's billiard, a square billiard with a central disk removed; another class of shapes investigated by Sinai, and proved by him to be classically chaotic, includes the odd region shown in Figure 1. Figure 2 gives some idea of how ergodicity arises. There are many mixed systems where chaos and integrability coexist, such as the *mushroom billiard*—a semicircle atop a rectangular foot (featured on the cover of the March 2006 issue of the *Notices* to accompany an article by Mason Porter and Steven Lansel).

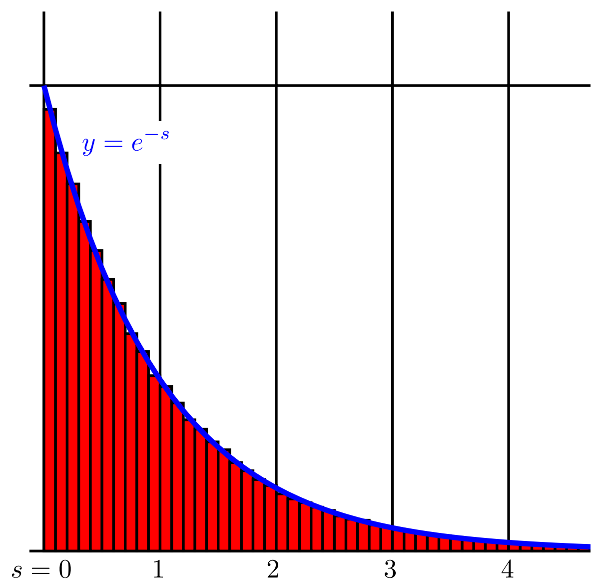

Figure 3. It is conjectured that the distribution of eigenvalues $\pi^2(m^2/a^2 + n^2/b^2)$ of a rectangle with sufficiently incommensurable sides a, b is that of a Poisson process. The mean is $4\pi/ab$ by simple geometric reasoning, in conformity with Weyl's asymptotic formula. Here are plotted the statistics of the gaps $\lambda_{i+1} - \lambda_i$ found for the first 250,000 eigenvalues of a rectangle with side/bottom ratio $\sqrt[4]{5}$ and area 4π, binned into intervals of 0.1, compared to the expected probability density e^{-s}.

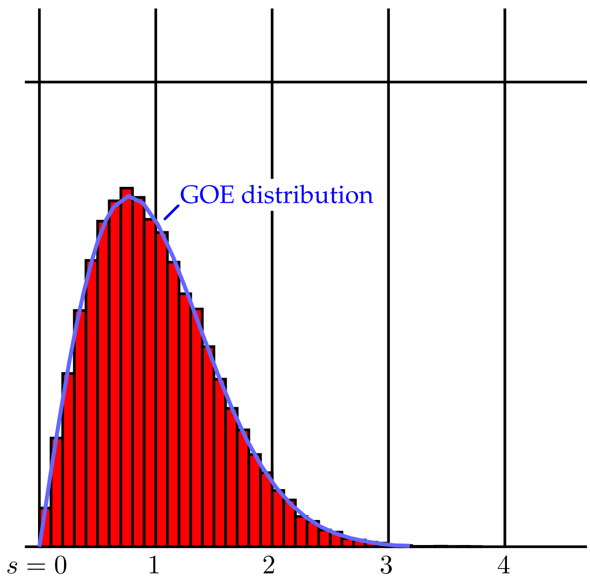

Figure 4. Plotted here are the normalized gaps between roughly 50,000 sorted eigenvalues for the domain Ω, computed by Alex Barnett, compared to the distribution of the normalized gaps between successive eigenvalues of a large random real symmetric matrix picked from the "Gaussian Orthogonal Ensemble", where the matrix entries are independent (save for the symmetry requirement) and the probability distribution is invariant under orthogonal transformations.

One way to see the effect of the classical dynamics is to study local statistics of the energy spectrum, such as the *level spacing distribution* $P(s)$, which is the distribution function of nearest-neighbor spacings $\lambda_{n+1} - \lambda_n$ as we run over all levels. In other words, the asymptotic proportion of such spacings below a given bound x is $\int_{-\infty}^{x} P(s)ds$. A dramatic insight of quantum chaos is given by the universality conjectures for $P(s)$:

- If the classical dynamics is *integrable*, then $P(s)$ coincides with the corresponding quantity for a sequence of *uncorrelated* levels (the Poisson ensemble) with the same mean spacing: $P(s) = ce^{-cs}$, $c = area/4\pi$ (Berry and Tabor, 1977).
- If the classical dynamics is *chaotic*, then $P(s)$ coincides with the corresponding quantity for the eigenvalues of a suitable ensemble of *random matrices* (Bohigas, Giannoni, and Schmit, 1984). Remarkably, a related distribution is observed for the zeros of Riemann's zeta function.

Not a single instance of these conjectures is known, in fact there are counterexamples, but the conjectures are expected to hold "generically", that is unless we have a good reason to think otherwise. A counterexample in the integrable case is the *square* billiard, where due to multiplicities in the spectrum, $P(s)$ collapses to a point mass at the origin. Deviations are also seen in the chaotic case in arithmetic examples. Nonetheless, empirical studies offer tantalizing evidence for the "generic" truth of the conjectures, as Figures 3 and 4 show.

Some progress on the Berry-Tabor conjecture in the case of the rectangle billiard has been achieved by Sarnak, by Eskin, Margulis, and Mozes, and by Marklof. However, we are still far from the goal even there. For instance, an implication of the conjecture is that there should be *arbitrarily large* gaps in the spectrum. Can you prove this for rectangles with aspect ratio $\sqrt[4]{5}$?

The behavior of $P(s)$ is governed by the statistics of the number $N(\lambda, L)$ of levels in windows whose location λ is chosen at random, and whose length L is of the order of the mean spacing between levels. Statistics for *larger* windows also offer information about the classical dynamics and are often easier to study. An important example is the *variance* of $N(\lambda, L)$, whose growth rate is believed to distinguish integrability from chaos [1] (in "generic" cases; there are arithmetic counterexamples). Another example is the *value distribution* of $N(\lambda, L)$, normalized to have mean zero and variance unity. It is believed that in the chaotic case the distribution is Gaussian. In the integrable case it has radically different behavior: For large L, it is a system-dependent, non-Gaussian distribution [2]. For smaller L, less is understood: In the case of the rectangle billiard, the distribution becomes Gaussian, as was proved recently by Hughes and Rudnick, and by Wigman.

Further Reading

[1] M. V. Berry, Quantum chaology (The Bakerian Lecture), *Proc. R. Soc. A* 413 (1987), 183-198.

[2] P. Bleher, Trace formula for quantum integrable systems, lattice-point problem, and small divisors, in *Emerging applications of number theory* (Minneapolis, MN, 1996), 1–38, IMA Vol. Math. Appl., **109**, Springer, New York, 1999.

[3] J. Marklof, Arithmetic Quantum Chaos, and S. Zelditch, Quantum ergodicity and mixing of eigenfunctions, in *Encyclopedia of mathematical physics*, Vol. 1, edited by J.-P. Françoise, G. L. Naber, and T. S. Tsun, Academic Press/Elsevier Science, Oxford, 2006.