WHAT IS... Property A?

Piotr Nowak and Guoliang Yu

Amenability of metric spaces usually manifests itself in the existence of large sets with relatively small boundary. Think of Euclidean n-space. Here the volume of the ball of radius r is proportional to r^n while the area of its boundary, the sphere of radius r, is proportional to r^{n-1}. Thus the area of the boundary is roughly $1/r$ of the volume of the ball, and the ratio tends to 0 as r increases. Such amenability properties are very useful in many contexts. Unfortunately, it usually does not take much to prevent them from being satisfied. A simple example of a space that is not amenable in this sense is the hyperbolic plane $\mathbb{H}^2$. Property A is a weak amenability-type condition that is much less restrictive than the one described above and is satisfied by many known metric spaces. It was introduced in [3] and turns out to be of great importance in many areas of mathematics. We start with the definition.

Definition ([3]). *A discrete metric space X has Property A if for every $\varepsilon > 0$ and every $R > 0$ there is a family $\{A_x\}_{x\in X}$ of finite subsets of $X \times \mathbb{N}$ and a number $S > 0$ such that*

(1) $\dfrac{\#(A_x \Delta A_y)}{\#(A_x \cap A_y)} < \varepsilon$ *whenever* $d(x, y) \le R$,

(2) $A_x \subseteq B(x, S) \times \mathbb{N}$ *for every* $x \in X$.

Here $A_x \Delta A_y = (A_x \setminus A_y) \cup (A_y \setminus A_x)$ denotes the symmetric difference. In simple words, these conditions mean that when points x and y are at most R apart, then the sets A_x and A_y are almost equal, but if the distance between x and y is at least $2S$, then A_x and A_y are disjoint. Property A is a large-scale geometric property, meaning that it is preserved by quasi-isometries—the copy of $\mathbb{N}$ appearing in the definition of the sets A_x is exactly to guarantee this invariance. We can thus say that a locally compact metric space has Property A if it is quasi-isometric to a discrete metric space with Property A.

Let us look at some examples. Observe that any finite metric space X has Property A trivially: simply take $A_x = X \times \{1\}$ for all $x \in X$, and the ratio in the above definition vanishes. Next in line is the set of integers $\mathbb{Z}$, with the usual Euclidean metric. To show Property A for $\mathbb{Z}$, given R and ε, choose an integer $r \ge 2R\varepsilon^{-1}$ and define A_x to be the ball of radius r centered at x for all $x \in \mathbb{Z}$.

The most interesting case is when X in the definition above is a finitely generated group G, equipped with a metric in the following way. Having fixed a finite set of generators, we declare the distance between two elements $g, h \in G$ to be the smallest number of generators and their inverses necessary to write $g^{-1}h$ as a word. Such a metric is called the *word length metric* on G. For example, if we consider $\mathbb{Z}$ as an additive group generated by a single element $\{1\}$, the word length metric is simply the usual metric on $\mathbb{Z}$ induced by absolute value.

In this setting the resemblance of Property A to amenability can be easily seen through Følner's criterion. A group G is called amenable if for every $\varepsilon > 0$ and every $R > 0$ there exists a finite set $F \subseteq G$ such that

$$\frac{\#(F \,\Delta\, gF)}{\#F} < \varepsilon,$$

whenever the distance of g to the identity element is less than R. Above gF is the translation of F by element g, and the sets F are called Følner sets.

It follows from the definition that amenable groups satisfy Property A: given $\varepsilon > 0$ and a corresponding Følner set F, one simply takes $A_g = gF$, and it can be easily verified that the sets A_g defined this way satisfy the required condition. In fact, close examination of definitions reveals that for groups, Property A is "non-equivariant" amenability—simply imagine what "equivariant Property A" would mean to end up exactly with amenability, modulo some simple calculations.

But Property A covers a much wider class of groups than that of amenable groups. Typical examples of non-amenable groups are the free groups on $n \ge 2$ generators $\mathbb{F}_n$, but as it turns out, free groups satisfy Property A. To give the proof recall that the Cayley graph of a free group is a tree. In this tree fix a geodesic ray γ that originates at the identity, and given $n \in \mathbb{N}$ for every $x \in \mathbb{F}_2$ define the set A_x to be the the unique geodesic segment of length $2n$ from x in the direction of the ray γ. It is not hard

Piotr Nowak is a Ph.D. candidate at Vanderbilt University. His email address is `piotr.nowak@vanderbilt.edu`.
Guoliang Yu is professor of mathematics at Vanderbilt University. His email address is `guoliang.yu@vanderbilt.edu`.

to check that for any ε there is an $n \in \mathbb{N}$ such that the required conditions are satisfied. This method of proof can be generalized to show that hyperbolic groups and the hyperbolic plane $\mathbb{H}^2$ mentioned earlier have Property A.

Another natural example is furnished by discrete linear groups, i.e., subgroups of the group $GL_n(F)$ of invertible $n \times n$ matrices over a field F. The fact that they satisfy Property A was proved by Guentner, Higson and Weinberger. The list of classes of groups for which Property A has been verified also includes one-relator groups, Coxeter groups, groups acting on finite dimensional CAT(0) cube complexes, and many more.

The original motivation for introducing Property A, see [3], was that it is a sufficient condition to coarsely embed a group into a Hilbert space. A coarse embedding is a natural notion of inclusion in large-scale geometry. We recall the definition.

Definition (Gromov)**.** *A function $f : X \to Y$ between metric spaces is a coarse embedding if*

$$\varphi_-(d_X(x_1, x_2)) \le d_Y(f(x_1), f(x_2)) \le \varphi_+(d_X(x_1, x_2))$$

for all $x_1, x_2 \in X$, where $\varphi_-, \varphi_+ : [0, \infty) \to [0, \infty)$ are nondecreasing functions and $\lim_{t\to\infty} \varphi_-(t) = \infty$.

In [3] the Novikov conjecture was proved under the assumption of coarse embeddability into Hilbert spaces. The Novikov conjecture is a rigidity statement about high-dimensional, compact, smooth manifolds. A compact manifold is said to be aspherical if its universal cover is contractible (a typical example is a torus T^n with $\mathbb{R}^n$ as its universal cover). The Borel conjecture claims that every aspherical manifold M is rigid in the sense that if another compact manifold N is homotopy equivalent to M, then N is actually homemorphic to M. In the case of aspherical manifolds, the Novikov conjecture is an infinitesimal version of the Borel conjecture. It asserts that the rational Pontryagin classes, i.e., certain characteristic classes associated to the tangent bundle of the manifold, are homotopy invariants. For more general manifolds the Novikov conjecture claims that higher signatures are homotopy invariants.

Thus the results of [3] yield the following implication:

if G has Property A then the Novikov conjecture is true for all closed manifolds with fundamental group G.

This theorem sparked significant interest in the notion of Property A, and as a result Higson and Roe characterized Property A in terms of existence of a topologically amenable action on some compact space. The notion of amenability for group actions was introduced by Zimmer in ergodic theory and later adapted to the topological setting by Anantharaman-Delaroche and Renault.

Another connection arose in the theory of C^*-algebras. The work of Guentner and Kaminker and subsequently Ozawa showed that, for groups, Property A is equivalent to exactness of $C_r^*(G)$, the reduced C^*-algebra of G. Exactness of C^*-algebras was introduced by Kirchberg, and a long-standing problem was whether $C_r^*(G)$ is exact for every G.

These results immediately prompted the question of whether there exist metric spaces and, more importantly, finitely generated groups, that do not have Property A. However this question turned out to be quite difficult, and still only a handful of examples is known. One of them is due to Gromov, who observed that a metric space constructed out of a sequence of expanders does not admit a coarse embedding into a Hilbert space and therefore cannot satisfy Property A. We refer to [2] for a description of expanders. For finitely generated groups the problem of finding examples that would not satisfy Property A is much harder. A natural idea is to try to find a group that would metrically contain, in its Cayley graph, a sequence of expanders. Such a group could not satisfy Property A for the same reason given earlier. Gromov implemented this idea in the realm of his random groups, but a search for more examples is under way. If found, such groups might shed some light on many problems in geometric group theory as well as for example index theory, where groups without Property A served as counterexamples to some versions of the Baum-Connes conjecture.

Another major question is to what degree Property A captures coarse embeddability into a Hilbert space. In particular, it was not known whether the two notions coincide. An example of a metric space distinguishing the two properties was given in [1] and can be described as follows. Take discrete cubes $\{0, 1\}^n$, $n = 1, 2, \ldots$. We define two points in a cube to be at distance k if they differ in exactly k coordinates. It is not hard to see that the disjoint union of these cubes (with the distance between cubes defined appropriately) embeds coarsely into a Hilbert space. However using the fact that the cube $\{0, 1\}^n$ has a structure of an amenable group—namely that of $\mathbb{Z}_2^n$—it was proved that such a disjoint union of cubes does not have Property A.

It would be extremely interesting to find more examples of finitely generated groups that do not satisfy Property A, especially ones that *do* embed coarsely into a Hilbert space. Such groups would have to exhibit completely new geometric phenomena. At present these issues are far from being understood.

Further Reading

[1] P. Nowak, Coarsely embeddable metric spaces without Property A, *J. Funct. Anal.* (1) **252** (2007), 126–136.

[2] P. Sarnak, What is…an expander?, *Notices of the AMS* **51** (2004), no. 7, 762–763.

[3] G. Yu, The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space, *Invent. Math.* (1) **139** (2000), 201–240.