WHAT IS...

a Period Domain?

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The notion of period domain goes back to the very beginnings of algebraic geometry, to the study of elliptic curves. These are compact Riemann surfaces of genus one, defined as the complex solutions of $y^2 = x^3 + ax + b$, plus one point at infinity. Such a surface E is a compact torus and so has a homology basis $\{\delta, \gamma\}$, where the intersection number of the two cycles is $\delta \cdot \gamma = 1$. Consider the differential one-form $\omega = dx/y$, which is holomorphic in local coordinates on E. The *period matrix* of E is given by the integrals

$$(1) \qquad (A, B) = \left(\int_\delta \omega, \int_\gamma \omega \right).$$

Multiplying ω by a suitable nonzero scalar, we may assume that its A period is one. Then a calculation, based on the fact that

$$(2) \qquad \sqrt{-1} \int_S \omega \wedge \bar{\omega} > 0,$$

shows that its B period has positive imaginary part. Consequently, the upper half plane $\mathcal{H} = \{ z = x + iy \in \mathbb{C} \mid y > 0 \}$ parametrizes the set of so-called normalized B periods. The upper half plane is the first example of a period domain.

An elliptic curve plus a homology marking, i.e., a choice of integer homology basis such that $\delta \cdot \gamma = 1$, determines a point in the period domain $\mathcal{H}$. Two normalized homology bases are related by an element of the group Γ of unimodular matrices with integer entries, and the two normalized B periods are related by the corresponding fractional linear transformation. If one has a family of elliptic curves E_t that depends holomorphically on t, then $B(t)$ is locally defined and varies holomorphically. The map $t \mapsto B(t)$ is the *period map*. Since $\mathcal{H}$ is biholomorphic to the unit disk, one finds, as a consequence of the uniformization theorem, that any nonconstant family of elliptic curves parametrized by the Riemann sphere must have at least three

singular fibers. The equivalence class of the normalized period modulo the action of the group Γ is intrinsically defined and lies in the quotient $\Gamma \backslash \mathcal{H}$. Thus, if E_t is a family of elliptic curves parametrized by a base S, one has a globally defined period map $S \to \Gamma \backslash \mathcal{H}$.

The notion of period domain is easily generalized to Riemann surfaces of higher genus, in which case the period matrix (A, B) has size g by $2g$. The role of the upper half plane is played by the Hermitian symmetric space of $Sp(2g, \mathbb{R})$: the Siegel upper half space of genus g, given by $g \times g$ complex symmetric matrices with positive definite imaginary part. Written $\mathcal{H}_g$, this is the space of normalized B periods. The group acting on it is $\Gamma = Sp(2g, \mathbb{Z})$.

To make the transition to algebraic manifolds of higher dimension, we think in terms of Hodge structures: the decomposition of the complex cohomology into the spaces $H^{p,q}$ spanned by closed differential forms expressible locally as a sum of terms $f\, dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$, where $z_1, \cdots, z_n$ are holomorphic local coordinates. For a projective algebraic manifold one has $H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}$ where $\overline{H^{p,q}} = H^{q,p}$. Such a decomposition, together with the lattice given by the integer cohomology modulo torsion, is a *Hodge structure of weight k*. For a Riemann surface the Hodge structure $H^1(X, \mathbb{C}) = H^{1,0} \oplus H^{0,1}$ is of weight one, and $H^{1,0}$ is identified with the row space of the period matrix. This space is subject to two important relations. One comes from the fact that for holomorphic differentials $\phi = f\, dz$ and $\psi = g\, dz$, the product $\phi \wedge \psi$ vanishes. The other comes from the fact that $i\phi \wedge \bar{\phi}$ is a positive multiple of the volume form. These are the *first and second Riemann bilinear relations*. A Hodge structure satisfying these relations is *polarized* (by cup product). In terms of normalized B periods, (1) B is symmetric, and (2) B has positive definite imaginary part. The Siegel upper half space parametrizes polarized Hodge structures of weight one.

General period domains are parameter spaces for polarized Hodge structures of weight k. The model is the subspace of the k-th cohomology of a complex projective algebraic manifold of dimension k which is annihilated by cup-product with the hyperplane class. Polarization is the generalization

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of the first and second Riemann bilinear relations. The resulting parameter space D can be represented, just as in the case of $\mathcal{H}_g = Sp(2g, \mathbb{R})/U(g)$, as a complex homogeneous space G/V, where G is a Lie group and V is a compact subgroup. However, V is rarely a maximal compact subgroup, and so D is rarely hermitian symmetric. Important special cases in which D is of weight $k > 1$ but is nonetheless hermitian symmetric are the period domains of K3 surfaces and of the cyclic cubic threefolds associated to cubic surfaces.

For period domains of weight $k > 1$, there is a differential relation not seen in the weight one case. To explain it, consider the subspaces $F^p = H^{k,0} \oplus \cdots \oplus H^{p,k-p}$. They define the *Hodge filtration* $F^k \subset F^{k-1} \subset \cdots$, denoted by $F^\bullet$. To give a Hodge decomposition is to give a Hodge filtration and *vice versa*. The Hodge filtration of a family of algebraic varieties that varies holomorphically with parameters also varies holomorphically. However, more is true. If t is a parameter on which $F^p(t)$ depends holomorphically, then the derivative satisfies

$$(3) \qquad \dot{F}^p(t) \subset F^{p-1}(t).$$

This relation is now known as *Griffiths transversality*.

More formally, let TD be the holomorphic tangent bundle of D. The relation (3) defines a holomorphic subbundle I to which period mappings coming from geometry are tangent. Mappings satisfying this differential relation are called *horizontal*. A general period map is just a horizontal holomorphic map. An immediate consequence of horizontality is that most Hodge structures do not come from geometry.

Curvature computations along the horizontal distribution imply that period maps defined on the unit disk are distance decreasing with respect to the Poincaré metric on the disk and the G-invariant metric on D. The distance-decreasing property of period maps from the punctured disk Δ^* to D forces them to extend across the origin. Thus a version of the Riemann removable singularity theorem holds. Period domains act, with respect to horizontal holomorphic mappings, as if they were bounded domains.

On the n-th cohomology of a family of nonsingular algebraic varieties over Δ^* is defined a *monodromy transformation T*. It controls the analytic continuation of the period map along a loop around the origin. The period mapping associated to the family over the punctured disk takes the form $\tau : \Delta^* \to \{T^i\}\backslash D$. Using the fact that T is an integral matrix and τ is distance-decreasing, one finds that the eigenvalues of T are m-th roots of unity. Passing to a finite covering of Δ^* we may assume that T is unipotent with logarithm N.

The distance decreasing properties of maps tangent to I make it possible to take limits of Hodge structures, just as one takes limits in calculus. The starting point is the asymptotic formula for a period map on the punctured disk,

$$(4) \qquad \phi(t) \sim \exp\left(\left(\frac{\log t}{2\pi\sqrt{-1}}\right)N\right)F_0^\bullet,$$

where the "limit filtration" $F_0^\bullet$, which lies in $\check{D}$, defines a so-called *mixed Hodge structure*. The previous relation, due to Schmid, is the starting point for the result that the index of unipotency of T^m is $n+1$, i.e., $(T^m - I)^{n+1} = 0$.

The boundary points for the limit Hodge filtration lie in the *compact dual* $\hat{D}$ of D, obtained by ignoring the positivity condition in the definition of polarization. For elliptic curves, $\hat{D}$ is just $\mathbb{P}^1$, and the limiting mixed Hodge structures added to compactify $\Gamma\backslash\mathcal{H}$ correspond to cusps of the fundamental domain of $SL_2(\mathbb{Z})$. It is a remarkable fact, encoded in the Clemens-Schmid exact sequence, that the limit mixed Hodge structure can largely be read from the geometry of the singular fiber.

The subbundle I usually generates TD under Lie bracket, as in the case of the contact distribution on the three-sphere or its holomorphic analogue, given in local coordinates by the null space of $\omega = dz - x\,dy$. As with the contact distribution, the dimension of integral submanifolds of I is smaller than the dimension of I, indeed, often much smaller. One may suspect that a nontrivial period mapping defined on a quasi-projective variety "comes from geometry". However, with the exception of weight one (abelian varieties) and K3 surfaces, almost nothing is known about this question.

We close with some observations of a more arithmetic character. First, the projective variety $\hat{D}$ is defined over $\mathbb{Q}$. Thus it makes sense to ask for the field of definition of $F^\bullet(t)$. If $F^\bullet$ is simple, then $End(F^\bullet) \otimes \mathbb{Q}$ is a division algebra whose center is a field k with $[k : \mathbb{Q}] \leq \dim H$. We say that the Hodge structure has CM type when the division algebra is commutative and equality holds. Equivalently, the *Mumford-Tate group* $M(F^\bullet)$ is an algebraic torus. The Mumford-Tate group is the $\mathbb{Q}$-subgroup of $Aut(H, \mathbb{Q})$ that fixes all the rational (p, p) classes ("Hodge classes") in the tensor algebra on H and its dual. The nature of Hodge structures of CM type, which have played an essential role in the weight one case, is just beginning to be explored in higher weight. The interface between Hodge theory, period domains, and arithmetic is one of the deepest and most promising areas for future work.

Further reading. Some foundational articles are those by P. A. Griffiths and W. Schmid in *Acta Math.* (1968), by P. Deligne in *Pub. Math. IHES* (1971), and the article on the singularities of the period mapping by W. Schmid in *Invent. Math.* (1973). Recent expository works include *Hodge Theory and Complex Algebraic Geometry I and II*, by C. Voisin, and *Period Domains and Period Mappings*, by J. Carlson, C. Peters, and S. Müller-Stach. The notes by B. Moonen on Mumford-Tate groups (1999) illuminate the material of the last paragraph.