WHAT IS...
a Woodin Cardinal?

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All mathematical statements can be expressed in the *language of set theory* (LST), whose variables are understood as ranging over sets, and whose only non-logical symbol $\in$ stands for the membership relation. The vast majority of mathematical proofs require no more than the axioms of Zermelo-Fraenkel set theory with Choice, or ZFC. In this way, set theory provides a foundation for all of mathematics. Nevertheless, a surprising number of basic questions about sets in general are not decided by the axioms of ZFC; moreover many of the more abstract questions of analysis, algebra, and topology are left similarly undecided. Perhaps the most famous of the undecided questions is Cantor's Continuum Problem: what is the cardinality of the set of all real numbers? Another more concrete such question is whether all sets of real numbers that are projective are Lebesgue measurable. (A set is projective if and only if it can be built up from a countable intersection of open sets by taking continuous images and complements finitely many times.)

The most fruitful way to extend ZFC so as to remove some of this incompleteness is to strengthen its axiom asserting that there are infinite sets. Large cardinal hypotheses do this.

For α an ordinal number, we define V_α by transfinite induction: $V_0 = \emptyset$, $V_{\alpha+1} = \{x \mid x \subseteq V_\alpha\}$, and $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for λ a limit ordinal. Thus V_α consists of those sets that can be built up in $< \alpha$ stages by taking sets of objects previously formed. Each V_α is transitive (i.e., if $x \in V_\alpha$, then $x \subseteq V_\alpha$), and $\alpha \leq \beta \Rightarrow V_\alpha \subseteq V_\beta$. It follows from the Axiom of Foundation of ZFC that every set is in some V_α. We write V for the union of all the V_α, the universe of all sets. V is not itself a set, but rather what is sometimes called a *proper class*.

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Large cardinal hypotheses attempt to capture the idea that there are more sets than one can possibly imagine, by means of the following informal *reflection principle*: suitable properties of V reflect to some V_α. For one example, V is infinite, and the ordinary Axiom of Infinity asserts that some V_α shares this property. For another, V is a model of second-order ZFC (also called *Kelley-Morse set theory*), so our informal principle leads to the assertion that some V_κ satisfies second-order ZFC. This is equivalent to κ being an inaccessible cardinal.

The stronger large cardinal hypotheses assert the existence of elementary embeddings $j : V \to M$ that are nontrivial, i.e., not the identity. Here M is a transitive class. Elementarity means that whenever $\varphi(v_1, \dots, v_n)$ is a formula of LST with the displayed free variables, and $a_1, \dots, a_n$ are sets, then

$$V \models \varphi[a_1, \dots, a_n] \Leftrightarrow M \models \varphi[j(a_1), \dots, j(a_n)].$$

(For N a transitive set or class, and $b_1, \dots, b_n \in N$, we say $N \models \varphi[b_1, \dots, b_n]$ if and only if $\varphi(v_1, \dots, v_n)$ is true when its quantifiers are understood as ranging over N, its variable v_i is understood as naming b_i, and the $\in$ symbol of LST is understood as standing for set-membership.) For such a j, the *critical point* of j, or $\mathrm{crit}(j)$, is the least ordinal κ such that $j(\kappa) \neq \kappa$. Since j is order-preserving, this implies $\kappa < j(\kappa)$. The reflection here occurs at κ: if $\varphi(\kappa)$ holds in V, and M resembles V enough that $M \models \varphi[\kappa]$, then $M \models \exists \alpha < v_1 \varphi(\alpha)[j(\kappa)]$, so by the elementarity of j, there is an $\alpha < \kappa$ such that $\varphi(\alpha)$ holds in V. The more M resembles V, the more reflection we have. By a result of K. Kunen, $M = V$ is impossible, but weaker conditions on M lead to plausible and useful principles.

For example, κ is *measurable* if and only if there is any transitive M and nontrivial embedding $j : V \to M$ with $\mathrm{crit}(j) = \kappa$. If there is such a j, M with $V_\beta \subseteq M$, we say κ is *β-strong*, and we say κ is *strong* if and only if κ is β-strong for all β. If there is such a j, M such that every λ-sequence of elements of M belongs to M, then we say κ is λ-*supercompact*, and we say κ is *supercompact* if and only if κ is λ-supercompact for all λ. We have listed these properties in order of increasing strength; indeed, if κ is strong, then $V_\kappa \models$ "there is a measurable cardinal", and if κ is supercompact, then $V_\kappa \models$ "there is a strong cardinal".

S. Ulam first isolated a property equivalent to measurability in 1930: κ is measurable if and only if there is a κ-additive, 2-valued measure defined on all subsets of κ that gives singletons measure 0. (To get a measure from an elementary embedding, set $\mu(A) = 1 \Leftrightarrow \kappa \in j(A)$, for $A \subseteq \kappa$. Conversely, one gets elementary embeddings from measures using the ultrapower construction.) Embeddings corresponding to stronger large cardinal properties can be captured in a similar way by systems of measures.

Let $\kappa < \delta$ be cardinals, and $A \subseteq V_\delta$; then κ is *A-strong in δ* if and only if for all $\beta < \delta$ there is an elementary $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $j(A) \cap V_\beta = A \cap V_\beta$. (The case $A = V_\delta$ implies κ is β-strong for all $\beta < \delta$.) We say δ is *A-Woodin* if and only if there is a $\kappa < \delta$ that is A-strong in δ, and we say δ is Woodin in case it is A-Woodin for all $A \subseteq V_\delta$. The hypothesis that there are Woodin cardinals is strictly between the existence of strong and supercompact cardinals in strength.

Woodin cardinals were discovered by W. H. Woodin in 1984. New techniques due to M. Foreman, M. Magidor, and S. Shelah had just shown that the Lebesgue measurability of projective sets of real numbers follows from large cardinal hypotheses much weaker than had been previously suspected. Woodin showed that in fact the existence of infinitely many Woodin cardinals implies all projective sets of reals are Lebesgue measurable. About a year later, in 1985, D. A. Martin and the author showed that the existence of infinitely many Woodin cardinals implies all projective subsets of the Baire space $\mathbb{N}^\mathbb{N}$ are *determined* (PD), a stronger regularity property that, by work of many people in the 1960s and 1970s, is the basis for a thorough and detailed structure theory for projective sets. (By itself, ZFC decides very little about the projective sets.)

Of course, it follows that any large cardinal hypothesis that implies there are infinitely many Woodin cardinals also implies PD. In fact, building on work of Martin, Woodin had already shown (just before the work of Foreman, Magidor, and Shelah) that one such hypothesis implies PD. The continuing importance of Woodin cardinals is related to the fact that they provide the minimal large cardinal hypothesis needed for PD. To make this precise, let us say a sentence φ belongs to the *language of second-order arithmetic* (LSA) if and only if φ refers only to natural numbers and sets of natural numbers, but not objects of higher type (like sets of sets of natural numbers). A set B is projective if and only if membership in B can be defined within LSA from some real parameter. Most questions about projective sets of reals can be phrased in LSA; for example, the Lebesgue measurability and determinacy of projective sets can be expressed using sentences in LSA. Then, by work of Martin, Woodin, and the author, we have that the following are equivalent:

(a) PD,
(b) for each n, every consequence in LSA of the theory ZFC plus "there are n Woodin cardinals" is true.

Thus PD is precisely the "instrumentalist's trace" of Woodin cardinals in the language of second-order arithmetic.

Underlying the proof that Woodin cardinals imply PD is a structure known as the *iteration tree*. Roughly speaking, an iteration tree on M is a tree of models with root M that is generated by a certain process. This process involves using a system of measures coding an embedding in one model to generate an embedding with domain in some other model, and thus equips the tree with commuting elementary embeddings from the models earlier on a given branch to those later on that branch. A simple example is an *alternating chain* on M, an iteration tree having two distinct branches, the "even" branch consisting of models M_n for n even (with $M_0 = M$), and the "odd" branch consisting of M_0 together with the M_n for n odd. If B is a subset of the Baire space, then an *alternating chain representation* of B is a continuous function $\mathcal{A}$ on $\mathbb{N}^\mathbb{N}$ such that for each $x \in \mathbb{N}^\mathbb{N}$, $\mathcal{A}(x)$ is an alternating chain on some V_δ, and $x \in B$ if and only if the direct limit along the even branch of $\mathcal{A}(x)$ is wellfounded. If B has an alternating chain representation, then B is determined. The proof of PD from Woodin cardinals goes by showing that if δ is Woodin, and there are infinitely many Woodin cardinals above δ, then every projective set has an alternating chain representation on V_δ.

In general, an iteration tree may have transfinite length. The construction of iteration trees, and the analysis of the properties of arbitrary iteration trees, is at the heart of many basic open problems in pure large cardinal theory. The extent to which δ is Woodin (that is, the complexity of those sets $A \subseteq V_\delta$ such that δ is A-Woodin) is mirrored in the complexity of the iteration trees one can generate on V_δ. This correspondence is behind the equivalence mentioned in the paragraph before last, and is one reason Woodin cardinals continue to be important in set theory.

References

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