

# BOUNDARY VALUE PROBLEMS FOR THE 2ND-ORDER SEIBERG-WITTEN EQUATIONS

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It is shown that the nonhomogeneous Dirichlet and Neuman problems for the 2nd-order Seiberg-Witten equation on a compact 4-manifold  $X$  admit a regular solution once the nonhomogeneous Palais-Smale condition  $\mathcal{H}$  is satisfied. The approach consists in applying the elliptic techniques to the variational setting of the Seiberg-Witten equation. The gauge invariance of the functional allows to restrict the problem to the Coulomb subspace  $\mathcal{C}_\alpha^c$  of configuration space. The coercivity of the  $\mathcal{S}W_\alpha$ -functional, when restricted into the Coulomb subspace, imply the existence of a weak solution. The regularity then follows from the boundedness of  $L^\infty$ -norms of spinor solutions and the gauge fixing lemma.

## 1. Introduction

Let  $X$  be a compact smooth 4-manifold with nonempty boundary. In our context, the Seiberg-Witten equations are the 2nd-order Euler-Lagrange equation of the functional defined in Definition 2.3. When the boundary is empty, their variational aspects were first studied in [3] and the topological ones in [1]. Thus, the main aim here is to obtain the existence of a solution to the nonhomogeneous equations whenever  $\partial X \neq \emptyset$ . The nonemptiness of the boundary inflicts boundary conditions on the problem. Classically, this sort of problem is classified according to its boundary conditions in *Dirichlet problem* ( $\mathcal{D}$ ) or *Neumann problem* ( $\mathcal{N}$ ).

Originally, the Seiberg-Witten equations were described in [8] as a pair of 1st-order PDE. The solutions of these equations were known as  $\mathcal{S}W_\alpha$ -monopoles, and their main achievement were to shed light on the understanding of the 4-dimensional differential topology, since new smooth invariants were defined by the topology of their moduli space of solutions (moduli gauge group). In the same article, Witten introduced a variational formulation for the equations and showed that its stable critical points turn out to be exactly the  $\mathcal{S}W_\alpha$ -monopoles. The variational aspects of the  $\mathcal{S}W_\alpha$ -equations were first explored in [3], where they proved that the functional satisfies the Palais-Smale condition and the solutions of the Euler-Lagrange (2nd-order) equations share the same important analytical properties as the  $\mathcal{S}W_\alpha$ -monopoles. Therefore, it is natural to ask if the equations fit into a Morse-Bott-Smale theory, where the lower number of critical points

is the Betti number of the configuration space. The topology of the configuration space was described in [1]. Besides, if the SW-theory is a Morse theory, another natural question is to argue about the existence of a Morse-Smale-Witten complex, as in [6]. In the last question, the  $\mathcal{S}^c W_\alpha$ -equations on manifolds endowed with tubular ends or boundary also demand attention. The analogy of the  $\mathcal{S}^c W_\alpha$ -equation's variational formulation, with the variational principle of the Ginzburg-Landau equation in superconductivity, further motivates the present study.

**1.1. Spin<sup>c</sup> structure.** The space of Spin<sup>c</sup> structures on  $X$  is identified with

$$\text{Spin}^c(X) = \{\alpha + \beta \in H^2(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}_2) \mid w_2(X) = \alpha \pmod{2}\}. \quad (1.1)$$

For each  $\alpha \in \text{Spin}^c(X)$ , there is a representation  $\rho_\alpha : \text{SO}_4 \rightarrow \text{Cl}_4$ , induced by a Spin<sup>c</sup> representation, and a pair of vector bundles  $(\mathcal{S}_\alpha^+, \mathcal{L}_\alpha)$  over  $X$  (see [4]). Let  $P_{\text{SO}_4}$  be the frame bundle of  $X$ , so

- (i)  $\mathcal{S}_\alpha = P_{\text{SO}_4} \times_{\rho_\alpha} V = \mathcal{S}_\alpha^+ \oplus \mathcal{S}_\alpha^-$ . The bundle  $\mathcal{S}_\alpha^+$  is the positive complex spinors bundle (fibers are Spin<sub>4</sub><sup>c</sup>-modules isomorphic to  $\mathbb{C}^2$ ),
- (ii)  $\mathcal{L}_\alpha = P_{\text{SO}_4} \times_{\det(\alpha)} \mathbb{C}$ . It is called the *determinant line bundle* associated to the Spin<sup>c</sup>-structure  $\alpha \cdot (c_1(\mathcal{L}_\alpha) = \alpha)$ .

Thus, for each  $\alpha \in \text{Spin}^c(X)$ , we associate a pair of bundles

$$\alpha \in \text{Spin}^c(X) \rightsquigarrow (\mathcal{L}_\alpha, \mathcal{S}_\alpha^+). \quad (1.2)$$

From now on, we considered on  $X$  a Riemannian metric  $g$  and on  $\mathcal{S}_\alpha$  a Hermitian structure  $h$ .

Let  $P_\alpha$  be the  $U_1$ -principal bundle over  $X$  obtained as the frame bundle of  $\mathcal{L}_\alpha$  ( $c_1(P_\alpha) = \alpha$ ). Also, we consider the adjoint bundles

$$\text{Ad}(U_1) = P_{U_1} \times_{\text{Ad}} U_1, \quad \text{ad}(\mathfrak{u}_1) = P_{U_1} \times_{\text{ad}} \mathfrak{u}_1, \quad (1.3)$$

where  $\text{Ad}(U_1)$  is a fiber bundle with fiber  $U_1$ , and  $\text{ad}(\mathfrak{u}_1)$  is a vector bundle with fiber isomorphic to the Lie algebra  $\mathfrak{u}_1$ .

**1.2. The main theorem.** Let  $\mathcal{A}_\alpha$  be (formally) the space of connections (covariant derivative) on  $\mathcal{L}_\alpha$ ,  $\Gamma(\mathcal{S}_\alpha^+)$  the space of sections of  $\mathcal{S}_\alpha^+$ , and  $\mathcal{G}_\alpha = \Gamma(\text{Ad}(U_1))$  the gauge group acting on  $\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+)$  as follows:

$$g \cdot (A, \phi) = (A + g^{-1}dg, g^{-1}\phi). \quad (1.4)$$

$\mathcal{A}_\alpha$  is an affine space with vector space structure, after fixing an origin, isomorphic to the space  $\Omega^1(\text{ad}(\mathfrak{u}_1))$  of  $\text{ad}(\mathfrak{u}_1)$ -valued 1-forms. Once a connection  $\nabla^0 \in \mathcal{A}_\alpha$  is fixed, a bijection  $\mathcal{A}_\alpha \leftrightarrow \Omega^1(\text{ad}(\mathfrak{u}_1))$  is exposed by  $\nabla^A \leftrightarrow A$ , where  $\nabla^A = \nabla^0 + A$ ,  $\mathcal{G}_\alpha = \text{Map}(X, U_1)$ , since  $\text{Ad}(U_1) \simeq X \times U_1$ . The curvature of a 1-connection form  $A \in \Omega^1(\text{ad}(\mathfrak{u}_1))$  is the 2-form  $F_A = dA \in \Omega^2(\text{ad}(\mathfrak{u}_1))$ .

*Definition 1.1.* (1) The configuration space of the  $\mathcal{D}$ -problem is

$$\mathcal{C}_\alpha^{\mathcal{D}} = \{(A, \phi) \in \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+) \mid (A, \phi) \big|_Y \stackrel{\text{gauge}}{\sim} (A_0, \phi_0)\}, \quad (1.5)$$

(2) the configuration space of the  $\mathcal{N}$ -problem is

$$\mathcal{C}_\alpha^{\mathcal{N}} = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+). \quad (1.6)$$

Although each boundary problem requires its own configuration space, the superscripts  $\mathcal{D}$  and  $\mathcal{N}$  will be used whenever the distinction is necessary, since most arguments work for both sort of problems. The gauge group  $\mathcal{G}_\alpha$  action on each of the configuration spaces is given by (1.4).

The Dirichlet ( $\mathcal{D}$ ) and Neumann ( $\mathcal{N}$ ) boundary value problems associated to the  $\mathcal{S}^c W_\alpha$ -equations are the following: we consider  $(\Theta, \sigma) \in \Omega^1(\text{ad}(u_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$  and  $(A_0, \phi_0)$  defined on the manifold  $\partial X$  ( $A_0$  is a connection on  $\mathcal{L}_\alpha \big|_{\partial X}$ ,  $\phi_0$  is a section of  $\Gamma(\mathcal{S}_\alpha^+ \big|_{\partial X})$ ). In this way, find  $(A, \phi) \in \mathcal{C}_\alpha^{\mathcal{D}}$  satisfying  $\mathcal{D}$  and  $(A, \phi) \in \mathcal{C}_\alpha^{\mathcal{N}}$  satisfying  $\mathcal{N}$ , where

(1)

$$\mathcal{D} = \begin{cases} d^* F_A + 4\Phi^*(\nabla^A \phi) = \Theta, \\ \Delta_A \phi + \frac{(|\phi|^2 + k_g)}{4} \phi = \sigma, \\ (A, \phi) \big|_{\partial X} \stackrel{\text{gauge}}{\sim} (A_0, \phi_0), \end{cases} \quad \mathcal{N} = \begin{cases} d^* F_A + 4\Phi^*(\nabla^A \phi) = \Theta, \\ \Delta_A \phi + \frac{(|\phi|^2 + k_g)}{4} \phi = \sigma, \\ i^*(\ast F_A) = 0, \quad \nabla_\nu^A \phi = 0, \end{cases} \quad (1.7)$$

(2) the operator  $\Phi^* : \Omega^1(\mathcal{S}_\alpha^+) \rightarrow \Omega^1(u_1)$  is locally given by

$$\Phi^*(\nabla^A \phi) = \frac{1}{2} \nabla^A (|\phi|^2) = \sum_i \langle \nabla_i^A \phi, \phi \rangle \eta_i, \quad (1.8)$$

and  $\eta = \{\eta_i\}$  is an orthonormal frame in  $\Omega^1(\text{ad}(u_1))$ ,

(3)  $i^*(\ast F_A) = F_4$ , where  $F_4 = (F_{14}, F_{24}, F_{34}, 0)$  is the local representation of the 4th component (normal to  $\partial X$ ) of the 2-form of curvature in the local chart  $(x, U)$  of  $X$ ;  $x(U) = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \|x\| < \epsilon, x_4 \geq 0\}$ , and  $x(U \cap \partial X) \subset \{x \in x(U) \mid x_4 = 0\}$ . Let  $\{e_1, e_2, e_3, e_4\}$  be the canonical base of  $\mathbb{R}^4$ , so  $\nu = -e_4$  is the normal vector field along  $\partial X$ .

**THEOREM 1.2 (main theorem).** *If the pair  $(\Theta, \sigma) \in L^{k,2} \oplus (L^{k,2} \cap L^\infty)$  satisfies the  $\mathcal{H}$ -Condition 3.1, then the problems  $\mathcal{D}$  and  $\mathcal{N}$  admit a  $C^r$ -regular solution  $(A, \phi)$ , whenever  $2 < k$  and  $r < k$ .*

## 2. Basic set up

**2.1. Sobolev spaces.** As a vector bundle  $E$  over  $(X, g)$  is endowed with a metric and a covariant derivative  $\nabla$ , we define the Sobolev norm of a section  $\phi \in \Omega^0(E)$  as

$$\|\phi\|_{L^{k,p}} = \sum_{|i|=0}^k \left( \int_X |\nabla^i \phi|^p \right)^{1/p}. \quad (2.1)$$

In this way, the  $L^{k,p}$ -Sobolev Spaces of sections of  $E$  is defined as

$$L^{k,p}(E) = \{\phi \in \Omega^0(E) \mid \|\phi\|_{L^{k,p}} < \infty\}. \quad (2.2)$$

In our context, in which we fixed a connection  $\nabla^0$  on  $\mathcal{L}_\alpha$ , a metric  $g$  on  $X$ , and a Hermitian structure on  $\mathcal{S}_\alpha$ , the Sobolev spaces on which the basic setting is made are the following:

- (i)  $\mathcal{A}_\alpha = L^{1,2}(\Omega^1(\text{ad}(u_1)))$ ;
- (ii)  $\Gamma(\mathcal{S}_\alpha^+) = L^{1,2}(\Omega^0(X, \mathcal{S}_\alpha^+))$ ;
- (iii)  $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+)$ ;
- (iv)  $\mathcal{G}_\alpha = L^{2,2}(X, U_1) = L^{2,2}(\text{Map}(X, U_1))$ . ( $\mathcal{G}_\alpha$  is an  $\infty$ -dimensional Lie group with Lie algebra  $\mathfrak{g} = L^{1,2}(X, u_1)$ ).

The above Sobolev spaces induce a Sobolev structure on  $\mathcal{C}_\alpha^{\mathcal{D}}$  and on  $\mathcal{C}_\alpha^{\mathcal{N}}$ . From now on, the configuration spaces will be denoted by  $\mathcal{C}_\alpha$  by ignoring the superscripts, unless needed.

The most basic analytical results needed to achieve the main result is the *gauge fixing lemma* (see [7]) and the estimate (2.3), both extended by Marini [5] to manifolds with boundary.

LEMMA 2.1 (gauge fixing lemma). *Every connection  $\hat{A} \in \mathcal{A}_\alpha$  is gauge equivalent, by a gauge transformation  $g \in \mathcal{G}_\alpha$  named Coulomb ( $\mathcal{C}$ ) gauge, to a connection  $A \in \mathcal{A}_\alpha$  satisfying*

- (1)  $d_\tau^{*f} A_\tau = 0$  on  $\partial X$ ,
- (2)  $d^* A = 0$  on  $X$ ,
- (3) *in the  $\mathcal{N}$ -problem, the connection  $A$  satisfies  $A_\nu = 0$  ( $\nu \perp \partial X$ ).*

COROLLARY 2.2. *Under the hypothesis of Lemma 2.1, there exists a constant  $K > 0$  such that the connection  $A$ , gauge equivalent to  $\hat{A}$  by the Coulomb gauge, satisfies the following estimates:*

$$\|A\|_{L^{1,p}} \leq K \cdot \|F_A\|_{L^p}. \quad (2.3)$$

*Notation.*  $*_f$  is the Hodge operator in the flat metric and the index  $\tau$  denotes tangential components.

**2.2. Variational formulation.** A global formulation for problems  $\mathcal{D}$  and  $\mathcal{N}$  is made using the Seiberg-Witten functional.

Definition 2.3. Let  $\alpha \in \text{Spin}^c(X)$ . The Seiberg-Witten functional  $\mathcal{S}^c W_\alpha : \mathcal{C}_\alpha \rightarrow \mathbb{R}$  is defined as

$$\mathcal{S}^c W_\alpha(A, \phi) = \int_X \left\{ \frac{1}{4} |F_A|^2 + |\nabla^A \phi|^2 + \frac{1}{8} |\phi|^4 + \frac{k_g}{4} |\phi|^2 \right\} dV_g + \pi^2 \alpha^2, \quad (2.4)$$

where  $k_g =$  scalar curvature of  $(X, g)$ .

*Remark 2.4.* The  $\mathcal{G}_\alpha$ -action on  $\mathcal{C}_\alpha$  has the following properties:

- (1) the  $\mathcal{F}^\circ W_\alpha$ -functional is  $\mathcal{G}_\alpha$ -invariant,
- (2) the  $\mathcal{G}_\alpha$ -action on  $\mathcal{C}_\alpha$  induces on  $T\mathcal{C}_\alpha$  a  $\mathcal{G}_\alpha$ -action as follows: let  $(\Lambda, V) \in T_{(A,\phi)}\mathcal{C}_\alpha$  and  $g \in \mathcal{G}_\alpha$ ,

$$g \cdot (\Lambda, V) = (\Lambda, g^{-1}V) \in T_{g \cdot (A,\phi)}\mathcal{C}_\alpha. \quad (2.5)$$

Consequently,  $d(\mathcal{F}^\circ W_\alpha)_{g \cdot (A,\phi)}(g \cdot (\Lambda, V)) = d(\mathcal{F}^\circ W_\alpha)_{(A,\phi)}(\Lambda, V)$ .

The tangent bundle  $T\mathcal{C}_\alpha$  decomposes as

$$T\mathcal{C}_\alpha = \Omega^1(\text{ad}(\mathbf{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+). \quad (2.6)$$

In this way, the 1-form  $d\mathcal{F}^\circ W_\alpha \in \Omega^1(\mathcal{C}_\alpha)$  admits a decomposition  $d\mathcal{F}^\circ W_\alpha = d_1\mathcal{F}^\circ W_\alpha + d_2\mathcal{F}^\circ W_\alpha$ , where

$$\begin{aligned} d_1(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} : \Omega^1(\text{ad}(\mathbf{u}_1)) &\longrightarrow \mathbb{R}, & d_1(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} \cdot \Lambda &= d(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} \cdot (\Lambda, 0), \\ d_2(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} : \Gamma(\mathcal{S}_\alpha^+) &\longrightarrow \mathbb{R}, & d_2(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} \cdot V &= d(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} \cdot (0, V). \end{aligned} \quad (2.7)$$

By performing the computations, we get

- (1) for every  $\Lambda \in \mathcal{A}_\alpha$ ,

$$d_1(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} \cdot \Lambda = \frac{1}{4} \int_X \text{Re} \{ \langle F_A, d_A \Lambda \rangle + 4 \langle \nabla^A(\phi), \Phi(\Lambda) \rangle \} dx, \quad (2.8)$$

where  $\Phi : \Omega^1(\mathbf{u}_1) \rightarrow \Omega^1(\mathcal{S}_\alpha^+)$  is the linear operator  $\Phi(\Lambda) = \Lambda(\phi)$ , with dual defined in (1.8),

- (2) for every  $V \in \Gamma(\mathcal{S}_\alpha^+)$ ,

$$d_2(\mathcal{F}^\circ W_\alpha)_{(A,\phi)} \cdot V = \int_X \text{Re} \left\{ \langle \nabla^A \phi, \nabla^A V \rangle + \left\langle \frac{|\phi|^2 + k_g}{4} \phi, V \right\rangle \right\} dx. \quad (2.9)$$

Therefore, by taking  $\text{supp}(\Lambda) \subset \text{int}(X)$  and  $\text{supp}(V) \subset \text{int}(X)$ , we restrict to the interior of  $X$ , and so, the gradient of the  $\mathcal{F}^\circ W_\alpha$ -functional at  $(A, \phi) \in \mathcal{C}_\alpha$  is

$$\text{grad}(\mathcal{F}^\circ W_\alpha)(A, \phi) = \left( d_A^* F_A + 4\Phi^*(\nabla^A \phi), \Delta_A \phi + \frac{|\phi|^2 + k_g}{4} \phi \right). \quad (2.10)$$

It follows from the  $\mathcal{G}_\alpha$ -action on  $T\mathcal{C}_\alpha$  that

$$\text{grad}(\mathcal{F}^\circ W_\alpha)(g \cdot (A, \phi)) = \left( d_A^* F_A + 4\Phi^*(\nabla^A \phi), g^{-1} \cdot \left( \Delta_A \phi + \frac{|\phi|^2 + k_g}{4} \phi \right) \right). \quad (2.11)$$

An important analytical aspect of the  $\mathcal{F}^\circ W_\alpha$ -functional is the coercivity lemma proved in [3].

LEMMA 2.5 (coercivity). *For each  $(A, \phi) \in \mathcal{C}_\alpha$ , there exist  $g \in \mathcal{G}_\alpha$  and a constant  $K_C^{(A, \phi)} > 0$ , where  $K_C^{(A, \phi)}$  depends on  $(X, g)$  and  $\mathcal{S}^\circ W_\alpha(A, \phi)$ , such that*

$$\|g \cdot (A, \phi)\|_{L^{1,2}} < K_C^{(A, \phi)}. \quad (2.12)$$

*Proof* (see [3, Lemma 2.3]). The gauge transform is the Coulomb one given in the Lemma 2.1.  $\square$

Considering the gauge invariance of the  $\mathcal{S}^\circ W_\alpha$ -theory, and the fact that the gauge group  $\mathcal{G}_\alpha$  is an infinite-dimensional Lie group, we cannot hope to handle the problem in general. From now on, we need to restrict the problem to the space, named Coulomb subspace,

$$\mathcal{C}_\alpha^{\mathfrak{C}} = \{(A, \phi) \in \mathcal{C}_\alpha; \|(A, \phi)\|_{L^{1,2}} < K_{\mathfrak{C}}^{(A, \phi)}\}. \quad (2.13)$$

The superscripts  $\mathfrak{D}$  and  $\mathcal{N}$  have been omitted here for simplicity, although each one should be taken in account according to the problem. These choices of spaces come from the nature of the  $\mathcal{G}_\alpha$  action on  $\mathcal{C}_\alpha$ , they are suggested by the gauge fixing lemma and the coercivity lemma (not shared by an actions in general).

### 3. Existence of a solution

**3.1. Nonhomogeneous Palais-Smale condition** — $\mathcal{H}$ . In the variational formulation, the problems  $\mathfrak{D}$  and  $\mathcal{N}$  (1.7) are written as

$$\begin{aligned} (\mathfrak{D}) &= \begin{cases} \text{grad}(\mathcal{S}^\circ W_\alpha)(A, \phi) = (\Theta, \sigma), \\ (A, \phi)|_{\partial X} \underset{\text{gauge}}{\sim} (A_0, \phi_0), \end{cases} \\ (\mathcal{N}) &= \begin{cases} \text{grad}(\mathcal{S}^\circ W_\alpha)(A, \phi) = (\Theta, \sigma), \\ i^*( * F_A) = 0, \quad \nabla_n^A \phi = 0. \end{cases} \end{aligned} \quad (3.1)$$

The equations in (1.7) may not admit a solution for any pair  $(\Theta, \sigma) \in \Omega^1(\text{ad}(u_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$ . In finite dimension, if we consider a function  $f : X \rightarrow \mathbb{R}$ , the analogous question would be to find a point  $p \in X$  such that, for a fixed vector  $u$ ,  $\text{grad}(f)(p) = u$ . This question is more subtle if  $f$  is invariant under a Lie group action on  $X$ . Therefore, we need a hypothesis about the pair  $(\Theta, \sigma) \in \Omega^1(\text{ad}(u_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$ .

*Condition 3.1* ( $\mathcal{H}$ ). Let  $(\Theta, \sigma) \in L^{1,2}(\Omega^1(\text{ad}(u_1))) \oplus (L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)) \cap L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$  be a pair such that there exists a sequence  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \subset \mathcal{C}_\alpha^{\mathfrak{C}}$  (2.13) with the following properties:

- (1)  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \subset L^{1,2}(\mathcal{A}_\alpha) \times (L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)) \cup L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$  and there exists a constant  $c_\infty > 0$  such that, for all  $n \in \mathbb{Z}$ ,  $\|\phi_n\|_\infty < c_\infty$ ,
- (2) there exists  $c \in \mathbb{R}$  such that, for all  $n \in \mathbb{Z}$ ,  $\mathcal{S}^\circ W_\alpha(A_n, \phi_n) < c$ ,
- (3) the sequence  $\{d(\mathcal{S}^\circ W_\alpha)_{(A_n, \phi_n)}\}_{n \in \mathbb{Z}} \subset (L^{1,2}(\Omega^1(\text{ad}(u_1))) \oplus L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)))^*$ , of linear functionals, converges weakly to

$$L_\Theta + L_\sigma : T^{\mathfrak{C}}\mathcal{C}_\alpha \longrightarrow \mathbb{R}, \quad (3.2)$$

where

$$L_{\Theta}(\Lambda) = \int_X \langle \Theta, \Lambda \rangle, \quad L_{\sigma}(V) = \int_X \langle \sigma, V \rangle. \quad (3.3)$$

**3.2. Strong convergence of  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$  in  $L^{1,2}$ .** As a consequence of Lemma 2.5, the sequence  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$  given by the  $\mathcal{H}$ -condition converges to a pair  $(A, \phi)$ ;

- (1) weakly in  $\mathcal{C}_{\alpha}$ ,
- (2) weakly in  $L^4(\mathcal{A}_{\alpha} \times \Gamma(\mathcal{S}_{\alpha}^+))$ ,
- (3) strongly in  $L^p(\mathcal{A}_{\alpha} \times \Gamma(\mathcal{S}_{\alpha}^+))$ , for every  $p < 4$ .

*Remark 3.2.* Let  $\{A_n\}_{n \in \mathbb{N}} \subset L^2$  be a converging sequence in  $L^2$  satisfying  $d^*A_n = 0$ , for all  $n \in \mathbb{N}$ , and let  $A = \lim_{n \rightarrow \infty} A_n \in L^2$ . So,  $d^*A = 0$ , once

$$|\langle d^*A, \rho \rangle| \leq |A - A_n|_{L^2} \cdot |d\rho|_{L^2}, \quad (3.4)$$

for all  $\rho \in \Omega^0(\text{ad}(u_1))$ .

**THEOREM 3.3.** *The limit  $(A, \phi) \in L^2(\mathcal{A}_{\alpha} \times \Gamma(\mathcal{S}_{\alpha}^+))$ , obtained as a limit of the sequence  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$ , is a weak solution of (1.7).*

*Proof.* The proof goes along the same lines as in the 2nd step in the proof of the compactness theorem in [3].

- (1) For every  $\Lambda \in \mathcal{A}_{\alpha}$ ,

$$d_1(\mathcal{S}^{\circ}W_{\alpha})_{(A_n, \phi_n)} \cdot \Lambda = \frac{1}{4} \int_X \text{Re} \{ \langle F_{A_n}, d_{A_n} \Lambda \rangle + 4 \langle \nabla^{A_n}(\phi_n), \Phi(\Lambda) \rangle \} dx + \int_{\partial X} \text{Re} \{ \Lambda \wedge *F_{A_n} \}, \quad (3.5)$$

where

- (a)  $\Phi : \Omega^1(u_1) \rightarrow \Omega^1(\mathcal{S}_{\alpha}^+)$  is the linear operator  $\Phi(\Lambda) = \Lambda(\phi)$ ; its dual is defined in (1.8). Assuming  $\phi \in L^{\infty}$  (Lemma 3.4), it follows that

$$\lim_{n \rightarrow \infty} d_1(\mathcal{S}^{\circ}W_{\alpha})_{(A_n, \phi_n)} \cdot \Lambda = d_1(\mathcal{S}^{\circ}W_{\alpha})_{(A, \phi)} \cdot \Lambda. \quad (3.6)$$

Therefore,  $d_1(\mathcal{S}^{\circ}W_{\alpha})_{(A, \phi)} \cdot \Lambda = \int_X \langle \Theta, \Lambda \rangle$ ,

- (b)  $\Lambda \wedge *F_A = -\langle \Lambda, F_4 \rangle dx_1 \wedge dx_2 \wedge dx_3$ . Since the above equation is true for all  $\Lambda$ , let  $\text{supp}(\Lambda) \subset \partial X$ , so  $F_4 = 0$  ( $\Rightarrow i^*(F_A) = 0$ ).

- (2) For every  $V \in \Gamma(\mathcal{S}_{\alpha}^+)$ ,

$$d_2(\mathcal{S}^{\circ}W_{\alpha})_{(A_n, \phi_n)} \cdot V = \int_X \text{Re} \left\{ \langle \nabla^{A_n} \phi_n, \nabla^{A_n} V \rangle + \left\langle \frac{|\phi_n|^2 + k_g}{4} \phi_n, V \right\rangle \right\} dx + \int_{\partial X} \text{Re} \{ \langle \nabla_y^{A_n} \phi_n, V \rangle \}. \quad (3.7)$$

Analogously, it follows that  $(A, \phi)$  is a weak solution of the equation

$$d_2(\mathcal{S}^{\circ}W_{\alpha})_{(A, \phi)} \cdot V = \int_X \langle \sigma, V \rangle. \quad (3.8)$$

So, in the  $\mathcal{N}$ -problem,  $\nabla_v^A \phi = 0$ . □

In order to pursue the strong  $L^{1,2}$ -convergence for the sequence  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$ , we obtain in the following an upper bound for  $\|\phi\|_{L^\infty}$ , whenever  $(A, \phi)$  is a weak solution.

LEMMA 3.4. *Let  $(A, \phi)$  be a solution of either  $\mathcal{D}$  or  $\mathcal{N}$  in (1.7), so the following hold.*

- (1) *If  $\sigma = 0$ , then there exists a constant  $k_{X,g}$ , depending on the Riemannian metric on  $X$ , such that*

$$\|\phi\|_{\infty} < k_{X,g} \text{vol}(X). \quad (3.9)$$

- (2) *If  $\sigma \neq 0$ , then there exist constant  $c_1 = c_1(X, g)$  and  $c_2 = c_2(X, g)$  such that*

$$\|\phi\|_{L^p} < c_1 + c_2 \|\sigma\|_{L^{3p}}^3. \quad (3.10)$$

*In particular, if  $\sigma \in L^\infty$ , then  $\phi \in L^\infty$ .*

*Proof.* Fix  $r \in \mathbb{R}$  and suppose that there is a ball  $B_{r^{-1}}(x_0)$ , around the point  $x_0 \in X$ , such that

$$|\phi(x)| > r, \quad \forall x \in B_{r^{-1}}(x_0). \quad (3.11)$$

Define

$$\eta = \begin{cases} \left(1 - \frac{r}{|\phi|}\right)\phi & \text{if } x \in B_{r^{-1}}(x_0), \\ 0 & \text{if } x \in X - B_{r^{-1}}(x_0). \end{cases} \quad (3.12)$$

So,

$$\begin{aligned} |\eta| &\leq |\phi|, \\ \nabla \eta &= r \frac{\langle \phi, \nabla \phi \rangle}{|\phi|^3} \phi + \left(1 - \frac{r}{|\phi|}\right) \nabla \phi \\ \Rightarrow |\nabla \eta|^2 &= r^2 \frac{\langle \phi, \nabla \phi \rangle^2}{|\phi|^4} + 2r \left(1 - \frac{r}{|\phi|}\right) \frac{\langle \phi, \nabla \phi \rangle^2}{|\phi|^3} + \left(1 - \frac{r}{|\phi|}\right)^2 |\nabla \phi|^2 \\ \Rightarrow |\nabla \eta|^2 &< r^2 \frac{|\nabla \phi|^2}{|\phi|^2} + 2r \left(1 - \frac{r}{|\phi|}\right) \frac{|\nabla \phi|^2}{|\phi|} + \left(1 - \frac{r}{|\phi|}\right)^2 |\nabla \phi|^2. \end{aligned} \quad (3.13)$$

Since  $r < |\phi|$ ,

$$|\nabla\eta|^2 < 4|\nabla\phi|^2. \tag{3.14}$$

Hence, by (3.13) and (3.14),  $\eta \in L^{1,2}$ . The directional derivative of  $\mathcal{G}^\alpha W_\alpha$  in direction  $\eta$  is given by

$$d(\mathcal{G}^\alpha W_\alpha)_{(A,\phi)}(0,\eta) = \int_X \left[ \langle \nabla^A \phi, \nabla^A \eta \rangle + \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) \right]. \tag{3.15}$$

By (2.9),

$$\int_X \left[ \langle \nabla^A \phi, \nabla^A \eta \rangle + \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) \right] = \int_X \left\langle \sigma, \left( 1 - \frac{r}{|\phi|} \right) \phi \right\rangle. \tag{3.16}$$

However,

$$\int_X \langle \nabla^A \phi, \nabla^A \eta \rangle = \int_X \left[ r \frac{\langle \phi, \nabla^A \phi \rangle^2}{|\phi|^3} + \left( 1 - \frac{r}{|\phi|} \right) |\nabla\phi|^2 \right] > 0. \tag{3.17}$$

So,

$$\int_X \frac{|\phi|^2 + k_g}{4} |\phi| (|\phi| - r) < \int_X \left\langle \sigma, \left( 1 - \frac{r}{|\phi|} \right) \phi \right\rangle < \int_X |\sigma| (|\phi| - r). \tag{3.18}$$

Hence,

$$\int_X (|\phi| - r) \left( \frac{|\phi|^2 + k_g}{4} |\phi| - |\sigma| \right) < 0. \tag{3.19}$$

Since  $r < |\phi(x)|$ , whenever  $x \in B_{r^{-1}}(x_0)$ , it follows that

$$(|\phi|^2 + k_g)|\phi| < 4|\sigma|, \quad \text{a.e. in } B_{r^{-1}}(x_0). \tag{3.20}$$

There are two cases to be analysed independently.

(1)  $\sigma = 0$ . In this case, we get

$$(|\phi|^2 + k_g)|\phi| < 0, \quad \text{a.e.} \tag{3.21}$$

The scalar curvature plays a central role here: if  $k_g \geq 0$ , then  $\phi = 0$ ; otherwise,

$$|\phi| \leq \max \{0, (-k_g)^{1/2}\}. \tag{3.22}$$

Since  $X$  is compact, we let  $k_{X,g} = \max_{x \in X} \{0, [-k_g(x)]^{1/2}\}$ , and so,

$$\|\phi\|_{\infty} < k_{X,g} \text{vol}(X). \tag{3.23}$$

(2) Let  $\sigma \neq 0$ . The inequality (3.20) implies that

$$|\phi|^3 + k_g |\phi| - 4|\sigma| < 0, \quad \text{a.e.} \tag{3.24}$$

Consider the polynomial

$$Q_{\sigma(x)}(w) = w^3 + k_g w - 4|\sigma(x)|. \tag{3.25}$$

An estimate for  $|\phi|$  is obtained by estimating the largest real number  $w$  satisfying  $Q_{\sigma(x)}(w) < 0$ .  $Q_{\sigma(x)}$  being monic implies that  $\lim_{w \rightarrow \infty} Q_{\sigma(x)}(w) = +\infty$ . So, either  $Q_{\sigma(x)} > 0$ , whenever  $w > 0$ , or there exists a root  $\rho \in (0, \infty)$ . The first case would imply that

$$Q_{\sigma(x)}(|\phi(x)|) > 0, \quad \text{a.e.}, \tag{3.26}$$

contradicting (3.20). By the same argument, there exists a root  $\rho \in (0, \infty)$  such that  $Q_{\sigma(x)}(w)$  changes its sign in a neighborhood of  $\rho$ . Let  $\rho$  be the largest root in  $(0, \infty)$  with this property. By the Corollary A.2, there exist constants  $c_1 = c_1(X, g)$  and  $c_2$  such that

$$|\rho| < c_1 + c_2 |\sigma(x)|^3. \tag{3.27}$$

Consequently,

$$|\phi(x)| < c_1 + c_2 |\sigma(x)|^3, \quad \text{a.e. in } B_{r^{-1}}(x_0) \tag{3.28}$$

and

$$\|\phi\|_{L^p} < C_1 + C_2 \|\sigma\|_{L^{3p}}^3 \quad \text{restricted to } B_{r^{-1}}(x_0), \tag{3.29}$$

where  $C_1, C_2$  are constants depending on  $\text{vol}(B_{r^{-1}}(x_0))$ . The inequality (3.29) can be extended over  $X$  by using a  $C^{\infty}$  partition of unity. Moreover, if  $\sigma \in L^{\infty}$ , then

$$\|\phi\|_{\infty} < C_1 + C_2 \|\sigma\|_{\infty}^3, \tag{3.30}$$

where  $C_1, C_2$  are constants depending on  $\text{vol}(X)$ . □

A sort of concentration lemma, proved in [3], can be extended as follows.

LEMMA 3.5. *Let  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$  be the sequence given by the  $\mathcal{H}$ -Condition 3.1. Then,*

$$\lim_{n \rightarrow \infty} \int_X \langle \Phi^*(\nabla^{A_n} \phi_n), A_n - A \rangle = 0. \tag{3.31}$$

*Proof.* By (1.8),

$$\begin{aligned}
 \lim_{n \rightarrow \infty} \int_X \langle \Phi^*(\nabla^{A_n} \phi_n), A_n - A \rangle &= \lim_{n \rightarrow \infty} \int_X \langle \nabla_i^{A_n} \phi_n, \phi_n \rangle \cdot \langle \eta_i, A_n - A \rangle, \\
 \lim_{n \rightarrow \infty} \int_X \langle \nabla_i^{A_n} \phi_n, \phi_n \rangle \cdot \langle \eta_i, A_n - A \rangle &\leq \lim_{n \rightarrow \infty} \int_X |\langle \nabla_i^{A_n} \phi_n, \phi_n \rangle|^2 \cdot \int_X |\langle \eta_i, A_n - A \rangle|^2 \\
 &\leq \lim_{n \rightarrow \infty} \left[ \int_X |\nabla_i^{A_n} \phi_n|^2 \cdot |\phi_n|^2 \right] \cdot \int_X |A_n - A|^2 \\
 &\leq \lim_{n \rightarrow \infty} c_\infty \cdot \left[ \int_X |\nabla_i^{A_n} \phi_n|^2 \right] \cdot \|A_n - A\|_{L^2}^2 \\
 &\leq \lim_{n \rightarrow \infty} c_\infty \cdot \|\phi_n\|_{L^{1,2}}^2 \cdot \|A_n - A\|_{L^2}^2 = 0.
 \end{aligned} \tag{3.32}$$

□

**THEOREM 3.6.** *Let  $(\Theta, \sigma)$  be a pair satisfying the  $\mathcal{H}$ -Condition 3.1. Then, the sequence  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$ , given by Condition 3.1, converges strongly to  $(A, \phi) \in \mathcal{C}_\alpha$ .*

*Proof.* From Theorem 3.3,  $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$  converges weakly in  $L^{1,2}$  to  $(A, \phi) \in \mathcal{C}_\alpha$ . The proof is splitted into 2 parts.

(1)  $\lim_{n \rightarrow \infty} \|A_n - A\|_{L^{1,2}} = 0$ . Let  $d^* : \Omega^1(\text{ad}(u_1)) \rightarrow \Omega^0(\text{ad}(u_1))$ . The operator  $d : \ker(d^*) \rightarrow \Omega^2(\text{ad}(u_1))$  being elliptic implies, by the fundamental elliptic estimate, that

$$\|A_n - A\|_{L^{1,2}} \leq c \|d(A_n - A)\|_{L^2} + \|A_n - A\|_{L^2}. \tag{3.33}$$

The first term in the right-hand side is controlled as follows:

$$\begin{aligned}
 \|dA_n - dA\|_{L^2}^2 &= \int_X \langle d(A_n - A), d(A_n - A) \rangle \\
 &= \int_X \langle dA_n, d(A_n - A) \rangle - \int_X \langle dA, d(A_n - A) \rangle \\
 &= \int_X \langle d^* F_{A_n}, A_n - A \rangle - \int_X \langle d^* F_A, A_n - A \rangle \\
 &= d(\mathcal{F}^\alpha W_\alpha)_{(A_n, \phi_n)}(A_n - A) - 4 \int_X \langle \Phi^*(\nabla^{A_n} \phi_n), A_n - A \rangle \\
 &\quad - d(\mathcal{F}^\alpha W_\alpha)_{(A, \phi)}(A_n - A) - 4 \int_X \langle \Phi^*(\nabla^A \phi), A_n - A \rangle + o(1) \\
 &= -4 \left\{ \int_X \langle \Phi^*(\nabla^{A_n} \phi_n), A_n - A \rangle + \int_X \langle \Phi^*(\nabla^A \phi), A_n - A \rangle \right\} \\
 &\quad + o(1), \quad \lim_{n \rightarrow \infty} o(1) = 0.
 \end{aligned} \tag{3.34}$$

Thus, it follows from Lemma 3.5 that  $\lim_{n \rightarrow \infty} \|A_n - A\|_{L^{1,2}} = 0$ , and consequently,  $A_n \rightarrow A$  strongly in  $L^4$ .

$$(2) \lim_{n \rightarrow \infty} \|\phi_n - \phi\|_{L^{1,2}} = 0.$$

$$\|\nabla^0 \phi_n - \nabla^0 \phi\|_{L^2}^2 = \overbrace{\int_X \langle \nabla^0 \phi_n, \nabla^0 (\phi_n - \phi) \rangle}^{(1)} - \overbrace{\int_X \langle \nabla^0 \phi, \nabla^0 (\phi_n - \phi) \rangle}^{(2)}. \quad (3.35)$$

The term (1) leads to

$$\begin{aligned} & \int_X \langle \nabla^0 \phi_n, \nabla^0 (\phi_n - \phi) \rangle \\ &= \int_X \langle (\nabla^{A_n} - A_n) \phi_n, (\nabla^{A_n} - A_n) (\phi_n - \phi) \rangle \\ &= \int_X \langle \nabla^{A_n} \phi_n, \nabla^{A_n} (\phi_n - \phi) \rangle - \int_X \langle \nabla^{A_n} \phi_n, A_n (\phi_n - \phi) \rangle \\ &\quad - \int_X \langle A_n \phi_n, \nabla^{A_n} (\phi_n - \phi) \rangle + \int_X \langle A_n \phi_n, A_n (\phi_n - \phi) \rangle \\ &= \overbrace{d(\mathcal{S}^{\circ}W_{\alpha})_{(A_n, \phi_n)}(\phi_n - \phi)}^{(11)} - \int_X \frac{|\phi_n|^2 + k_g}{4} \langle \phi_n, \phi_n - \phi \rangle \\ &\quad - \overbrace{\int_X \langle \nabla^{A_n} \phi_n, A_n (\phi_n - \phi) \rangle}^{(12)} - \overbrace{\int_X \langle A_n \phi_n, \nabla^{A_n} (\phi_n - \phi) \rangle}^{(13)} \\ &\quad + \overbrace{\int_X \langle A_n \phi_n, A_n (\phi_n - \phi) \rangle}^{(14)}. \end{aligned} \quad (3.36)$$

The term (2) in (3.35) leads to similar terms named (21), (22), (23), and (24). We analyze each one of the above-obtained overbraced terms.

(a) Terms (11) and (21):

$$\begin{aligned} & d(\mathcal{S}^{\circ}W_{\alpha})_{(A_n, \phi_n)}(\phi_n - \phi) - \int_X \frac{|\phi_n|^2 + k_g}{4} \langle \phi_n, \phi_n - \phi \rangle + o(1) \\ &= \langle \sigma, \phi_n - \phi \rangle - \int_X \frac{|\phi_n|^2 + k_g}{4} |\phi_n - \phi|^2 - \int_X \frac{|\phi_n|^2 + k_g}{4} \langle \phi, \phi_n - \phi \rangle + o(1) \\ &\leq \langle \sigma, \phi_n - \phi \rangle - \int_X \frac{|\phi_n|^2 + k_g}{4} \langle \phi, \phi_n - \phi \rangle + o(1) \\ &\leq \|\sigma\|_{L^2}^2 \cdot \|\phi_n - \phi\|_{L^2}^2 + \left\| \frac{|\phi_n|^2 + k_g}{4} \right\|_{L^2}^2 \cdot \|\phi\|_{\infty} \cdot \|\phi_n - \phi\|_{L^2}^2 + o(1), \end{aligned} \quad (3.37)$$

where  $\lim_{n \rightarrow \infty} o(1) = 0$ . By the similarity between (11) and (21), we conclude the boundedness of term (22).

(b) Terms (12) and (22):

(i) term (12):

$$\begin{aligned}
 & \int_X \langle \nabla^{A_n} \phi_n, A_n(\phi_n - \phi) \rangle \\
 &= \int_X \langle \nabla^{A_n} \phi_n, (A_n - A)(\phi_n - \phi) \rangle + \int_X \langle \nabla^{A_n} \phi_n, A(\phi_n - \phi) \rangle \\
 &\leq \int_X |\nabla^{A_n} \phi_n|^2 \cdot \int_X |A_n - A|^4 \cdot \int_X |\phi_n - \phi|^4 \\
 &\quad + \int |\nabla^{A_n} \phi_n|^2 \cdot \int_X |A(\phi_n - \phi)|^2,
 \end{aligned} \tag{3.38}$$

(ii) term (22)

$$\int_X \langle \nabla^A \phi, A(\phi_n - \phi) \rangle \leq \int_X |\nabla^A \phi|^2 \cdot \int_X |A(\phi_n - \phi)|^2. \tag{3.39}$$

The term  $\int_X |\nabla^A \phi|^2$  is bounded by Proposition 4.1 and  $A \in C^0$  by Theorem 4.4.

(c) Term {(13)-(23)}:

$$\begin{aligned}
 & \int_X \langle A_n \phi_n, \nabla^{A_n}(\phi_n - \phi) \rangle - \int_X \langle A \phi, \nabla^A(\phi_n - \phi) \rangle \\
 &= \int_X \langle (A_n - A) \phi_n, \nabla^{A_n}(\phi_n - \phi) \rangle + \overbrace{\int_X \langle A \phi_n, \nabla^{A_n}(\phi_n - \phi) \rangle}^{(i)} \\
 &\quad - \int_X \langle (A_n - A) \phi, \nabla^A(\phi_n - \phi) \rangle - \overbrace{\int_X \langle A_n \phi, \nabla^A(\phi_n - \phi) \rangle}^{(ii)}.
 \end{aligned} \tag{3.40}$$

In each of the last two lines above, the first terms are bounded by  $\|A_n - A\|_{L^4}$ , while the term {(i)-(ii)} can be written as

$$\begin{aligned}
 & \int_X \langle (A - A_n) \phi_n, \nabla^{A_n}(\phi_n - \phi) \rangle + \int_X \langle A_n(\phi_n - \phi), \nabla^{A_n}(\phi_n - \phi) \rangle \\
 &\quad + \int_X \langle A_n \phi, \overbrace{(\nabla^{A_n} - \nabla^A)}^{(A_n - A)}(\phi_n - \phi) \rangle.
 \end{aligned} \tag{3.41}$$

So, it is also bounded by  $\|A_n - A\|_{L^4}$ .

(d) Term {(14)-(24)}:

$$\begin{aligned}
 & \int_X \langle A_n \phi_n, A_n(\phi_n - \phi) \rangle - \int_X \langle A \phi, A(\phi_n - \phi) \rangle \\
 &= \int_X \langle A_n \phi_n, (A_n - A)(\phi_n - \phi) \rangle + \int_X \langle (A_n - A) \phi_n, A(\phi_n - \phi) \rangle \\
 &\quad + \int |A(\phi_n - \phi)|^2.
 \end{aligned} \tag{3.42}$$

Since  $A \in C^0$ , it follows that  $\lim_{n \rightarrow \infty} \|A(\phi_n - \phi)\|^2 = 0$ . □

**4. Regularity of the solution  $(A, \phi)$**

Let  $\beta = \{e_i; 1 \leq i \leq 4\}$  be an orthonormal frame fixed on  $TX$  with the following properties; for all  $i, j \in \{1, 2, 3, 4\}$ :

- (1)  $[e_i, e_j] = 0$ ,
- (2)  $\nabla_{e_i} e_j = 0$  ( $\nabla =$  Levi-Civita connection on  $X$ ).

Let  $\beta^* = \{dx_1, \dots, dx_n\}$  be the dual frame induced on  $\mathcal{S}_\alpha^*$ . From the 2nd property of the frame  $\beta$ , it follows that  $\nabla_{e_i} dx^j = 0$  for all  $i, j \in \{1, 2, 3, 4\}$ . For the sake of simplicity, let  $\nabla_{e_i}^A = \nabla_i^A$ . Therefore,  $\nabla^A : \Omega^0(\text{ad}(u_1)) \rightarrow \Omega^1(\text{ad}(u_1))$  is given by

$$\begin{aligned} \nabla^A \phi &= \sum_l (\nabla_l^A \phi) dx_l \implies |\nabla^A \phi|^2 = \sum_l |\nabla_l^A \phi|^2, \\ (\nabla^A)^2 &= \sum_{k,l} (\nabla_k^A \nabla_l^A \phi) dx_l \wedge dx_k \implies |(\nabla^A)^2|^2 = \sum_{k,l} |\nabla_k^A \nabla_l^A \phi|^2. \end{aligned} \tag{4.1}$$

In this setting, the 2 form of curvature of the connection  $A$  is given by

$$(F_A)_{kl} = F_{kl} = \nabla_l^A \nabla_k^A - \nabla_k^A \nabla_l^A. \tag{4.2}$$

In order to compute the operator  $\Delta_A = (\nabla^A)^* \nabla^A : \Omega^0(\mathcal{S}_\alpha^+) \rightarrow \Omega^0(\mathcal{S}_\alpha^+)$ , let  $*$  :  $\Omega^i(\mathcal{S}_\alpha) \rightarrow \Omega^{4-i}(\mathcal{S}_\alpha)$  be the Hodge operator and consider the identity

$$(\nabla^A)^* = - * \nabla^A * : \Omega^1(\mathcal{S}_\alpha^+) \rightarrow \Omega^0(\mathcal{S}_\alpha^+). \tag{4.3}$$

Hence,

$$\Delta_A \phi = - \sum_k \nabla_k^A \nabla_k^A \phi. \tag{4.4}$$

In this way,

$$\begin{aligned} |\Delta_A \phi|^2 &= \sum_{k,l} \langle \nabla_k^A \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle \\ &= \sum_{k,l} [\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) - \langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \nabla_l^A \phi \rangle] \\ &= \sum_{k,l} [\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) - \langle \nabla_k^A \phi, \nabla_l^A \nabla_k^A \nabla_l^A \phi \rangle - \langle \nabla_k^A \phi, F_{lk} \nabla_l^A \phi \rangle] \\ &= \sum_{k,l} [\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) - \nabla_l^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle)] \\ &\quad + \sum_{k,l} [\langle \nabla_l^A \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle + \langle \nabla_k^A \phi, F_{lk} \nabla_l^A \phi \rangle] \\ &= \sum_{k,l} [\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) - \nabla_l^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle)] + \sum_{k,l} |\nabla_k^A \nabla_l^A \phi|^2 \\ &\quad + \sum_{k,l} [\langle F_{kl} \phi, \nabla_k^A \nabla_l^A \phi \rangle + \langle \nabla_k^A \phi, F_{kl} \nabla_l^A \phi \rangle] \end{aligned} \tag{4.5}$$

and so,

$$\begin{aligned}
 |(\nabla^A)^2 \phi|^2 &\leq |\Delta_A \phi|^2 + \sum_{k,l} \{ |\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) | \} + \sum_{k,l} \{ |\nabla_l^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle) | \} \\
 &+ \sum_{k,l} \{ | \langle F_{kl} \phi, \nabla_k^A \phi \nabla_l^A \phi \rangle | \} + \sum_{k,l} \{ | \langle \nabla_k^A \phi, F_{kl} \nabla_l^A \phi \rangle | \}.
 \end{aligned} \tag{4.6}$$

Now, by applying the inequalities

$$\left( \sum_i a_i \right)^r \leq K_r \cdot \sum_i |a_i|^r, \quad \sqrt{\sum_{i=1}^n a_i} \leq \sum_{i=1}^n \sqrt{a_i} \tag{4.7}$$

to (4.6), we get

$$\begin{aligned}
 |(\nabla^A)^2 \phi|^p &\leq K_p \cdot |\Delta_A \phi|^p + K_p \cdot \sum_{k,l} \{ |\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) |^{p/2} \} \\
 &+ K_p \sum_{k,l} \{ |\nabla_l^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle) |^{p/2} \} \\
 &+ \sum_{k,l} \{ | \langle F_{kl} \phi, \nabla_k^A \phi \nabla_l^A \phi \rangle |^{p/2} \} + \sum_{k,l} \{ | \langle \nabla_k^A \phi, F_{kl} \nabla_l^A \phi \rangle |^{p/2} \}.
 \end{aligned} \tag{4.8}$$

After integrating, it follows that

$$\begin{aligned}
 k_1 \cdot \|(\nabla^A)^2 \phi\|_{L^p}^p &\leq \|\Delta_A \phi\|_{L^p}^p + k_2 \cdot \|\nabla^A \phi\|_{L^p}^p + k_3 \cdot \|F_A(\phi)\|_{L^p}^p \\
 &+ k_4 \cdot \|F_A(\nabla^A \phi)\|_{L^p}^p + k_5 \cdot \sum_{k,l} \int_X \{ |\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) |^{p/2} \} \\
 &+ k_6 \sum_{k,l} \int_X \{ |\nabla_l^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle) |^{p/2} \}.
 \end{aligned} \tag{4.9}$$

The boundedness of the right-hand side of (4.9) results from the analysis of each term.

**PROPOSITION 4.1.** *Let  $(A, \phi) \in \mathcal{C}_\alpha$  be a solution of equations in (1.7). If  $\sigma \in L^\infty$ , then*

- (1)  $\nabla^A \phi \in L^2$ ,
- (2)  $\Delta_A \phi \in L^2$ .

*Proof.* (1)  $\nabla^A \phi \in L^2$ :

$$\begin{aligned}
 \langle \Delta_A \phi, \phi \rangle + \left( \frac{|\phi|^2 + k_g}{4} \right) |\phi|^2 &= \langle \sigma, \phi \rangle \\
 \implies |\nabla^A \phi|^2 + \left( \frac{|\phi|^2 + k_g}{4} \right) |\phi|^2 &= \langle \sigma, \phi \rangle \leq \frac{1}{\epsilon^2} |\sigma|^2 + \epsilon^2 |\phi|^2.
 \end{aligned} \tag{4.10}$$

Therefore,

$$|\nabla^A \phi|^2 < \frac{1}{\epsilon^2} |\sigma|^2 + \left( \epsilon^2 - \frac{k_g}{4} \right) |\phi|^2 - \frac{|\phi|^4}{4}. \tag{4.11}$$

From Lemma 3.4, there exists a polynomial  $p$ , with coefficients depending on  $(X, g)$  and  $\epsilon$ , such that

$$\|\nabla^A \phi\|_{L^2}^2 < p(\|\sigma\|_\infty). \quad (4.12)$$

So,  $\nabla^A \phi \in L^2$ .

(2)  $\Delta_A \phi \in L^2$ :

$$\langle \Delta_A \phi, \Delta_A \phi \rangle + \frac{|\phi|^2 + k_g}{4} \langle \phi, \Delta_A \phi \rangle = \langle \sigma, \Delta_A \phi \rangle; \quad (4.13)$$

let  $0 < \epsilon < 1$ ,

$$\begin{aligned} |\Delta_A \phi|^2 + \frac{|\phi|^2 + k_g}{4} |\nabla^A \phi|^2 &= \langle \sigma, \Delta_A \phi \rangle < \frac{1}{\epsilon^2} |\sigma|^2 + \epsilon^2 |\Delta_A \phi|^2, \\ (1 - \epsilon^2) |\Delta_A \phi|^2 + \frac{|\phi|^2 + k_g}{4} |\nabla^A \phi|^2 &< \frac{1}{\epsilon^2} |\sigma|^2. \end{aligned} \quad (4.14)$$

By the boundedness of the term

$$\int_X |\phi|^2 \cdot |\nabla^A \phi|^2 < \|\phi\|_\infty^2 \cdot \|\nabla^A \phi\|_{L^2}^2, \quad (4.15)$$

one deduces the existence of a polynomial  $q$ , with coefficients depending on  $\epsilon$  and  $(X, g)$ , such that

$$\|\Delta_A \phi\|_{L^2} < q(\|\sigma\|_\infty). \quad (4.16)$$

□

**PROPOSITION 4.2.** *Let  $(A, \phi)$  be solutions of the  $\mathcal{S}^q W_\alpha$ -equations, where  $(\Theta, \sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)$ , then  $F_A \in L^q$ , for all  $q < \infty$ .*

*Proof.* By (1.8),  $\Phi^*(\nabla^A \phi) = (1/2)\nabla^A(|\phi|^2)$ , and so,

$$d^* F_A + 4\Phi^*(\nabla^A \phi) = \Theta \implies \|d^* F_A\|_{L^2}^2 \leq \|\phi\|_{L^{1,2}}^2 + \|\Theta\|_{L^2}^2. \quad (4.17)$$

There are two cases to be analysed.

(1)  $F_A$  is harmonic. Since the Laplacian defined on  $u_1$ -forms is an elliptic operator, the fundamental inequality for elliptic operators asserts that there exists a constant  $C_k$  such that

$$\|F_A\|_{L^{k+2,2}} \leq \|\Delta F_A\|_{L^{k,2}} + C_k \|F_A\|_{L^2}. \quad (4.18)$$

Consequently,  $F_A$  being harmonic implies, for all  $k \in \mathbb{N}$ , that

$$\|F_A\|_{L^{k,2}} \leq C_k \|F_A\|_{L^2} \implies F_A \in C^\infty. \quad (4.19)$$

(2)  $F_A$  is not harmonic. In this case, since  $\Theta \in L^{1,2}$ ,  $\phi \in L^\infty$  and

$$\Delta_A F_A = d(\langle \phi, \nabla^A \phi \rangle) + d\Theta = \langle \phi, F_A(\phi) \rangle + d\Theta, \quad (4.20)$$

it follows that  $F_A \in L^{2,2}$ . Therefore, by the Sobolev embedding theorem,  $F_A \in L^q$ , for all  $q < \infty$ .  $\square$

**PROPOSITION 4.3.** *Let  $(A, \phi)$  be solutions of the  $\mathcal{S}^c W_\alpha$ -equations, where  $(\Theta, \sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)$ , then  $(\nabla^A)^2 \phi \in L^p$ , for all  $1 < p < 2$ .*

*Proof.* In (4.9), we must take care of the last terms.

(1)  $F(\nabla^A \phi) \in L^p$ , for all  $1 < p < 2$ . By Young's inequality,

$$\|F(\nabla^A \phi)\|_{L^p} \leq \|F_A\|_{L^{2p/(2-p)}} \cdot \|\nabla^A \phi\|_{L^2}. \tag{4.21}$$

(2) There is no contribution from the divergent terms, since

$$\int_x \{ |\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) |^{p/2} \} \leq [\text{vol}(X)]^{(2-p)/p} \int_x \{ |\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) | \}. \tag{4.22}$$

In the same way,

$$\begin{aligned} \sum_{k,l} \int_x \{ |\nabla_k^A (\langle \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi \rangle) |^{p/2} \} &= 0, \\ \sum_{k,l} \int_x \{ |\nabla_l^A (\langle \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi \rangle) |^{p/2} \} &= 0. \end{aligned} \tag{4.23}$$

The estimates above applied to (4.9) implies that

$$\begin{aligned} \|(\nabla^A)^2 \phi\|_{L^p} &\leq k_1 \|\Delta_A \phi\|_{L^p}^p + k_2 \|\nabla^A \phi\|_{L^p}^p + k_3 \|\nabla^A \phi\|_{L^p}^p \\ &+ k_4 \|F_A(\phi)\|_{L^p}^p + k_5 \|F_A\|_{L^{p/(2-p)}} \cdot \|\nabla^A \phi\|_{L^p}^p. \end{aligned} \tag{4.24}$$

$\square$

Thus,  $\phi \in L^{2,p}$ , for all  $1 < p < 2$ . Considering that  $\sigma \in L^{1,2}$ , the bootstrap argument applied on (1.7) implies that  $\phi \in L^{3,p}$ , for every  $k \geq 2$  and  $1 < p < 2$ . Hence, by Sobolev embedding theorem,  $\phi \in C^0$ .

**THEOREM 4.4.** *Let  $(A, \phi)$  be a solution of the  $\mathcal{S}^c W_\alpha$ -equations, where  $(\Theta, \sigma) \in L^{k,2}(\Omega^1(\text{ad}(u_1))) \oplus (L^{k,2}(\Gamma(\mathcal{S}_\alpha^+)) \cap L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$ , then  $(A, \phi) \in L^{k+2,p} \times (L^{k+2,2} \cap L^\infty)$ , for all  $1 < p < 2$ . Moreover, if  $k > 2$ , then  $(A, \phi) \in C^r \times C^r$ , for all  $r < k$ .*

*Proof.* (1) If  $\Theta \in L^{k,2}$ , then by Proposition 4.2  $F_A \in L^{k+1,2}$ . Consequently, by Corollary 2.2,  $A \in L^{k+2,2}$ .

(2) The Sobolev class of  $\phi$  is obtained by the bootstrap argument.  $\square$

## Appendix

### Estimates for solutions of 3rd-degree equation

Let  $p, q \in \mathbb{R}$  and consider the equation

$$x^3 + px + q = 0. \tag{A.1}$$

PROPOSITION A.1. *The solutions of (A.1) are given in [2] by*

$$x_1 = z_1 + z_2, \quad x_2 = z_1 + \lambda z_2, \quad y_3 = z_1 + \lambda^2 z_2, \quad (\text{A.2})$$

where

$$z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{D}}, \quad z_2 = \sqrt[3]{-\frac{q}{2} - \sqrt[2]{D}}, \quad D = \frac{p^3}{27} + \frac{q^2}{4}, \quad (\text{A.3})$$

and  $\lambda \in \mathbb{C}$  satisfies  $\lambda^3 = 1$ .

COROLLARY A.2. *Let  $p$  and  $q$  be negative real numbers. So, the solutions of (A.1) are estimated according to the following cases:*

(1)  $D \geq 0$ :

$$|x_i| \leq \frac{8}{3} + \frac{1}{3}|q| + \frac{1}{12}q^2 + \frac{1}{81}p^3, \quad (\text{A.4})$$

(2)  $D < 0$ :

$$|x_i| \leq 3 + \frac{1}{6}q^2 + \frac{1}{81}|p|^3. \quad (\text{A.5})$$

*Proof.* Since

$$|x_i| \leq |z_1| + |z_2|, \quad (\text{A.6})$$

it is enough to estimate  $|z_1|$  and  $|z_2|$ . The basics identities needed are the following: suppose  $x \geq 0$ , whence

$$\sqrt[3]{x} \leq 1 + \frac{1}{2}x, \quad \sqrt[3]{x} \leq 1 + \frac{1}{3}x. \quad (\text{A.7})$$

(1)  $D \geq 0$ . In this case,  $z_1, z_2 \in \mathbb{R}$  and

$$|z_1| = \sqrt[3]{\left| -\frac{q}{2} + \sqrt[2]{D} \right|} \leq 1 + \frac{1}{3} \left| -\frac{q}{2} + \sqrt[2]{D} \right| \leq \frac{4}{3} + \frac{1}{6}|q| + \frac{1}{6}D. \quad (\text{A.8})$$

Thus,

$$|z_1| \leq \frac{4}{3} + \frac{1}{6}|q| + \frac{1}{24}q^2 + \frac{1}{162}p^3. \quad (\text{A.9})$$

The same estimate can be obtained for  $|z_2|$ . Hence,

$$|x_i| \leq \frac{8}{3} + \frac{1}{3}|q| + \frac{1}{12}q^2 + \frac{1}{81}p^3. \quad (\text{A.10})$$

(2)  $D \leq 0$ . In this case,  $z_1, z_2 \in \mathbb{C} - \mathbb{R}$ . Since  $D \in \mathbb{R}$ , we can write  $\sqrt[3]{D} = i\sqrt[3]{|D|}$  and

$$z_1 = \sqrt[3]{-\frac{1}{2}q + i\sqrt[3]{D}}, \quad z_2 = \sqrt[3]{-\frac{1}{2}q - i\sqrt[3]{D}}. \tag{A.11}$$

Therefore,

$$\begin{aligned} |z_i|^2 &= \sqrt[3]{\frac{q^2}{4} + |D|} < 1 + \frac{1}{12}q^2 + \frac{1}{3}|D| \leq 1 + \frac{1}{6}q^2 + \frac{1}{81}|p|^3, \\ |z_i| &< \frac{3}{2} + \frac{1}{12}q^2 + \frac{1}{162}|p|^3. \end{aligned} \tag{A.12}$$

Hence,

$$|x_i| < 3 + \frac{1}{6}q^2 + \frac{1}{81}|p|^3. \tag{A.13}$$

□

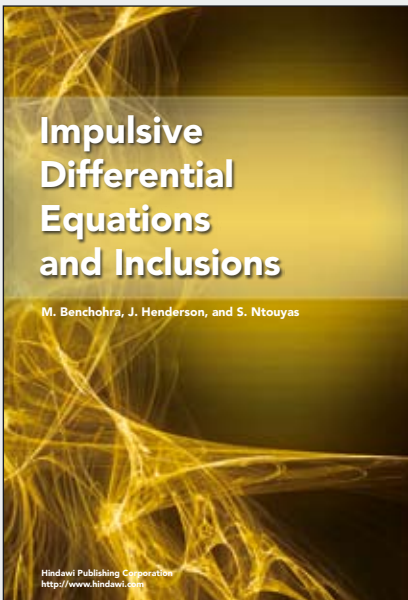
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# Impulsive Differential Equations and Inclusions

M. Benchohra, J. Henderson, and S. Ntouyas



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