# VARIATIONAL PRINCIPLE FOR THE SEIBERG-WITTEN EQUATIONS

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Abstract. Originally, the Seiberg-Witten equations were described to be dual to the Yang-Mills equation. However, there is a Variational Principle from which the $SW$-equation and some of their analytical properties can be studied.

## 1. **Introduction**

In november of 1994, Edward Witten gave a lecture at MIT about $N = 2$ Supersymmetric Quantum Field Theory and the ideas concerning the $S$-duality developed in a joint work with Seiberg in [15]. In order to please the mathematicians in the audience, he applied the new ideas to the Yang-Mills Theory to show them a new pair of $1^{st}$-order PDE, named $SW$-monopole equation, and conjectured that the $SW$-theory is dual to the Yang-Mills theory; the duality being at the quantum level, since the expectation values of dual theories are equals. In topology, this means that fixed a 4-manifold its Seiberg-Witten invariants are equal to the Donaldson invariants. After 10 years, it is believed the conjecture is true, some tour the force has been done in [11] to prove it, but in its generality it is still an open question. This new pair of equations has a simpler analytical nature than the Yang-Mills equations. Even though the open question, and the fact that the physical meaning of the Seiberg-Witten equations ($\mathcal{SW}_\alpha$-eq.) is yet to be discovered, the mathematical usefulness of the equations is rather deep and efficient to understand one of the most basic phenomenon of differential topology in four dimension, namely, the existence of non-equivalent differential smooth structures on the same underlying topological manifold. It has not been efficient enough to solve either the smooth Poincaré conjecture in dimension 4 or the 11/8-conjecture, but they have been very useful to improve the understanding of the relation between $2^{nd}$-homology classes and smooth structure, the sympletic structures on 4-manifolds and to give a construction of a large number of non-equivalent smooths structures on a compact smooth 4-manifold based on the isotopic classes of knots in $S^3$ [5]. Also, using the $SW$-theory, the Thom conjecture was proved in [9] and some results in [13] were obtained in a much easier way than the one using $YM$-theory. Most of the simplicity coming from the $SW$-theory, as compared with the $YM$-theory, came from the fact that the strutuctural group in $SW$-theory is the abelian $U_1$, wherea in $YM$-theory the group is the non-abelian $SU_2$.

During the 80's and earlies 90's, the Yang-Mills Theory was used to define a set of smooth invariants on 4-manifolds known as Donaldson invariants [2]. The Donaldson invariants sheded new light on the theory of smooth 4-manifolds, e.g.,

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the existence of a non enumerable set of exotics $\mathbb{R}^4$. However, the Donaldson Theory relies on a difficult analysis mostly carried out in the work of Karen Uhlenbeck [19] and also of Clifford Taubes [7]. Late in the 80's, Witten, in [21], showed that the Donaldson invariants could be described as the expectation values of a Topological Quantum Field Theory. The Yang-Mills functional was first described in 1954, by Yang and Mills [22], aiming to give a general framework in which the most basic nature's interactions would fit in. The configuration space of the Yang-Mills theory is the space of connections associated to a principal $G$-bundle (G=strutural group) $P$ over a 4-manifold $X$. The Yang-Mills functional is invariant by a conformal diffeomorphism of $X$ and by the Gauge Group, the group of automorphism of $P$. The Euler-Lagrange equation of the Yang-Mills functional is known as Yang-Mills equation and written as $d^*F_A = 0$ ($2^{nd}$-order PDE). Whenever the theory is on a principal $SU_2$-bundle, the stable critical points of the Yang-Mills functional are named *Instantons* and satisfies the anti-self dual equations ($asd$) $F_A^+ = 0$, a $1^{st}$-order PDE. Thanks to the conformal invariance of the $YM$-functional, any instanton in $\mathbb{R}^4$ can be lifted to a solution on $S^4$. Once $P$ is fixed, the set of instantons in $S^4$ is a smooth manifold of dimension $d = 8k - 3$, where $k = c_2(P)$ is the $2^{nd}$-Chern class of $P$. The $asd$-equation has some analogy with the Cauchy-Riemann equations for the Harmonic equation.

As mentioned before, the Seiberg-Witten equations came out of a duality principle and not of a variational one. However, in [20] Witten introduced a functional without making any use of it in order to derive the equations or understand their analytical properties. It turns out that the stable critical points of the this functional are the solutions of the $SW$-monopole equation; it is an open question to find on a smooth manifold a sufficient condition to the existence of a $SW$-monopole.

An outstanding difference between the Yang-Mills and $SW$-functional is the fact that the $SW$-functional is not invariant by conformal diffeomorphism of the 4-manifold and the only solution in $R^4$ is the trivial one $(0, 0)$, as proved in 5.1.

The $SW$-equations and the $YM$-equation became elliptic by considering them moduli the gauge invariance. In fact, the analytical properties depend on a special slice of the action defined in 3.15.

The variational setting of the Seiberg-Witten equation were explored in [8] and [4]. The maim purpose of these notes is to introduce and describe the Variational Principle and some of the analytical consequences to the $SW$-equations. A interesting feature for the Variational setting would be to give an interpretation, in terms of the geometry of the configuration space, to the $SW$-invariants. The article is organised as follows;

**:** section 1 - Introduction
**:** section 2 - Background
**:** section 3 - Variational Principle
**:** section 4 - Main Estimate
**:** section 5 - $\mathcal{H}$-Condition and Palais-Smale Condition
**:** section 6 - Homotopy Type of the Configuration Space
**:** section 7 - Dirchlet and Neuman Problems associated to the $SW$-equation.

It is a great pleasure to contribute to the volume in honour of the $70^{th}$-birthday of Professor Djairo Figueiredo, whose contribution to the Brazilian mathematical community has been outstanding.

## 2. **Background**

2.1. **Spin$^c$ Structure.** The space of $Spin^c$ structures on X is identified with

$$Spin^c(X) = \{\alpha + \beta \in H^2(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}_2) \mid w_2(X) = \alpha (mod \;\; 2)\}.$$

For each $\alpha \in Spin^c(X)$, there is a representation $\rho_\alpha : SO_4 \to \mathbb{C}l_4$, induced by a $Spin^c$ representation, and a pair of vector bundles $({\mathcal{S}_\alpha}^+, \mathcal{L}_\alpha)$ over X (see [10]). Let $P_{SO_4}$ be the frame bundle of $X$, so

- $\mathcal{S}_\alpha = P_{SO_4} \times_{\rho_\alpha} V = {\mathcal{S}_\alpha}^+ \oplus {\mathcal{S}_\alpha}^-$.
  The bundle ${\mathcal{S}_\alpha}^+$ is the positive complex spinors bundle (fibers are $Spin^c_4-modules$ isomorphic to $\mathbb{C}^2$)
- $\mathcal{L}_\alpha = P_{SO_4} \times_{det(\alpha)} \mathbb{C}$.
  It is called the *determinant line bundle* associated to the $Spin^c$-struture $\alpha$. $(c_1(\mathcal{L}_\alpha) = \alpha)$

Thus, for each $\alpha \in Spin^c(X)$ we associate a pair of bundles

$$\alpha \in Spin^c(X) \quad \rightsquigarrow \quad (\mathcal{L}_\alpha, {\mathcal{S}_\alpha}^+).$$

From now on, we considered on $X$ a Riemannian metric $g$ and on $\mathcal{S}_\alpha$ an hermitian structure $h$.

Let $P_\alpha$ be the $U_1$-principal bundle over X obtained as the frame bundle of $\mathcal{L}_\alpha$ $(c_1(P_\alpha) = \alpha)$. Also, we consider the adjoint bundles

$$Ad(U_1) = P_{U_1} \times_{Ad} U_1 \quad ad(\mathfrak{u}_1) = P_{U_1} \times_{ad} \mathfrak{u}_1,$$

where $Ad(U_1)$ is a bundle with fiber $U_1$, and $ad(\mathfrak{u}_1)$ is a vector bundle with fiber isomorphic to the Lie Algebra $\mathfrak{u}_1$.

Let $\mathcal{A}_\alpha$ be (formally) the space of connections (covariant derivative) on $\mathcal{L}_\alpha$, $\Gamma(\mathcal{S}^+_\alpha)$ the space of sections of $\mathcal{S}^+_\alpha$ and $\mathcal{G}_\alpha = \Gamma(Ad(U_1))$ the gauge group acting on $\mathcal{A}_\alpha \times \Gamma(\mathcal{S}^+_\alpha)$ as follows:

$$g.(A, \phi) = (A + g^{-1}dg, g^{-1}\phi). \tag{2.1}$$

$\mathcal{A}_\alpha$ is an affine space which vector space structure, after fixing an origin, is isomorphic to the space $\Omega^1(ad(\mathfrak{u}_1))$ of $ad(\mathfrak{u}_1)$-valued 1-forms. Once a connection $\nabla^0 \in \mathcal{A}_\alpha$ is fixed, a bijection $\mathcal{A}_\alpha \leftrightarrow \Omega^1(ad(\mathfrak{u}_1))$ is explicited by $\nabla^A \leftrightarrow A$, where $\nabla^A = \nabla^0 + A$. $\mathcal{G}_\alpha = Map(X, U_1)$, since $Ad(U_1) \simeq X \times U_1$. The curvature of a 1-connection form $A \in \Omega^1(ad(\mathfrak{u}_1))$ is the 2-form $F_A = dA \in \Omega^2(ad(\mathfrak{u}_1))$.

2.2. **Seiberg-Witten Monopole Equation.** Since we are in dimension 4, the vector bundle $\Omega^2(ad(\mathfrak{u}_1))$ splits as

$$\Omega^2_+(ad(\mathfrak{u}_1)) \oplus \Omega^2_-(ad(\mathfrak{u}_1)), \tag{2.2}$$

where (+) is the seld-dual component and (-) the anti-self-dual.

The $1^{st}$-order $SW$-monopole equations are defined over the configuration space $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma({\mathcal{S}_\alpha}^+)$ as

$$\begin{cases} D_A^+(\phi) = 0, \\ F_A^+ = \sigma(\phi), \end{cases} \tag{2.3}$$

where

- $D_A^+$ is the $Spinc^c$-Dirac operator defined on $\Gamma(\mathcal{S}_\alpha^+)$,
- The quadratic form $\sigma : \Gamma(\mathcal{S}_\alpha^+) \to End^0(S_\alpha^+)$ given by

$$\sigma(\phi) = \phi \otimes \phi^* - \frac{|\ \phi\ |^2}{2}.I \tag{2.4}$$

performs the coupling of the $ASD$-equation with the $Dirac^c$ operator. Locally, for $\phi = (\phi_1, \phi_2)$, the quadratic form takes the value

$$\sigma(\phi) = \begin{pmatrix} \frac{|\phi_1|^2 - |\phi_2|^2}{2} & \phi_1.\bar{\phi_2} \\ \phi_2.\bar{\phi_1} & \frac{|\phi_2|^2 - |\phi_1|^2}{2} \end{pmatrix}.$$

The set of solutions of ( 2.3), known as $SW$-monopoles space, is the space $\mathcal{F}^{-1}(0)$, where $\mathcal{F}_\alpha : \mathcal{C}_\alpha \to \Omega^2_+(X) \oplus \Gamma(\mathcal{S}_\alpha{}^-)$ is a map defined by

$$\mathcal{F}_\alpha(A, \phi) = (F_A^+ - \sigma(\phi), D_A^+(\phi)).$$

The $\mathcal{SW}_\alpha$-equations are $\mathcal{G}_\alpha$-invariant and the map $\mathcal{F}$ is a Fredholm map up to the gauge equvalence.

**Definition 2.2.1.** *A $\mathcal{SW}_\alpha$-monopole is a solution $(A, \phi)$ of the $\mathcal{SW}_\alpha$-monopole equation such that $\phi \neq 0$. Solution of type $(A, 0)$ cames from an anti-self-dual connection $A$.*

## 3. **Variational Principle**

3.1. **Sobolev Spaces.** As a vector bundle E over (X,g) is endowed with a metric and a covariant derivative $\nabla$, we define the Sobolev norm of a section $\phi \in \Omega^0(E)$ as

$$||\ \phi\ ||_{L^{k,p}} = \sum_{|i|=0}^{k} (\int_X |\ \nabla^i \phi\ |^p)^{\frac{1}{p}}.$$

In this way, the $L^{k,p}$-Sobolev Spaces of sections of E is defined as

$$L^{k,p}(E) = \{\phi \in \Omega^0(E) \mid \quad ||\ \phi\ ||_{L^{k,p}} < \infty\}.$$

In our context, in which we fixed a connection $\nabla^0$ on $\mathcal{L}_\alpha$, a metric $g$ on $X$ and an hermitian structure on $\mathcal{S}_\alpha$, the Sobolev Spaces on which the basic setting is made are the following;

- $\mathcal{A}_\alpha = L^{1,2}(\Omega^1(ad(\mathfrak{u}_1))))$;
- $\Gamma(\mathcal{S}_\alpha{}^+) = L^{1,2}(\Omega^0(X, \mathcal{S}_\alpha{}^+))$;
- $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha{}^+)$;
- $\mathcal{G}_\alpha = L^{2,2}(X, U_1) = L^{2,2}(Map(X, U_1))$.
  ($\mathcal{G}_\alpha$ is an $\infty$-dimensional Lie Group which Lie algebra is $\mathfrak{g} = L^{1,2}(X, \mathfrak{u}_1)$).

The Dirichlet ($\mathcal{D}$) and Neuman ($\mathcal{N}$) problems requiry their own configurations spaces $\mathcal{C}_\alpha^{\mathcal{D}}$ and $\mathcal{C}_\alpha^{\mathcal{N}}$, respectively. From now on, both the configuration spaces will be denoted by $\mathcal{C}_\alpha$ by ignoring the superscripts, unless if it needed be.

The most basic analytical results is the *Gauge Fixing Lemma* (Uhlenbeck - [19]) and the estimate 3.1, both extended by Marini,A. [12] to manifolds with boundary; it gives a clue to define a suitable slice of the $\mathcal{G}_\alpha$-action.

**Lemma 3.1.1.** *(Gauge Fixing Lemma) - Every connection $\hat{A} \in \mathcal{A}_\alpha$ is gauge equivalent, by a gauge transformation $g \in \mathcal{G}_\alpha$ named Coulomb ($\mathfrak{C}$) gauge, to a connection $A \in \mathcal{A}_\alpha$ satisfying*

1. $d_\tau^{*_f} A_\tau = 0$ *on* $\partial X$,
2. $d^* A = 0$ *on* $X$.
3. *In the $\mathcal{N}$-problem, the connection $A$ satisfies $A_\nu = 0$ ($\nu \perp \partial X$).*

**Corollary 3.1.2.** *Under the hypothesis of 3.1.1, there exists a constant $K > 0$ such that the connection $A$, gauge equivalent to $\hat{A}$ by the Coulomb gauge, satisfies the following estimates:*

$$|| A ||_{L^{1,p}} \leq K. || F_A ||_{L^p} \tag{3.1}$$

**notation:** $*_f$ is the Hodge operator in the flat metric and the index $\tau$ denotes tangencial components.

3.2. **Variational Formulation.** The most natural functional to be considered is

$$SW(A,\phi) = \frac{1}{2}\int_X \{| F_A^+ - \sigma(\phi) |^2 + | D_A^+(\phi) |^2\} dv_g. \tag{3.2}$$

Clearly, the $\mathcal{SW}_\alpha$-monopoles are the stable critical points. The next set of identities are applied to expand the functional 3.2;

**Proposition 3.2.1.** *For each $\alpha \in Spin^c(X)$, let $\mathcal{L}_\alpha$ be the determinant line bundle associated to $\alpha$ and $(A,\phi) \in \mathcal{C}_\alpha$. Also, assume that $k_g$=scalar curvature of (X,g). Then,*

1. $< F_A^+, \sigma(\phi) >= \frac{1}{2} < F_A^+.\phi, \phi >$.
2. $< \sigma(\phi), \sigma(\phi) >= \frac{1}{4} | \phi |^4$.
3. *Weitzenböck formula*
$$D^2\phi = \nabla^*\nabla\phi + \frac{k_g}{4}\phi + \frac{F_A}{2}.\phi.$$
4. $\sigma(\phi)\phi = \frac{|\phi|^2}{2}\phi$.
5. *The intersection form of $X$ $Q_X : H^2(X,\mathbb{R}) \times H^2(X,\mathbb{R}) \to \mathbb{R}$ is given by*
$$Q(\omega,\eta) = \int_X \omega \wedge \eta, \quad \alpha^2 = Q(\alpha,\alpha).$$
6. $\int_X | F_A^+ |^2 \, dv_g = \int_X \frac{1}{2} | F_A |^2 \, dv_g + 2\pi^2\alpha^2$.

As a consequence, a new functional turns up into the scenario;

**Definition 3.2.2.** *For each $\alpha \in Spin^c(X)$, the Seiberg-Witten Functional is the functional $\mathcal{SW}_\alpha : \mathcal{C}_\alpha \to \mathbb{R}$ given by*

$$\mathcal{SW}_\alpha(A,\phi) = \int_X \{\frac{1}{4} \mid F_A \mid^2 + \mid \nabla^A \phi \mid^2 + \frac{1}{8} \mid \phi \mid^4 + \frac{1}{4} k_g \mid \phi \mid^2\} dv_g + \pi^2\alpha^2, \quad (3.3)$$

*where $k_g$= scalar curvature of (X,g).*

**Remark 3.2.3.**

1. Since X is compact and $\| \phi \|_{L^4} < \| \phi \|_{L^{1,2}}$, the functional is well defined on $\mathcal{C}_\alpha$ ,
2. The $\mathcal{SW}_\alpha$-functional *(3.3)* being Gauge invariant induces a functional $\mathcal{SW}_\alpha : \mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha) \to \mathbb{R}$.
3. The $\mathcal{SW}_\alpha$-functional is bounded below by 0, and it is equal to 0 if and only if either there exists a $\mathcal{SW}_\alpha$-monopole or a anti-self-dual $U_1$-connection.

It is an open question to find a sufficient condition to prove that there exist a $\mathcal{SW}_\alpha$-monopole on a 4-manifold $X$. If $X$ is sympletic, Taubes [16] proved that whenever $\alpha$ is the the canonical class then there is a $\mathcal{SW}_\alpha$-monopole. As a consequence of the main estimate 4.0.2, there is a non-existence result for $\mathcal{SW}_\alpha$-monopoles on manifolds whose scalar curvature is non-negative. A necessary condition for the existence of a $\mathcal{SW}_\alpha$-monopole is the following estimate;

**Proposition 3.2.4.** *Let $\alpha \in Spin^c(X)$ and $(A,\phi)$ be a $\mathcal{SW}_\alpha$-monopole. So,*

$$\alpha^2 \leq \frac{2}{\pi^2} v_X.(k^-_{g,X})^4 \quad (3.4)$$

*Proof.*

$$\mathcal{SW}_\alpha(A,\phi) - \int_X k_g \mid \phi \mid^2 = \frac{1}{4} \| F_A \|^2_{L^2} + \| \nabla^A \phi \|^2_{L^2} + \frac{1}{8} \| \phi \|^4_{L^4} .$$

Therefore, taking $k^-_{g,X} = \min_{x\in X} k_g(x)$,

$$\mathcal{SW}_\alpha(A,\phi) + (-k^-_{g,X}) \| \phi \|^2_{L^2} \geq \frac{1}{4} \| F_A \|^2_{L^2} + \| \nabla^A \phi \|^2_{L^2} + \frac{1}{8} \| \phi \|^4_{L^4},$$

and,

$$\| \phi \|^4_{L^4} \leq 8\mathcal{SW}_\alpha(A,\phi) + 8(k^-_{g,X})^2 \| \phi \|^2_{L^2},$$

where $k^-_{X,g} = \sqrt{\max\{0, -k^-_{X,g}\}}$. It follows from inequality ( 7.1) that

$$\| \phi \|^4_{L^2} \leq 8v_X \mathcal{SW}_\alpha(A,\phi) + 8v_X.(k^-_{g,X})^2 \| \phi \|^2_{L^2} .$$

So,

$$\| \phi \|^4_{L^2} - 8v_X.(k^-_{g,X})^2 \| \phi \|^2_{L^2} - 8v_X \mathcal{SW}_\alpha(A,\phi) \leq 0. \quad (3.5)$$

Let's consider the quadratic function $f : \mathbb{R} \to \mathbb{R}$

$$f(x) = x^2 - 8v_X.(k^-_{g,X})^2 x - 8v_X \mathcal{SW}_\alpha(A,\phi).$$

If the inequality $f(x) \leq 0$ is not satisfied by any $x \in \mathbb{R}$, then we would have $\phi = 0$. The discriminant of $f$ is

$$\triangle = 32v_X\left(2v_X.(k^-_{g,X})^4_{g,X} + \mathcal{SW}_\alpha(A,\phi)\right).$$

The inequality ( 3.5) admits solution if and only if $\triangle \geq 0$, since $f``(x) > 0$ and $\min_{x\in\mathbb{R}} f(x) = -\frac{\triangle}{4}$. Therefore,

$$\mathcal{SW}_\alpha(A,\phi) \geq -2v_X.(k^-_{g,X})^4$$

Since $\mathcal{SW}_\alpha(A,\phi) = -\pi^2\alpha^2$, the lower upper bound of the $\mathcal{SW}_\alpha$-functional is

$$\mathcal{SW}_\alpha(A,\phi) \geq \max\{-\pi^2\alpha^2, -2v_X.(k^-_{g,X})^4\}.$$

In this way, if $(A,\phi)$ is a $\mathcal{SW}_\alpha$-monopole satisfying the $1^{st}$-order $\mathcal{SW}_\alpha$-equation then $\mathcal{SW}_\alpha(A,\phi) = -\pi^2\alpha^2$, where

$$\alpha^2 \leq \frac{2}{\pi^2}v_X.(k^-_{g,X})^4.$$

The $L^2$-norm of a spinor field turns out to be bounded, as shown the identity below;

$$4v_X.(k^-_{g,X})^2 - 2\sqrt{4v_X\left[2v_X(k^-_{g,X})^4 + \mathcal{SW}_\alpha(A,\phi)\right]} \leq \|\phi\|^2_{L^2} \leq \tag{3.6}$$

$$\leq 4v_X.(k^-_{g,X})^2 + 2\sqrt{4v_X\left[2v_X.(k^-_{g,X})^4 + \mathcal{SW}_\alpha(A,\phi)\right]} \tag{3.7}$$

□

The Euler-Lagrange equations of the $\mathcal{SW}_\alpha$-functional *(3.3)* are

$$\Delta_A\phi + \frac{|\phi|^2}{4}\phi + \frac{k_g}{4}\phi = 0, \tag{3.8}$$

$$d^*F_A + 4\Phi^*(\nabla^A\phi) = 0, \tag{3.9}$$

where $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1(\mathcal{S}_\alpha{}^+)$. The dual operator $\Phi^* : \Omega^1(\mathcal{S}_\alpha{}^+) \to \Omega^1(\mathfrak{u}_1)$ is locally, in a orthonormal basis $\{\eta^i\}_{1\leq i\leq 4}$ of $T^*X$, written as

$$\Phi^*(\nabla^A\phi) = \sum_{i=1}^4 < \nabla^A_i\phi, \phi > \eta^i, \quad \text{where} \quad \nabla^A_i = \nabla^A_{X_i} \quad (\eta_i(X_j) = \delta_{ij}). \tag{3.10}$$

The equations above are refered as the $\mathcal{SW}_\alpha$-equations.

**Remark 3.2.5.** The $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ has the following properties;

1. the $\mathcal{SW}_\alpha$-functional is $\mathcal{G}_\alpha$-invariant.
2. the $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ induces on $T\mathcal{C}_\alpha$ a $\mathcal{G}_\alpha$-action as follows: let $(\Lambda, V) \in T_{(A,\phi)}\mathcal{C}_\alpha$ and $g \in \mathcal{G}_\alpha$,

$$g.(\Lambda, V) = (\Lambda, g^{-1}V) \in T_{g.(A,\phi)}\mathcal{C}_\alpha.$$

Consequently, $d(\mathcal{SW}_\alpha)_{g.(A,\phi)}(g.(\Lambda, V)) = d(\mathcal{SW}_\alpha)_{(A,\phi)}(\Lambda, V)$.

The tangent bundle $T\mathcal{C}_\alpha$ decomposes as

$$T\mathcal{C}_\alpha = \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+).$$

In this way, the 1-form $d\mathcal{SW}_\alpha \in \Omega^1(\mathcal{C}_\alpha)$ can be decomposed as
$d\mathcal{SW}_\alpha = d_1\mathcal{SW}_\alpha + d_2\mathcal{SW}_\alpha$, where

$$d_1(\mathcal{SW}_\alpha)_{(A,\phi)} : \Omega^1(ad(\mathfrak{u}_1)) \to \mathbb{R}, \quad d_1(\mathcal{SW}_\alpha)_{(A,\phi)}.\Lambda = d(\mathcal{SW}_\alpha)_{(A,\phi)}.(\Lambda, 0)$$

$$d_2(\mathcal{SW}_\alpha)_{(A,\phi)} : \Gamma(\mathcal{S}_\alpha^+) \to \mathbb{R}, \quad d_2(\mathcal{SW}_\alpha)_{(A,\phi)}.V = d(\mathcal{SW}_\alpha)_{(A,\phi)}.(0, V).$$

By performing the computations, we get

(1) for every $\Lambda \in \mathcal{A}_\alpha$,

$$d_1(\mathcal{SW}_\alpha)_{(A,\phi)}.\Lambda = \frac{1}{4}\int_X Re\{< F_A, d_A\Lambda > +4 < \nabla^A(\phi), \Phi(\Lambda) >\}dx, \tag{3.11}$$

where $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1({\mathcal{S}_\alpha}^+)$ is the linear operator $\Phi(\Lambda) = \Lambda(\phi)$, which dual is defined in 3.10,

(2) for every $V \in \Gamma(\mathcal{S}_\alpha^+)$,

$$d_2(\mathcal{SW}_\alpha)_{(A,\phi)}.V = \int_X Re\{< \nabla^A\phi, \nabla^A V > + < \frac{\mid \phi \mid^2 + k_g}{4}\phi, V >\}dx. \tag{3.12}$$

Therefore, by taking $supp(\Lambda) \subset int(X)$ and $supp(V) \subset int(X)$, we restrict to the interior of $X$, and so, the gradient of the $\mathcal{SW}_\alpha$-functional at $(A, \phi) \in \mathcal{C}_\alpha$ is

$$grad(\mathcal{SW}_\alpha)(A, \phi) = (d_A^* F_A + 4\Phi^*(\nabla^A\phi), \triangle_A\phi + \frac{\mid \phi \mid^2 + k_g}{4}\phi). \tag{3.13}$$

It follows from the $\mathcal{G}_\alpha$-action on $T\mathcal{C}_\alpha$ that

$$grad(\mathcal{SW}_\alpha)(g.(A, \phi)) = \left(d_A^* F_A + 4\Phi^*(\nabla^A\phi), g^{-1}.(\triangle_A\phi + \frac{\mid \phi \mid^2 + k_g}{4}\phi)\right). \tag{3.14}$$

3.3. **Coercivity of the $\mathcal{SW}_\alpha$-Functional.** An important analytical aspect of the $\mathcal{SW}_\alpha$-functional is the Coercivity Lemma proved in [8];

**Lemma 3.3.1. Coercivity** - *For each $(A, \phi) \in \mathcal{C}_\alpha$, there exists $g \in \mathcal{G}_\alpha$ and a constant $K_{\mathfrak{C}}^{(A,\phi)} > 0$, where $K_{\mathfrak{C}}^{(A,\phi)}$ depends on $(X, g)$ and $\mathcal{SW}_\alpha(A, \phi)$, such that*

$$\| g.(A, \phi) \|_{L^{1,2}} < K_{\mathfrak{C}}^{(A,\phi)}.$$

*Proof.* lemma 2.3 in [8]. The gauge transform is the Coulomb one given in the Gauge Fixing Lemma 3.1.1. □

Considering the gauge invariance of the $\mathcal{SW}_\alpha$-theory, and the fact that the gauge group $\mathcal{G}_\alpha$ is a infinite dimensional Lie Group, we can't hope to handle any analytical question in general, we need to work on a slice for the action. So forth, we restrict the problem to the space, named Coulomb Subspace,

$$\mathcal{C}_\alpha^{\mathfrak{C}} = \{(A,\phi) \in \mathcal{C}_\alpha; || (A,\phi) ||_{L^{1,2}} < K_{\mathfrak{C}}^{(A,\phi)}\}, \tag{3.15}$$

## 4. **Main Estimate**

In order to pursue the strong $L^{1,2}$-convergence for the sequence $\{(A_n,\phi_n)\}_{n\in\mathbb{Z}}$, given by condition 5.0.3, next we obtain an upper bound for $|| \phi ||_{L^\infty}$, whenever $(A,\phi)$ is a weak solution . Due to its fundamental importance, it is named the *Main Estimate.*

**Lemma 4.0.2.** *Let $(A,\phi)$ be a solution of either $\mathcal{D}$ or $\mathcal{N}$ in 7.1, so*

(1) *If $\sigma = 0$, then there exists a constant $k_{X,g}$, depending on the Riemannian metric on X, such that*

$$|| \phi ||_\infty < k_{X,g}, \quad k_{X,g} = \sqrt{\max_{x\in X}\{0, -k_g(x)\}} \tag{4.1}$$

(2) *If $\sigma \neq 0$, then there exist constant $c_1 = c_1(X,g)$ and $c_2 = c_2(X,g)$ such that*

$$|| \phi ||_{L^p} < c_1 + c_2 \,|| \sigma ||_{L^{3p}}^3 \,. \tag{4.2}$$

*In particular, if $\sigma \in L^\infty$ then $\phi \in L^\infty$*

*Proof.* Fix $r \in \mathbb{R}$ and suppose that there is a ball $B_{r^{-1}}(x_0)$, around the point $x_0 \in X$, such that

$$| \phi(x) | > r, \quad \forall x \in B_{r^{-1}}(x_0).$$

Define

$$\eta = \begin{cases} \left(1 - \frac{r}{|\phi|}\right)\phi, & \text{if} \quad x \in B_{r^{-1}}(x_0), \\ 0, \quad \text{if} & x \in X - B_{r^{-1}}(x_0) \end{cases}$$

So,

$$| \eta | \leq | \phi | \tag{4.3}$$

$$\nabla\eta = r\frac{<\phi,\nabla\phi>}{|\phi|^3}\phi + (1 - \frac{r}{|\phi|})\nabla\phi$$

$$\Rightarrow |\nabla\eta|^2 = r^2\frac{<\phi,\nabla\phi>^2}{|\phi|^4} + 2r(1 - \frac{r}{|\phi|})\frac{<\phi,\nabla\phi>^2}{|\phi|^3} + (1 - \frac{r}{|\phi|})^2 \,|\nabla\phi|^2$$

$$\Rightarrow |\nabla\eta|^2 < r^2\frac{|\nabla\phi|^2}{|\phi|^2} + 2r(1 - \frac{r}{|\phi|})\frac{|\nabla\phi|^2}{|\phi|} + (1 - \frac{r}{|\phi|})^2 \,|\nabla\phi|^2 \,.$$

Since $r < |\phi|$,

$$|\nabla\eta|^2 < 4\,|\nabla\phi|^2 \,. \tag{4.4}$$

Hence, by 4.3 and 4.4, $\eta \in L^{1,2}$.
The directional derivative of $\mathcal{SW}_\alpha$ in direction $\eta$ is given by

$$d(\mathcal{SW}_\alpha)_{(A,\phi)}(0,\eta) = \int_X [< \nabla^A\phi, \nabla^A\eta > + \frac{\mid \phi \mid^2 + k_g}{4} \mid \phi \mid (\mid \phi \mid - r)].$$

By 3.12),

$$\int_X [< \nabla^A\phi, \nabla^A\eta > + \frac{\mid \phi \mid^2 + k_g}{4} \mid \phi \mid (\mid \phi \mid - r)] = \int_X < \sigma, (1 - \frac{r}{\mid \phi \mid})\phi > .$$

However,

$$\int_X < \nabla^A\phi, \nabla^A\eta >= \int_X [r\frac{< \phi, \nabla^A\phi >^2}{\mid \phi \mid^3} + (1 - \frac{r}{\mid \phi \mid}) \mid \nabla\phi \mid^2] > 0.$$

So,

$$\int_X \frac{\mid \phi \mid^2 + k_g}{4} \mid \phi \mid (\mid \phi \mid - r) < \int_X < \sigma, (1 - \frac{r}{\mid \phi \mid})\phi >< \int_X \mid \sigma \mid (\mid \phi \mid - r).$$

Hence,

$$\int_X (\mid \phi \mid - r) \left( \frac{\mid \phi \mid^2 + k_g}{4} \mid \phi \mid - \mid \sigma \mid \right) < 0.$$

Since $r <\mid \phi(x) \mid$, whenever $x \in B_{r^{-1}}(x_0)$, it follows that

(4.5) $$(\mid \phi \mid^2 + k_g) \mid \phi \mid < 4 \mid \sigma \mid, \quad \text{almost everywhere in} \quad B_{r^{-1}}(x_0).$$

There are two cases to be analysed independently;

(1) $\sigma = 0$.
In this case, we get

(4.6) $$(\mid \phi \mid^2 + k_g) \mid \phi \mid < 0, \quad \text{almost everywhere.}$$

The scalar curvature plays a central role here: if $k_g \geq 0$ then $\phi = 0$; otherwise,

$$\mid \phi \mid \leq max\{0, (-k_g)^{1/2}\}.$$

Since X is compact, we let $k_{X,g} = \sqrt{\max_{x\in X}\{0, -k_g(x)\}}$, and so,

$$\mid\mid \phi \mid\mid_\infty < k_{X,g}.$$

(2) Let $\sigma \neq 0$.
The inequality 4.5 implies that

$$\mid \phi \mid^3 + k_g \mid \phi \mid - 4 \mid \sigma \mid < 0 \quad \text{a.e.}$$

Consider the polynomial

$$Q_{\sigma(x)}(w) = w^3 + k_g w - 4 \mid \sigma(x) \mid .$$

A estimate for $\mid \phi \mid$ is obtained by estimating the largest real number $w$ satisfying $Q_{\sigma(x)}(w) < 0$. $Q_{\sigma(x)}$ being monic implies that $\lim_{w\to\infty} Q_{\sigma(x)}(w) = +\infty$. So, either $Q_{\sigma(x)} > 0$, whenever $w > 0$, or there exist a root $\rho \in (0, \infty)$. The first case would imply that

$$Q_{\sigma(x)}(\mid \phi(x) \mid) > 0, \quad a.e.,$$

contradicting 4.5. By the same argument, there exists a root $\rho \in (0, \infty)$ such that $Q_{\sigma(x)}(w)$ chances its sign in a neighboorhood of $\rho$. Let $\rho$ be the largest root in $(0, \infty)$ with this property. There exist constants $c_1 = c_1(X, g)$ and $c_2$ such that

$$\mid \rho \mid < c_1 + c_2 \mid \sigma(x) \mid^3 .$$

Consequently,

$$\mid \phi(x) \mid < c_1 + c_2 \mid \sigma(x) \mid^3, \quad \text{a.e. in } B_{r^{-1}}(x_0) \tag{4.7}$$

and

$$\mid\mid \phi \mid\mid_{L^p} < C_1 + C_2 \mid\mid \sigma \mid\mid^3_{L^{3p}}, \quad \text{restricted to } B_{r^{-1}}(x_0) \tag{4.8}$$

where $C_1, C_2$ are constants depending on $vol(B_{r^{-1}}(x_0))$.
The inequality 4.8 can be extended over X by using a $C^\infty$ partition of unity. Moreover, if $\sigma \in L^\infty$, then

$$\mid\mid \phi \mid\mid_\infty < C_1 + C_2 \mid\mid \sigma \mid\mid^3_\infty, \tag{4.9}$$

where $C_1, C_2$ are constants depending on $vol(X)$.

□

## 5. $\mathcal{H}$-Condition and Palais-Smale Condition

In the variational formulation, the problems $\mathcal{D}$ and $\mathcal{N}$ ( 7.1) are written as

(5.1)
$$(\mathcal{D}) = \begin{cases} grad(\mathcal{SW}_\alpha)(A, \phi) = (\Theta, \sigma), \\ (A, \phi) \mid_{\partial X} \overset{\text{gauge}}{\sim} (A_0, \phi_0). \end{cases} \quad (\mathcal{N}) = \begin{cases} grad(\mathcal{SW}_\alpha)(A, \phi) = (\Theta, \sigma), \\ i^*(*F_A) = 0, \nabla^A_n \phi = 0, \end{cases}$$

The equations in 5.1 may not admit a solution for any pair $(\Theta, \sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}^+_\alpha)$. In finite dimension, if we consider a function $f : X \to \mathbb{R}$, the analogous question would be to find a point $p \in X$ such that, for a fixed vector $u$, $grad(f)(p) = u$. This question is more subtle if $f$ is invariant under a Lie group action on $X$. Therefore, we need the hypothesis below on the pair $(\Theta, \sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}^+_\alpha)$;

**Condition 5.0.3.** **($\mathcal{H}$)** - *Let* $(\Theta, \sigma) \in L^{1,2}(\Omega^1(ad(\mathfrak{u}_1))) \oplus (L^{1,2}(\Gamma(\mathcal{S}^+_\alpha)) \cap L^\infty(\Gamma(\mathcal{S}^+_\alpha)))$ *be a pair such that there exists a sequence* $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \subset \mathcal{C}^{\mathfrak{c}}_\alpha$ *( 3.15) with the following properties;*

1. $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}} \subset L^{1,2}(\mathcal{A}_\alpha) \times (L^{1,2}(\Gamma(\mathcal{S}^+_\alpha)) \cup L^\infty(\Gamma(\mathcal{S}^+_\alpha)))$ *and there exists a constant* $c_\infty > 0$ *such that, for all* $n \in \mathbb{Z}$, $\mid\mid \phi_n \mid\mid_\infty < c_\infty$.
2. *there exists* $c \in \mathbb{R}$ *such that, for all* $n \in \mathbb{Z}$, $\mathcal{SW}_\alpha(A_n, \phi_n) < c$,
3. *the sequence* $\{d(\mathcal{SW}_\alpha)_{(A_n, \phi_n)}\}_{n \in \mathbb{Z}} \subset \left(L^{1,2}(\Omega^1(ad(\mathfrak{u}_1))) \oplus L^{1,2}(\Gamma(\mathcal{S}^+_\alpha))\right)^*$, *of linear functionals, converges weakly to*

$$L_\Theta + L_\sigma : T\mathcal{C}_\alpha \to \mathbb{R},$$

*where*

$$L_\Theta(\Lambda) = \int_X < \Theta, \Lambda >, \quad L_\sigma(V) = \int_X < \sigma, V > .$$

As a consequency of 3.3.1, the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ given by the $\mathcal{H}$-condition has the following properties;

(1) converges to a pair $(A, \phi)$ weakly in $\mathcal{C}_\alpha$,
(2) converges to a pair $(A, \phi)$ weakly in $L^4(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}^+_\alpha))$,
(3) converges to a pair $(A, \phi)$ strongly in $L^p(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}^+_\alpha))$, for every $p < 4$.
(4) The limit $(A, \phi) \in L^2(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}^+_\alpha))$, obtained as a limit of the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, is a weak solution of 7.1 ( [4]).

It turns out that the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, given by the $\mathcal{H}$-condition 5.0.3, converges strongly. The proof in [4] use the main estimate 4.0.2 and the fact that

$$\lim_{n\to\infty} \int_X < \Phi^*(\nabla^{A_n}\phi_n), A_n - A >= 0.$$

**Theorem 5.0.4.** [4] *Let $(\Theta, \sigma)$ be a pair satisfying the $\mathcal{H}$- condition 5.0.3. Then, the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, given by 5.0.3, converges strongly to $(A, \phi) \in \mathcal{C}_\alpha$.*

**Corollary 5.0.5.** [8] *The $\mathcal{SW}_\alpha$-functional satisfies the Palais-Smale Condition.*

**5.1 - $\mathcal{SW}_\alpha$-Monopoles in $X = \mathbb{R}^4$**

It cames out of the Main Estimate that in $\mathbb{R}^4$ the only solution, up to gauge equivalence, to the $\mathcal{SW}_\alpha$-equations is $(0, 0)$. The scalar curvature of $\mathbb{R}^4$ being $k_g = 0$ implies, by the Main Estimate,, that any solution has type $(A, 0)$, where $d^*F_A = 0$. However, by Hodge theory $F_A = 0$, so the only solution, up to gauge equivalence, is $(0, 0)$.

## 6. Homotopy Type of the Configuration Space

In this section, let's consider $X$ a boundaryless smooth 4-manifold. Considering that the $\mathcal{SW}_\alpha$-functional satisfies the Palais-Smale condition it is natural to ask about the existence of non-stable critical points. This is achieved once we know the homotopy type of $\mathcal{C}_\alpha$. The embeddeding of the Jacobian Torus

$$i : T^{b_1(X)} = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})} \hookrightarrow \mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha), \quad b_1(X) = dim_{\mathbb{R}} H^1(X, \mathbb{R}),$$

defined in 6.0.6, and the variational formulation of the $\mathcal{SW}_\alpha$-equations together give us a interpretation to the topology of $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$;

**Proposition 6.0.6.** *Let X be a closed, smooth 4-manifold. The solutions of $d^*F_A = 0$, module the $\mathcal{G}_\alpha$-action, define the Jacobian Torus*

$$T^{b_1(X)} = \frac{H^1(X, \mathbb{R})}{H^1(X, \mathbb{Z})}, \quad b_1(X) = dim_{\mathbb{R}} H^1(X, \mathbb{Z}).$$

*Proof.* Let's recall that $\alpha = \frac{i}{2\pi}\int F_A$. The equation $d^*F_A = 0$ implies that $F_A$ is an harmonic 2-form, and by Hodge theory, it is the only one. Let $A$ and $B$ be solutions and consider $B = A + b$, so,

$$d^*F_B + d^*F_A + d^*db = 0 \quad \Rightarrow db = 0,$$

from where we can associate $B \rightsquigarrow b \in H^1(X,\mathbb{R})$ (and $F_B = F_A$).

If a connection $B_1$ is gauge equivalent to $B_2$, then there exists $g \in \mathcal{G}_\alpha$ such that $B = A + g^{-1}dg$ and $F_B = F_A$. However, the 1-form $g^{-1}dg \in H^1(X,\mathbb{Z})$. Consequently, if $b_1, b_2$ are the respectives elements in $H^1(X,\mathbb{R})$, then $b_2 = b_1$ in $\frac{H^1(X,\mathbb{R})}{H^1(X,\mathbb{Z})}$. □

The relation among the critical set of $\mathcal{SW}_\alpha$-functional and the homotopy of $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ is described by the next result;

**Theorem 6.0.7.** [3] *Let $X$ be a closed smooth 4-manifold endowed with a riemannian metric $g$ which scalar curvature is $k_g$. Let*

1. *If $k_g \geq 0$, then the gradient flow of the $\mathcal{SW}_\alpha$-functional defines an homotopy equivalence among $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ and $i(T^{b_1(X)})$.*
2. *If $k_g < 0$, then $\mathcal{A}_\alpha \times_{\mathcal{G}_\alpha} \Gamma(S^+_\alpha)$ has the same homotopy type of $T^{b_1(X)}$.*

$(A,0)$ is a solution of the Monopole $\mathcal{SW}_\alpha$-equation (minimum for $\mathcal{SW}_\alpha$) whenever $F^+_A = 0$. It is known ( [2]) that if $b^+_2 > 1$, then such solutions do not exists for a dense set of the space of metrics on X. Therefore,

1. As a consequence of 3.2.4, only for a finite number of classes $\alpha \in Spin^c(X)$ there exists a $\mathcal{SW}_\alpha$-monopole attaining the minimum.
2. If $\alpha \in Spin^c(X)$ is none of the classes considered in the previous item, then

$$\inf_{(A,\phi)\in\mathcal{A}_\alpha\times_{\mathcal{G}_\alpha}\Gamma(S^+_\alpha)} \mathcal{SW}_\alpha(A,\phi) > 0.$$

## 7. Dirichlet and Neuman Problems associated to the $\mathcal{SW}_\alpha$-Equation

The Dirichlet $(\mathcal{D})$ and Neumann $(\mathcal{N})$ boundary value problems associated to the $\mathcal{SW}_\alpha$-equations are the following: Let's consider $(\Theta,\sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}^+_\alpha)$ and $(A_0,\phi_0)$ defined on the manifold $\partial X$ ($A_0$ is a connection on $\mathcal{L}_\alpha \mid_{\partial X}$, $\phi_0$ is a section of $\Gamma(\mathcal{S}^+_\alpha \mid_{\partial X})$). In this way, find $(A,\phi) \in \mathcal{C}^{\mathcal{D}}_\alpha$ satisfying $\mathcal{D}$ and $(A,\phi) \in \mathcal{C}^{\mathcal{N}}_\alpha$ satisfying $\mathcal{N}$, where

$$\mathcal{D} = \begin{cases} d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta, \\ \Delta_A\phi + \frac{(|\phi|^2+k_g)}{4}\phi = \sigma, \\ (A,\phi)\mid_{\partial X} \overset{\text{gauge}}{\sim} (A_0,\phi_0), \end{cases} \qquad \mathcal{N} = \begin{cases} d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta, \\ \Delta_A\phi + \frac{(|\phi|^2+k_g)}{4}\phi = \sigma, \\ i^*(*F_A) = 0, \nabla^A_\nu\phi = 0, \end{cases} \tag{7.1}$$

where $i^*(*F_A) = F_4$, $F_4 = (F_{14}, F_{24}, F_{34}, 0)$ is, localy, the $4^{th}$-component (normal to $\partial X$) of the 2-form of curvature in the local chart $(x,U)$ of $X$;
$x(U) = \{x = (x_1,x_2,x_3,x_4) \in \mathbb{R}^4; \| x \| < \epsilon, x_4 \geq 0\}$, and
$x(U \cap \partial X) \subset \{x \in x(U) \mid x_4 = 0\}$. Let $\{e_1,e_2,e_3,e_4\}$ be the canonical base of $\mathbb{R}^4$, so $\nu = -e_4$ is the normal vector field along $\partial X$.

The existence of a strong solution follows from 5.0.4. Basically, the regularity follows from the corollary 3.1.2 and from the Main Estimate 4.0.2, as proved in [4];

**Theorem 7.0.8.** [4] *If the pair* $(\Theta, \sigma) \in L^{k,2} \oplus (L^{k,2} \cap L^\infty)$ *satisfies the* $\mathcal{H}$*-condition 5.0.3, then the problems* $\mathcal{D}$ *and* $\mathcal{N}$ *admit a* $C^r$*-regular solution* $(A, \phi)$*, whenever* $2 < k$ *and* $r < k$.

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