SYMMETRIC SPACES OF RIEMANN SURFACES

10 de julho de 2008

Sumário

Capítulo 1

Jacobi and Picard Varieties

This chapter is devoted to the description of the space of complex line bundles $\xi \in H^1(M, \mathcal{O}^*)$ with $c_1(\xi) = 0$. Let's start with sheavf the exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O} \xrightarrow{e} \mathcal{O}^* \longrightarrow 0, \tag{1.1}$$

where $e(f) = exp(2\pi i f)$. The associated exact sequence of cohomology groups is

$$0 \longrightarrow \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \xrightarrow{e_*} H^1(M, \mathcal{O}^*) \xrightarrow{c_1} \mathbb{Z} \longrightarrow 0, \tag{1.2}$$

where $c_1(\xi) \in \mathbb{Z}$ is the 1^{st}-Chern Class of the complex line bundle ξ. In this way,

$$\frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} = \{\xi \in H^1(M, \mathcal{O}^*) \mid c_1(\xi) = 0\} \tag{1.3}$$

Also consider the sheaf exact sequence

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{O} \xrightarrow{d} \mathcal{O}^{1,0} \longrightarrow 0, \tag{1.4}$$

which associated exact sequence of cohomology groups is

$$\begin{array}{llll} 0 \longrightarrow H^0(M,\mathbb{C}) & \longrightarrow H^0(M,\mathcal{O}) & \longrightarrow H^0(M,\mathcal{O}^{1,0}) \\ \xrightarrow{\partial_d} H^1(M,\mathbb{C}) & \longrightarrow H^1(M,\mathcal{O}) & \longrightarrow H^1(M,\mathcal{O}^{1,0}) \\ \xrightarrow{\partial_d} H^2(M,\mathbb{C}) & \longrightarrow H^2(M,\mathcal{O}) = 0. & \end{array} \tag{1.5}$$

(i) M closed $\Rightarrow H^0(M, \mathcal{O}) \simeq \mathbb{C}$.

(ii) $H^1(M, \mathcal{O}^{1,0}) \simeq H^0(M, \mathcal{O}) \simeq \mathbb{C}$, by Serre's duality.

(iii) $H^2(M, \mathcal{O}) = 0$

Thus, the exact sequence 1.5 becames

$$0 \longrightarrow H^0(M, \mathcal{O}^{1,0}) \xrightarrow{\partial_d} H^1(M,\mathbb{C}) \longrightarrow H^1(M,\mathcal{O}) \longrightarrow 0 \tag{1.6}$$

Therefore, $H^1(M,\mathcal{O}) = \frac{H^1(M,\mathbb{C})}{\partial_d H^0(M,\mathcal{O}^{1,0})}$ and

$$\frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} = \frac{H^1(M,\mathbb{C})}{H^1(M,\mathbb{Z}) + \partial_d H^0(M,\mathcal{O}^{1,0})} \tag{1.7}$$

From basic topology, it is known that $H^1(M,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$ and $H^1(M,\mathbb{C}) \simeq \mathbb{C}^{2g}$. Since $H^0(M,\mathcal{O}^{1,0}) \simeq \mathbb{C}^g$ is the g dimensional $\mathbb{C}$-vector space of Abelian differentials on M and ∂_d is a monomorphism, so $H^1(M,\mathcal{O}) = \frac{H^1(M,\mathbb{C})}{\partial_d H^0(M,\mathcal{O}^{1,0})} \simeq \mathbb{C}^g$ is also a g-dimensional $\mathbb{C}$-vector space[1]. In order to describe the space $\frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \simeq \frac{\mathbb{C}^g}{H^1(M,\mathbb{Z})}$ it is necessary to understand how $H^1(M,\mathbb{Z})$ embedds into $H^1(M,\mathcal{O})$.

Lemma 1.1. *Consider M a closed Riemann surface and $\partial_d : H^0(M,\mathcal{O}^{1,0}) \to H^1(M,\mathbb{C})$ the boundary homomorphism in the exact sequence 1.6. Let $\phi \in H^0(M,\mathcal{O}^{1,0})$ be an element such that $\partial_d\phi \in H^1(M,\mathbb{R}) \subset H^1(M,\mathbb{C})$, then $\phi = 0$.*

Corollary 1.1. *Let $\{\alpha_1,\beta_1,\dots,\alpha_g,\beta_g\}$ be a basis of $H^1(M,\mathbb{Z})$ and $v \to \tilde{v}$ the projection $H^1(M,\mathbb{C}) \to H^1(M,\mathcal{O})$. Then $\{\tilde{\alpha}_1,\tilde{\beta}_1,\dots,\tilde{\alpha}_g,\tilde{\beta}_g\}$ is a basis of $H^1(M,\mathcal{O})$.*

Consequently,

$$\frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \simeq \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}};$$

i.e., the space $\frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})}$ is a complex g-torus.

Definition 1.1. Let M be a closed Riemann surface;

i. The set $\mathcal{P}_M = \{\xi \in H^1(M,\mathcal{O}^*) \mid c_1(\xi) = 0\}$ is the Picard Variety of M.

ii. The g-complex torus $\mathcal{J}_M = \frac{H^1(M,\mathcal{O})}{H^1(M,\mathbb{Z})} \simeq \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}$ is the Jacobian variety of M.

The difference between $\mathcal{P}_M$ and $\mathcal{J}_M$ is subtle from the discussion. Basically, one described in terms of complex line bundles and the other one is obtained from a lattice.

Theorem 1.1. *The group of complex linear bundles over a closed Riemann surfaces has the natural structure*

$$H^1(M,\mathcal{O}^*) \simeq \mathcal{P}_M \oplus \mathbb{Z}.$$

[1] From Serre's duality this was already known

1.0.1 Lattices and Complex Torus

Definition 1.2. A lattice $\mathcal{L} \subset \mathbb{R}^n$ is a subgroup of $\mathbb{R}^n$ isomorphic to $\mathbb{Z}^r$. In terms of a $\mathbb{R}$-basis $\{u_1, \ldots, u_r\}$,

$$\mathcal{L} = \{\sum_{i=1}^{r} n_i u_i \mid n_i \in \mathbb{Z}\}.$$

Let's consider in $\mathbb{C}^g \overset{\mathbb{R}}{\simeq} \mathbb{R}^{2g}$ the lattice $\mathcal{L} \simeq \mathbb{Z}^{2g}$. Topologically, $\mathbb{C}^g \diagup \mathcal{L} = (\mathbb{R}\diagup\mathbb{Z})^{2g} \overset{diffeo}{\simeq} T^{2g}$ is a real $2g$-torus. Since the universal covering of T^{2g} is $\mathbb{R}^{2g}$, there is a covering map $p : \mathbb{C}^g \to \mathbb{C}^g \diagup \mathcal{L}$ inducing on $\mathbb{C}^g \diagup \mathcal{L}$ a natural complex analytic structure and also an analytical Lie group structure. These structures depend on $\mathcal{L}$ as showed next;

Lemma 1.2. *Let V, V' be g-dimensional complex vector spaces and $\mathcal{L}, \mathcal{L}'$ lattices groups of rank $2g$. The complex tori $V/\mathcal{L}, V'/\mathcal{L}'$ are holomorphic equivalente iff there exist a complex linear isomorphism $F : V \to V'$ such that $F(\mathcal{L}) = \mathcal{L}'$.*

Demonstração. If there exists a complex linear isomorphism $F : V \to V'$, such that $F(\mathcal{L}) = \mathcal{L}'$, it is evident that the tori are holomorphic equivalent. Let's assume that exists only a complex analytic homeomorphism $f : V/\mathcal{L} \to V'/\mathcal{L}'$. Consider the covering maps $\pi : V \to V/\mathcal{L}$ and $\pi' : V' \to \mathcal{L}'$. The composition $f \circ \pi : V \to V'/\mathcal{L}'$ yields an analytical local hmeomorphism. Since V and V' are simply connected, the map $f \circ \pi$ factors through an analytical local homeomorphism $F : V \to V'$ as in the diagram below. Therefore, for any $u \in \mathcal{L}$ there will exist an element $u' \in \mathcal{L}'$ such that $F(p+u) = F(p)+u'$, for all $p \in V$. In terms of a coordinate system $z = (z_1, \ldots, z_g)$ for V and $w = (w_1, \ldots, w_g)$ for V', the mapping F will be given by a g-tupla $w_i = F_i(z)$. Differentiating the relation satisfied by F, it follows that $\frac{\partial F_i}{\partial z_j}(p+u) = \frac{\partial F_i}{\partial z_j}(p)$ for all $p \in V$ and $u \in \mathcal{L}$. In this way, the functions $\frac{\partial F_i}{\partial z_j} : V \to \mathbb{C}$ are thus invariants under $\mathcal{L}$, and each one induces a function on the compact complex manifold $V/\mathcal{L}$. So, they are all constants. Consequently, F is linear. Since F is locally a homeomorphism it must be non-singular and clearly $F(\mathcal{L}) = \mathcal{L}'$. □

1.0.2 Marked Surfaces

Let M be a compact Riemann surface of genus $g > 0$, and let $\widetilde{M}$ be its universal covering;

(i) $\widetilde{M}$ inherits from M a complex analytic structure.

(ii) $\widetilde{M} \overset{homeo}{\simeq} D^2 = \{z \in \mathbb{C}; \mid z \mid < 1\}$

(iii) Let $\pi : \widetilde{M} \to M$ be the covering map and $G = \{T : \widetilde{M} \to \widetilde{M} \mid \pi \circ T = \pi\}$ the group of deck transformations. Thus, $M = \widetilde{M}\diagup G$.

By fixing points $p_0 \in M$ and $z_0 \in \widetilde{M}$, it is possible do describe:

1. an isomorphism $G \simeq \pi_1(M, p_0)$.

2. a presentation $\pi_1(M,p_0) =< A_1, B_1, \dots, A_g, B_g : C_1 \dots C_g = 1 >$, where, for all $1 \le i \le g$, A_i, B_i represent homotopy classes of closed paths in M and $C_i = [\alpha_i, \beta_i]$ is the commutator.

3. $H^1(M, \mathbb{Z}) =< A_1, B_1, \dots, A_g, B_g >\simeq \mathbb{Z}^{2g}$.

4. the complement $V = M \setminus \cup_{i=1}^{g} (A_i \cup B_i) \hookrightarrow M$ is simply connected and lifts homeomorphically to a number of opens subsets $\widetilde{V}_g \subset \widetilde{M}$, $g \in G$. Besides, $G(V_g) \cap V_g = \varnothing$, for all $g \in G$, and $D^2 = \overline{\cup_{g\in G} g.\widetilde{V}_g}$.

Definition 1.3. Once the choice of p_0, z_0, A_i, B_i $(1 \le i \le g)$ is made on M, we say that $M(p_0, z_0; A_i, B_i)$ is a marked Rieman surface.

All possible markings arise from a given marking by applying suitable orientation preserving isotopies on M.

1.0.3 Jacobi Variety $\mathcal{J}_M$

As described before, $\mathcal{J}_M$ is the g-torus $H^1(M, \mathcal{O}) / H^1(M, \mathbb{Z}) \simeq \mathbb{C}^g / \mathbb{Z}^{2g}$, where $H^1(M, \mathcal{O}) = H^1(M, \mathbb{C}) / \partial_d H^0(M, \mathcal{O}^{1,0})$. Moreover, $H^1(M, \mathbb{C}) = Hom(H_1(M, \mathbb{C}), \mathbb{C}) = Hom(\pi_1(M, p_0), \mathbb{C})$. Let $\beta = \{A_1, B_1, \dots, A_g, B_g\}$ be a basis for $H_1(M, \mathbb{Z})$ and $\beta^{1,0} = \{w_1, \dots, w_g\}$ a basis for $H^0(M, \mathcal{O}^{1,0})$. From corollary **??**, the basis β projects into a $\mathbb{R}$-basis $\widetilde{\beta} = \{\widetilde{A}_1, \widetilde{B}_1, \dots, \widetilde{A}_g, \widetilde{B}_g\} \subset H^1(M, \mathcal{O})$, which means that $\widetilde{\beta}$ generates a lattice $\mathcal{L}_\beta \subset \mathbb{C}^g$ and $\mathcal{J}_M = \mathbb{C}^g / \mathcal{L}_\beta$.

In order to describe the Jacobi variety $\mathcal{J}_M$ let's study the Abelian differentials

$$H^0(M, \mathcal{O}^{1,0}) = \{w \in \Omega^{1,0}(M, \mathbb{C}) \mid w = f_\alpha dz_\alpha, \overline{\partial} f_\alpha = 0\}, \tag{1.8}$$

where $\mathcal{A}_M = \{(U_\alpha, z_\alpha) \mid \alpha \in \Lambda\}$ is a holomorphic atlas. Note that whenever $w \in H^0(M, \mathcal{O}^{1,0})$

$$dw = \partial w + \overline{\partial} w = \overline{\partial}(f_\alpha dz_\alpha) = \overline{\partial} f_\alpha d\bar{z}_\alpha \wedge dz_\alpha = 0,$$

so $w \in H^1_{DR}(M, \mathbb{C})$. Any $w \in H^0(M, \mathcal{O}^{1,0})$ can be viewd as a $\pi_1(M, p_0)$-invariant holomorphic 1-form in $\Omega^{1,0}(\widetilde{M})$, which is also denoted w. Since $\widetilde{M}$ is simply connected, the closedness condition on w implies it is exact, i.e., there exits $W \in H^0(M, \mathcal{O})$ such that $w = dW$. W is called an Abelian integral for M and it is determined up to a constant. Consider $W(z_0) = 0$, so

$$W(z) = \int_{z_0}^{z} w. \tag{1.9}$$

The $\pi_1(M, p_0)$-invariance of w implies that

$$d[W(g(z)) - W(z)] = w(g(z)) - w(z) = 0, \ \forall T \in \pi_1(M, p_0).$$

Therefore, $W(T(z)) = W(z) - \widetilde{\omega}(T)$, for some $\widetilde{\omega}(T) \in \mathbb{C}$.

Proposition 1.1. $\widetilde{\omega} \in Hom(G, \mathbb{C})$. *In particular* $\widetilde{\omega}(I) = 1$ *and* $\widetilde{\omega}(T^{-1}) = -\widetilde{\omega}(T)$.

Demonstração. Let $S, T \in G$,

$$W(ST(z)) = W(T(z)) - \widetilde{\omega}(S) = W(z) - \widetilde{\omega}(T) - \widetilde{\omega}(S) = W(z) - \widetilde{\omega}(ST).$$

Hence, $\widetilde{\omega}(ST) = \widetilde{\omega}(S) + \widetilde{\omega}(T)$. □

Definition 1.4. $\widetilde{\omega} \in Hom(\pi_1(M, p_0), \mathbb{C})$ is the period class of the abelian 1-form $w \in H^0(M, \mathcal{O}^{1,0})$;

$$\widetilde{\omega}(T) = -\int_{z_0}^{T(z_0)} w; \ g \in G. \tag{1.10}$$

Proposition 1.2. *The homomorphism* $H^0(M, \mathcal{O}^{1,0}) \to Hom(G, \mathbb{C})$, *given by* $w \to \widetilde{\omega}$ *is a monomorphism*

Demonstração. Suppose it is not. Let $w_1, w_2 \in H^0(M, \mathcal{O}^{1,0})$ and suppose that $\widetilde{\omega}_1 = \widetilde{\omega}_2$. Thus, $(W_1 - W_2)(g(z)) = (W_2 - W_1)(z)$ and consequently, $W_2 - W_1$ is a holomorphic function on $\widetilde{M}/\pi_1(M, p_0) = M$, hence is a constant function on M. Since $W_1(z_0) = W_2(z_0) = 0$, then $W_2(z) = W_1(z)$ for all $z \in \widetilde{M}$. □

Fixed a marking on M, a basis can be described for the group $H^1(M, \mathbb{C})$ by considering the linear functionals

$$\begin{aligned} a_i^*(A_j) = \delta_{ij}, \ a_i^*(B_j) = 0, \\ b_i^*(A_j) = 0, \ b_i^*(B_j) = \delta_{ij}. \end{aligned} \tag{1.11}$$

Thus, $\beta^* = \{a_1^*, b_1^*, \ldots, a_g^*, b_g^*\}$ is a basis for $H^1(M, \mathbb{C})$ and given $h \in Hom(H_1(M, \mathbb{Z}), \mathbb{C})$ we have $h = \sum_{i=1}^{g} (h(A_i)a_i^* + h(B_i)b_i^*)$. The explicit isomorphism $H^1(M, \mathbb{C}) \simeq \mathbb{C}^{2g}$ is given by

$$h \to (h(A_1), \ldots, h(A_g), h(B_1), \ldots, h(B_g)).$$

In this way, fixed a marking $M(p_0, z_0; A_i, B_i)$ for M and a basis $\{w_1, \ldots, w_g\}$ for $H^0(M, \mathcal{O}^{1,0})$, the subspace $\partial_d H^0(M, \mathcal{O}^{1,0}) \subset H^1(M, \mathbb{C})$ is spanned by the basis $\{\widetilde{\omega}_1, \ldots, \widetilde{\omega}_g\}$, where

$$\widetilde{\omega}_i = (\widetilde{\omega}_i(A_1), \ldots, \widetilde{\omega}_i(A_g), \widetilde{\omega}_i(B_1), \ldots, \widetilde{\omega}_i(B_g)).$$

Using the basis above, the monomorphism $\partial_d : H^0(M, \mathcal{O}^{1,0}) \to H^1(M, \mathbb{C})$ is represented by a $2g \times g$ matrix $\begin{pmatrix} \Omega' \\ \Omega'' \end{pmatrix}$, where $\Omega' = \{\widetilde{\omega}_i(A_j)\}$ and $\Omega'' = \{\widetilde{\omega}(B_j)\}$ are $g \times g$ matrices.

Theorem 1.2. *The period matrices* Ω', Ω'' *satisfy the following conditions;*

1. *Riemann's Identity*

$$\Omega'.(\Omega'')^t - \Omega''.(\Omega')^t = 0 \tag{1.12}$$

2. *Riemann's Inequality - The matrix*

$$i\Omega'.(\overline{\Omega}'')^t - i\Omega''.(\overline{\Omega}')^t \tag{1.13}$$

defines a positive definite hermitian product.

Corollary 1.2. *The matrices Ω', Ω'' are non-singular.*

Demonstração. □

Corollary 1.3. *There is a canonical basis $\beta = \{w_1, \ldots, w_g\}$ for $H^0(M, \mathcal{O}^{1,0})$ such that the associated period matrix has the form $\begin{pmatrix} I \\ \Omega \end{pmatrix}$, $I = id, \Omega \in GL(g, \mathbb{C})$.*

From theorem 1.2, the matrix Ω defined in corollary 1.3 is symmetric and $Im(\Omega)$ is positive definite.

The motivation to describe the lattice $H^1(M, \mathbb{Z}) \subset H^1(M, \mathcal{O})$ is the Jacobian map $\widetilde{\mathcal{J}} : \widetilde{M} \to \mathbb{C}^g$ defined by

$$\widetilde{\mathcal{J}}(z) = (W_1(z), \ldots, W_g(z)) \tag{1.14}$$

Note that $\widetilde{\mathcal{J}}(T(z)) - \widetilde{\mathcal{J}}(z) = (\widetilde{\omega}_1(T), \ldots, \widetilde{\omega}_g(T))$, for all $T \in \pi_1(M, p_0)$. Thus, consider the vectors $u_T = (\widetilde{\omega}_1(T), \ldots, \widetilde{\omega}_g(T))$. For our purpose, thanks to the fact that $Hom(\pi_1(M, p_0), \mathbb{C}) = H^1(M, \mathbb{C})$, each group element $T \in \pi_1(M, p_0)$ can be represented by a series $T = \sum_{i=1}^{g}[m_i A_i + n_i B_i]$. Therefore,

$$u_T = \sum_{i=1}^{g} (m_i u_{A_i} + n_i u_{B_i}) \, .$$

Definition 1.5. With respect to a marking $M(p_0, z_0; A_i, B_i)$ for M and a basis $\{w_1, \ldots, w_g\}$ for $H^0(M, \mathcal{O}^{1,0})$, the lattice $H^1(M, \mathbb{Z}) \subset H^1(M, \mathcal{O})$, denoted by $\mathcal{L}_\beta$, is

$$\mathcal{L}_\beta = \{\sum_{i=1}^{g} (m_i u_{A_i} + n_i u_{B_i}) \mid m_i, n_i \in \mathbb{Z}\}.$$

The Jacobian Torus of M is then $\mathcal{J}_M = \mathbb{C}^g \diagup \mathcal{L}_\beta$.

Therefore, the map $\widetilde{\mathcal{J}} : \widehat{M} \to \mathbb{C}^g$ induces a map $\mathcal{J} : M \to \mathcal{J}_M(M)$, the truly Jacobian map.

Remark 1. Consider in $\mathbb{C}^g$ the $g \times 2g$ matrix $\Lambda = (u_{A_1} \, \ldots \, u_{A_g} \; u_{B_1} \, \ldots \, u_{B_g})$, which colunms are the generators of $\mathcal{L}_\beta$. A change of basis in $\mathbb{C}^g$ corresponds to a change $\Lambda \to M.\Lambda$, where $M \in GL(g, \mathbb{C})$. The lattice is preserved by a multiplication $\Lambda \to |lambda.N$, $N \in GL(2g, \mathbb{Z})$. Thus, Λ, Λ' represent the same complex torus iff $\Lambda' = M\Lambda N$, where $M \in GL(g, \mathbb{C})$ and $N \in GL(2g, \mathbb{Z})$. Here, it is interesting to use the canonical basis for $H^0(M, \mathcal{O}^{1,0})$. In the canonical basis, $\Lambda = \begin{pmatrix} I \\ \Omega \end{pmatrix}$. To proceed further, suppose that

$\Lambda = (I, \Omega)$ and $\Lambda' = (I, \Omega')$ represent the same complex torus, so that $\Lambda' = M\Lambda N$. Decompose N into $g \times g$ matrix blocks

$$N = \begin{pmatrix} A & B \\ C & D \end{pmatrix};$$

then $(I, \Omega') = (M(A + \Omega C), M(B + \Omega D))$. Hence, (I, ω') and (I, Ω) represent the same complex torus iff $M^{-1} = A + \Omega C$ and $\Omega' = (A + \Omega C)^{-1}(B + \Omega D)$. Besides, $Im(\Omega')$ is non-singular. It is illustrative to consider the case $g = 1$ where $\Lambda = (1, \lambda)$ where $\lambda \in \mathbb{C}$ and $Im(\lambda) \neq 0$. Thus, $(1, \lambda')$ and $(1, \lambda)$ define the same torus iff there exits a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{Z})$ such that

$$\lambda' = \frac{a\lambda + c}{b\lambda + d}.$$

This argument classify the complex analytic structures on the torus $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$.

1.0.4 Abel's Theorem

The Picard variety $\mathcal{P}_M = \{\xi \in H^1(M, \mathcal{O}^*) \mid c_1(\xi) = 0\}$ is isomorphic to the Jacobian variety $\mathcal{J}_M$. The difference is the way both are used in the theory; whenever complex line bundles are used the theory might be referring to $\mathcal{P}_M$ and whenever the g-torus is used then $\mathcal{J}_M$ is being used. In order to have a clear understanding of the isomorphism $\mathfrak{J} : \mathcal{P}_M \to \mathcal{J}_M$ associating to a complex line bundle $\xi \in \mathcal{P}_M$ a point $\mathfrak{J}(\xi) \in \mathcal{J}_M$, there is the classical Abel's theorem. In order to describe $\mathfrak{J}$, consider the point bundles ζ_p, ζ_q associated to the points $p, q \in M$, respectively, and the map

$$\begin{aligned} \Phi : & M \times M \to \mathcal{P}_M \\ & (p, q) \to \zeta_p \zeta_q^{-1}. \end{aligned} \tag{1.15}$$

Let $\tau : [0, 1] \to M$ be any C^∞ path such that $\tau(0) = p,\ \tau(1) = q$. As before, consider $\{w_1, \ldots, w_g\}$ a basis for $H^0(M, \mathcal{O}^{1,0})$, the complex number $v_i = \int_\tau w_i$ independs on τ. Define the vector

$$v_{pq} = (v_1, \ldots, v_g) = (\int_\tau w_1, \ldots, \int_\tau w_g) \in \mathbb{C}^g. \tag{1.16}$$

Theorem 1.3. *(Abel's Theorem) Let $p, q \in M$ and ζ_p, ζ_q be their respectives point line bundles. Thus, the map $\mathfrak{J} : \mathcal{P}_M \to \mathcal{J}_M$ is given by*

$$\mathfrak{J}(\zeta_p \zeta_q^{-1}) = v_{pq}.$$

Moreover, $\mathfrak{J}$ is a holomorph isomorphism, i.e.,

$$\mathfrak{J}(\zeta_{p_1}.\zeta_{q_1}^{-1}.\zeta_{p_2}, \zeta_{q_2}^{-1}) = \mathfrak{J}(\zeta_{p_1}.\zeta_{q_1}^{-1}) + \mathfrak{J}(\zeta_{p_2}.\zeta_{q_2}^{-1}).$$

Corollary 1.4. *For any points $p_1,\dots,p_r,q_1,\dots,q_r \in M$, $\zeta_{p_1}\dots\zeta_{p_r} = \zeta_{q_1}\dots\zeta_{q_r}$ iff for all family of C^∞-arcs $\tau_i : [0,1] \to M$, $\tau_i(0) = p_i, \tau_i(1) = p_i$, the vector*

$$\sum_{i=1}^{r} v_{p_iq_i} = \sum_{i=1}^{r}(\int_{\tau_i} w_1,\dots,\int_{\tau_i} w_g) \in \mathcal{L}_\beta.$$

Demonstração. Since $\zeta_{p_1}\dots\zeta_{p_r}.\zeta_{q_1}^{-1}\dots\zeta_{q_r}^{-1} = (\zeta_{p_1}\zeta_{q_1}^{-1})\dots(\zeta_{p_r}.\zeta_{q_r}^{-1}) = 1$, let's consider the family of C^∞-arcs $\tau_i : [0,1] \to M$, $\tau_i(0) = p_i, \tau_i(1) = q_i$. Then

$$\mathfrak{J}(\zeta_{p_1}\dots\zeta_{p_r}.\zeta_{q_1}^{-1}\dots\zeta_{q_r}^{-1}) = \sum_{i=1}^{r}\mathfrak{J}(\zeta_{p_i}.\zeta_{q_i}^{-1}) = \sum_i v_{p_iq_i}.$$

Therefore, $\mathfrak{J}(\zeta_{p_1}\dots\zeta_{p_r}.\zeta_{q_1}^{-1}\dots\zeta_{q_r}^{-1}) = 1$ iff $\sum_i v_{p_iq_i} \in \mathcal{L}_\beta$. □

1.0.5 Subvariety J_1

As seen in last section, a basis $\{w_1,\dots,w_g\}$ for $H^0(M,\mathcal{O}^{1,0})$ and a base point $z_0 \in M$ define the Jacobian map $\mathcal{J} : M \to \mathcal{J}_M$ by

$$\mathcal{J}_M(z) = (W_1(z),\dots,W_g(z)), \quad W_i(z) = \int_{z_0}^{z} w_i.$$

Proposition 1.3. *Let M be a Riemann surface with genus $g > 0$. Then, $\mathcal{J} : M \to \mathcal{J}_M$ is a complex analytic homeomorphism between M and a complex analytic submanifold $J_1 \subset \mathcal{J}_M$.*

Demonstração. Note that $(d\mathcal{J})_z : T_zM \to T_{\mathcal{J}(z)}\mathcal{J}_M$ is given by $(d\mathcal{J})_z.u = (w_1(z).u,\dots,w_g(z).u)$ and so is non-singular because $\{w_1,\dots,w_g\}$ is a basis and can not have a commom zero. So, $\mathcal{J}$ is locally a holomorphic diffeomorphism. The compactness of M implies that $\mathcal{J}$ is an open map. In order to prove that $\mathcal{J}$ is $1-1$, suppose $\mathcal{J}(p) = \mathcal{J}(q)$ and let $\tau : [0,1] \to M$ be a C^∞-arc connecting p to q. So, the vector $v_{pq} \in \mathcal{L}_\beta$, consequently $\zeta_p = \zeta_q$. Since $g > 0$, it follows that $p = q$. □

The Jacobian map leads to an useful epimorphism from the $\mathbb{Z}$-module of divisors $H^0(M,\mathcal{D}) = \{\sum_p \nu_p p \mid \nu_p = 0\ \forall p$, but finitely many $p's\}$ to the abelian group $\mathcal{J}_M(M)$. Just recall that in $H^0(M,\mathcal{D})$ there is a equivalent relation: $D' \simeq D$ if there exists a meromorphic function $f : M \to \mathbb{C}$ such that $D' - D = D(f)$. The epimorphism

$$\begin{aligned} &H^0(M,\mathcal{D}) \to H^1(M,\mathcal{O}^*) \\ &D = \sum_p \nu_p p \to \xi_D = \prod_p \xi_p^{\nu_p}, \quad c_1(\xi_D) = \mid D \mid = \sum_p \nu_p. \end{aligned}$$

induces a isomorphism $H^0(M,\mathcal{D})/\sim H^1(M,\mathcal{O}^*)$.

Proposition 1.4. *For a Riemann surface M with genus $g > 0$, there exist a epimorphism $\mathcal{J}_* : H^0(M,\mathcal{D}) \to \mathcal{P}_M$, given by*

$$\mathcal{J}_*(D) = \xi_D.\zeta_{z_0}^{-|D|},$$

which kernel is $Ker(\mathcal{J}_*) = \{D \in H^0(M, \mathcal{D}); D \sim\mid D \mid z_0\}$, *where* $z_0 \in M$ *is the base point.*

Demonstração. Since $\mathcal{J}_*(z_0) = 0$, $\mid D- \mid D \mid .z_0 \mid= 0$ implies that $\xi_D.\zeta_{z_0}^{-|D|} \in \mathcal{P}_M$. It is an epimorphism because $H^0(M, \mathcal{D}) \to H^1(M, \mathcal{O}^*)$ is an epimorphism. Also, $\mathcal{J}_*(D- \mid D \mid .z_0) = \mathcal{J}_*(D)$ for all D. So, there exists $p_i^+, p_i^- \in M$ such that $D- \mid D \mid .z_0 = \sum_i(p_i^+ - p_i^-)$. Let $\tau_i : [0,1] \to M$ be C^∞-arcs such that $\tau_i(0) = p_i^-, \tau_i(1) = p_i^+$, so $\mathcal{J}_*(D- \mid D \mid .z_0)$ is represented in $\mathcal{J}_M(M)$ by the vector $v = \sum_i v_{p_i^- p_i^+}$. Then $\mathcal{J}_*(D) = 0$ precisely when $\xi_D = \zeta_{z_0}^{|D|}$, equivalently, when $v \in \mathcal{L}_\beta$, hence $D- \mid D \mid .z_0 \sim 0$. □

Corollary 1.5. $H^1(M, \mathcal{O}^*) \overset{iso}{\simeq} \mathbb{Z} \oplus \mathcal{J}_M$.

Demonstração. It is enough to note that $Ker(\mathcal{J}_*) \simeq \mathbb{Z}$. Then the results follow from the exact sequence

$$0 \longrightarrow Ker(\mathcal{J}_*) \longrightarrow H^0(M, \mathcal{D}) \xrightarrow{\mathcal{J}_*} \mathcal{J}_M \longrightarrow 0.$$

□

Remark 2. Let $D = z - z_0 \in H^0(M, \mathcal{D})$, thus $\mathcal{J}_*(z) = \mathcal{J}_*(z - z_0) = \zeta_p.\zeta_{z_0}^{-1}$. In this way, we can define $J_1 = \{\xi \in \mathcal{P}_M \mid \xi = \zeta_z.\zeta_{z_0}^{-1},\ z \in M\}$. Note that if $\xi = \zeta_z\zeta_{z_0}^{-1}$, then the bundle $\xi.\zeta_{z_0}$ has one holomorphic section (assume $g > 0$) because $\gamma(\zeta_z) = 1$. Thus

$$J_1 = \{\xi \in \mathcal{P}_M(M) \mid \gamma(\xi.\zeta_{z_0}) = 1\}.$$

1.0.6 Subvarieties $J_r \subset \mathcal{P}_M$

1.0.7 Group Structure of $\mathcal{J}_M$ and Consequences