EXERCÍCIOS - TRANSFORMAÇÕES LINEARES - SOLUÇÕES

1- $D: P_3(\mathbb{R}) \to P_1(\mathbb{R})$

$D(p(x)) = p''(x)$

$Ker(D) = ?$

$p''(x) = 0$

$D(a + bx + cx^2 + dx^3) = (b + 2cx + 3dx^2)'$

$= 2c + 6dx \qquad \Rightarrow c = d = 0$

$Ker(D) = \{p(x) = a + bx \ / \ a, b \in \mathbb{R}\}$

Base : $\{1, x\}$

2 (a) $T: \mathbb{R}^5 \to \mathbb{R}^4$ pode ser injetora. FALSO

$5 = \dim(\mathbb{R}^5) = \dim(Ker(T)) + \dim(Im(T))$

$\dim(Ker(T)) = 5 - \underbrace{\dim(Im(T))}_{\leq 4}$

$\therefore \dim(Ker(T)) \geq 1 \Rightarrow T$ não pode ser injetora

(b) $T: \mathbb{R}^3 \to \mathbb{R}^5$, com $\dim(Im(T)) = 3$ é injetora. VERDADEIRA

$3 = \dim(Ker(T)) + 3 \Rightarrow \dim(Ker(T)) = 0 \Rightarrow T$ é injetora!

c) $T: \mathbb{R}^n \to \mathbb{R}^m$, $T(0) = 0$, então T é linear. FALSO

Ex: $T(x_1, x_2) = (x_1 x_2, x_1 x_2, x_1 x_2)$

$T(0) = 0$, mas $T(\lambda(x_1, x_2)) = (\lambda^2 x_1 x_2, \lambda^2 x_1 x_2, \lambda^2 x_1 x_2) =$

$\lambda^2 (x_1 x_2, x_1 x_2, x_1 x_2) \neq \lambda T(x_1, x_2)$

d) Se $T: U \to V$ é injetora, então $\exists\ w \neq 0, w \in U$ tq $T(w) = 0$. FALSO

Como T é injetora $Ker(T) = \{0\}$, logo $\nexists\ w \in U$ tq $T(w) = 0$.

e) Se $T: V \to V$ possui inversa, então $\dim(Ker(T)) = \dim(V)$. FALSO

Se T possui inversa, então T é injetora e sobrejetora

$\to \dim(Ker(T)) = \{0\}$

$\dim(Im(T)) = \dim(V)$

3 a) $T: \mathbb{R}^2 \to P_2(\mathbb{R})$ dada por $T(a,b) = a + ax + ax^2$

$Ker(T)$: $a + ax + ax^2 = 0 \to a = 0$, b qqueer.

$\boxed{Ker(T) = \{(0,b) / b \in \mathbb{R}\}}$, base: $\{(0,1)\}$

$Im(T)$: $a + ax + ax^2 = a(1 + x + x^2)$

$= \{a(1+x+x^2) / a \in \mathbb{R}\}$, base $\{1+x+x^2\}$.

b) $T: M_{2x2}(\mathbb{R}) \to \mathbb{R}$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$.

$Ker(T) = 0$ $\quad a + d = 0 \to a = -d$. $\therefore \begin{pmatrix} -d & b \\ 0 & d \end{pmatrix}$

$Ker(T) = \left\{\begin{pmatrix} -d & b \\ c & d \end{pmatrix} / b, c, d \in \mathbb{R}\right\}$, base $\left\{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right\}$

$Im(T)$: $\mathbb{R}$.

c) $T: M_{2x2}(\mathbb{R}) \to P_2(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = (a + b + c) + dx^2$

$Ker(T)$: $(a+b+c) + dx^2 = 0 \to a + b + c = 0$, $\boxed{d = 0}$

$\boxed{a = -b - c}$

$\begin{pmatrix} -b-c & b \\ c & 0 \end{pmatrix} = \begin{pmatrix} -b & b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -c & 0 \\ c & 0 \end{pmatrix}$

$Ker(T) = \left\{\begin{pmatrix} -b-c & b \\ c & \end{pmatrix} / b, c \in \mathbb{R}\right\}$, base $\left\{\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\right\}$.

$Im(T) : (a+b+c)+dx^2 \qquad Im(T) = \{ k + k' x^2 \,/\, k, k' \in \mathbb{R} \}$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 1$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = x^2$

Base : $\{ 1, x^2 \}$.

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$T : \mathbb{R}^3 \to \mathbb{R}^3$

$T(x,y,z) = (z,x,y)$

$(T \circ T)(x,y,z) = T(z,x,y) = (y,z,x)$

$(T^2 \circ T)(x,y,z) = (T \circ T \circ T)(x,y,z) = T(y,z,x) = (x,y,z)$

$\therefore \; T \circ T \circ T = I \quad , \quad T^2 \circ T = I$

$\to \; T^{-1} = T^2$

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$T : M_{2\times2}(\mathbb{R}) \to M_{2\times2}(\mathbb{R}) \quad , \quad T\begin{pmatrix} a+2c & b+2d \\ 3c-a & 3d-b \end{pmatrix}$

$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} \to T^{-1}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$

$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} \to T^{-1}\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} \to T^{-1}\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$

$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \to T^{-1}\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \beta\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \gamma\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} + \xi\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}$

$$\begin{cases} \alpha + 2\gamma = a \\ \beta + 2\xi = b \\ -\alpha + 3\gamma = c \\ -\beta + 3\xi = d \end{cases} \qquad \boxed{\frac{a+c}{5} = \gamma} \qquad \alpha = \frac{3a-2c}{5}$$

$$\boxed{\frac{b+d}{5} = \xi} \qquad \beta = \frac{3b-2d}{5}$$

$$\therefore T^{-1}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = T^{-1}\left(\frac{3a-2c}{5}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \frac{3b-2d}{5}\begin{pmatrix} 0 & +1 \\ 0 & -1 \end{pmatrix} + \frac{a+c}{5}\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} + \frac{b+d}{5}\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix} \right)$$

$$= \left(\frac{3a-2c}{5}\right) T^{-1}\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} + \left(\frac{3b-2d}{5}\right)\begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} + \left(\frac{a+c}{5}\right) T^{-1}\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} + \left(\frac{b+d}{5}\right) T^{-1}\begin{pmatrix} 0 & 2 \\ 0 & 3 \end{pmatrix}$$

$$= \frac{3a-2c}{5}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \left(\frac{3b-2d}{5}\right)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{(a+c)}{5}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{(b+d)}{5}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \frac{1}{5}\begin{pmatrix} 3a-2c & 3b-2d \\ a+c & b+d \end{pmatrix} //$$

6 $T: \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$, dada por $T(p(x)) = (p(0), p(1), p(-1))$.

T é inversível?

$Ker(T) = \{p \in \mathcal{P}_2(\mathbb{R}) / T(p) = (0,0,0)\} \to (p(0), p(1), p(-1)) = (0,0,0)$

$p(0) = 0$
$p(1) = 0$
$p(-1) = 0$

p é um pol. de grau 2 com 3 raízes distintas!

Logo, $p(x) = 0 \ \forall x$

(um pol. de grau 2 possui no máx 2 raízes!)

$\to Ker(T) = \{0\} \to$ T é injetora!

$\therefore 3 = \dim(Ker(T)) + \dim(Im(T)) \to \dim(Im(T)) = 3 = \dim(\mathbb{R}^3)$

Logo, T é sobrejetora e (como tbém é injetora) portanto inversível!

Seja $\mathcal{B} = \{1, x, x^2\}$ uma base de $\mathcal{P}_2(\mathbb{R})$

$T(1) = (1,1,1) \to T^{-1}(1,1,1) = 1$

$T(x) = (0,1,-1) \qquad T^{-1}(0,1,-1) = x$

$T(x^2) = (0,1,1) \qquad T^{-1}(0,1,1) = x^2$

Seja $(a,b,c) \in \mathbb{R}^3$, temos que

$$(a,b,c) = a(1,1,1) + \frac{b-c}{2}(0,1,-1) + \frac{b+c-2a}{2}(0,1,1)$$

$$T^{-1}(a,b,c) = a\,T^{-1}(1,1,1) + \frac{b-c}{2}T^{-1}(0,1,-1) + \frac{b+c-2a}{2}T^{-1}(0,1,1)$$

$$= a \cdot 1 + \left(\frac{b-c}{2}\right)x + \left(\frac{b+c-2a}{2}\right)x^2 //$$

7) $T: \mathbb{R}^3 \to \mathbb{R}^3$

$Im(T) = span\{(2,1,1),(1,-1,2)\}$

Sol:

Como $Im(T)$ é gerada por 2 vetores l.i, temos que $\dim(Im(T)) = 2 \Rightarrow \dim(ker(T)) = 1$.

seja $\mathcal{B} = \{e_1, e_2, e_3\}$ a base canônica de $\mathbb{R}^3$

suponha que

$$\left.\begin{aligned} T(e_1) &= (2,1,1) \\ T(e_2) &= (1,-1,2) \\ T(e_3) &= (0,0,0) \end{aligned}\right\} \text{Desta forma, } \dim(ker(T)) = 1 \text{ e } \dim(Im(T)) = 2.$$

$$\begin{aligned} T(x,y,z) &= T(xe_1 + ye_2 + ze_3) \\ &= xT(e_1) + yT(e_2) + zT(e_3) \\ &= x(2,1,1) + y(1,-1,2) + z(0,0,0) \\ &= (2x+y, x-y, x+2y) \end{aligned}$$

Outras soluções são por ex:

$$\left\{\begin{aligned} T(e_1) &= (0,0,0) \\ T(e_2) &= (2,1,1) \\ T(e_3) &= (1,-1,2) \end{aligned}\right. \quad \text{ou} \quad \begin{aligned} T(e_1) &= (1,-1,2) \\ T(e_2) &= (0,0,0) \\ T(e_3) &= (2,1,1) \end{aligned} \quad \text{e etc.}$$

8) $T: \mathbb{R}^4 \to \mathbb{R}^4$

$ker(T) = span\{(1,1,0,0),(0,0,1,1)\}$.

sol: $\dim(ker(T)) = 2 \to \dim(Im(T)) = 4 - 2 = 2$.

Temos que $T(1,1,0,0) = (0,0,0,0)$ e

$T(0,0,1,1) = (0,0,0,0)$

Precisamos encontrar dois vetores em $\mathbb{R}^4$ que sejam l.i com $(1,1,0,0)$ e $(0,0,1,1)$.