Exercises

1. Show that $\mathbb{Q}$ is countably infinite.
2. Show that the maps f and g of Examples 1 and 2 are bijections.
3. Let X be the two-element set $\{0, 1\}$. Show there is a bijective correspondence between the set $\mathcal{P}(\mathbb{Z}_+)$ and the cartesian product X^ω.
4. (a) A real number x is said to be ***algebraic*** (over the rationals) if it satisfies some polynomial equation of positive degree

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$$

with rational coefficients a_i. Assuming that each polynomial equation has only finitely many roots, show that the set of algebraic numbers is countable.

(b) A real number is said to be ***transcendental*** if it is not algebraic. Assuming the reals are uncountable, show that the transcendental numbers are uncountable. (It is a somewhat surprising fact that only two transcendental numbers are familiar to us: e and π. Even proving these two numbers transcendental is highly nontrivial.)

5. Determine, for each of the following sets, whether or not it is countable. Justify your answers.
   (a) The set A of all functions $f : \{0, 1\} \to \mathbb{Z}_+$.
   (b) The set B_n of all functions $f : \{1, \dots, n\} \to \mathbb{Z}_+$.
   (c) The set $C = \bigcup_{n\in\mathbb{Z}_+} B_n$.
   (d) The set D of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$.
   (e) The set E of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$.
   (f) The set F of all functions $f : \mathbb{Z}_+ \to \{0, 1\}$ that are "eventually zero." [We say that f is ***eventually zero*** if there is a positive integer N such that $f(n) = 0$ for all $n \geq N$.]
   (g) The set G of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually 1.
   (h) The set H of all functions $f : \mathbb{Z}_+ \to \mathbb{Z}_+$ that are eventually constant.
   (i) The set I of all two-element subsets of $\mathbb{Z}_+$.
   (j) The set J of all finite subsets of $\mathbb{Z}_+$.
6. We say that two sets A and B ***have the same cardinality*** if there is a bijection of A with B.

(a) Show that if $B \subset A$ and if there is an injection

$$f : A \longrightarrow B,$$

then A and B have the same cardinality. [*Hint:* Define $A_1 = A$, $B_1 = B$, and for $n > 1$, $A_n = f(A_{n-1})$ and $B_n = f(B_{n-1})$. (Recursive definition again!) Note that $A_1 \supset B_1 \supset A_2 \supset B_2 \supset A_3 \supset \cdots$. Define a bijection $h : A \to B$ by the rule

$$h(x) = \begin{cases} f(x) & \text{if } x \in A_n - B_n \text{ for some } n, \\ x & \text{otherwise.]} \end{cases}$$