# An introduction to Hopf algebras:
## A categorical approach

**Marcelo Muniz S. Alves** [1]

**Eliezer Batista**[2]

**XXIII Brazilian Algebra Meeting**

Maringá, July 27th to August 1st, 2014

[1]Universidade Federal do Paraná, mmunizbr@gmail.com
[2]Universidade Federal de Santa Catarina, eliezer1968@gmail.com

# Contents

# Chapter 1

# Monoidal Categories

## 1.1 Monoidal Categories

We begin by laying the basis for the study of bialgebras and Hopf algebras from a categorical viewpoint. The simplest model of a monoidal category to have in mind is the one of vector spaces with the tensor product; a richer one is the category of representations of a finite group with the tensor product of representations, whose structure already points towards the category of modules over a bialgebra. The main references for this chapter are [8], [9] and [11].

Let $\mathcal{C}$ be a category. Recall that the category $\mathcal{C} \times \mathcal{C}$ has as objects the pairs $(X, Y)$ of objects of $\mathcal{C}$, and that a morphism $(X, Y) \to (X', Y')$ is just a pair of morphisms $(f, g)$, with $f : X \to X'$ and $g : Y \to Y'$ in $\mathcal{C}$. Composition is done coordinatewise, and clearly the identity of $(X, Y)$ in $\mathcal{C} \times \mathcal{C}$ is the morphism $(\mathrm{id}_X, \mathrm{id}_Y)$. This is extended in the canonical manner for the cartesian product of any finite number of categories (see Appendix A for other notations and results on categories).

Let $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ be a functor. A basic ingredient for a monoidal category is the concept of associativity for $\otimes$, which cannot be simply the equality $(X \otimes Y) \otimes Z = (X \otimes Y) \otimes Z$, since this is already false in the case of the tensor product of vector spaces. Turning back to this case, what we do have for the tensor product of vector spaces is an isomorphism $(U \otimes V) \otimes W \simeq (U \otimes V) \otimes W$ which "behaves well" with respect to linear maps: the isomorphism is natural in $U, V, W$.

The general setting is the following: the functor $\otimes$ produces two functors from $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ to $\mathcal{C}$,

$$(\_ \otimes \_) \otimes \_ : \begin{array}{ccc} \mathcal{C} \times \mathcal{C} \times \mathcal{C} & \to & \mathcal{C} \\ (X, Y, Z) & \mapsto & (X \otimes Y) \otimes Z \end{array}, \qquad \_ \otimes (\_ \otimes \_) : \begin{array}{ccc} \mathcal{C} \times \mathcal{C} \times \mathcal{C} & \to & \mathcal{C} \\ (X, Y, Z) & \mapsto & X \otimes (Y \otimes Z) \end{array}$$

and the associativity for $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is defined in terms of a natural isomorphism

$$a : (\_ \otimes \_) \otimes \_ \to \_ \otimes (\_ \otimes \_)$$

Recall that this means that $a$ is in fact a collection of isomorphisms

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

indexed by $(X, Y, Z) \in \mathcal{C} \times \mathcal{C} \times \mathcal{C}$, which is natural on $(X, Y, Z)$: for every morphism $(f, g, h) : (X, Y, Z) \to (X', Y', Z')$, the following diagram commutes.

$$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "a_{X,Y,Z}"] \arrow[d, "(f \otimes g) \otimes h"'] & X \otimes (Y \otimes Z) \arrow[d, "f \otimes (g \otimes h)"'] \\ (X' \otimes Y') \otimes Z' \arrow[r, "a_{X',Y',Z'}"] & X' \otimes (Y' \otimes Z') \end{tikzcd}$$

An **associativity constraint** for the pair $(\mathcal{C}, \otimes)$ is a natural isomorphism of functors

$$a : (\_ \otimes \_) \otimes \_ \to \_ \otimes (\_ \otimes \_)$$

satisfying the **pentagon axiom**

$$\begin{tikzcd}[column sep=small]
 & (W\otimes X)\otimes(Y\otimes Z) \arrow[dr, "a_{W,X,Y\otimes Z}"] & \\
((W\otimes X)\otimes Y)\otimes Z \arrow[ur, "a_{W\otimes X,Y,Z}"] \arrow[d, "a_{W,X,Y}\otimes Z"] & & W\otimes(X\otimes(Y\otimes Z)) \\
(W\otimes(X\otimes Y))\otimes Z \arrow[rr, "a_{W,X\otimes Y,Z}"'] & & W\otimes((X\otimes Y)\otimes Z) \arrow[u, "W\otimes a_{X,Y,Z}"']
\end{tikzcd} \tag{1.1}$$

(we will generally denote the identity of an object and the object by the same symbol, with exception to one or other ocasion where this convention would lead to confusion)

The triple $(\mathcal{C},\otimes,a)$ with $a$ satisfying (1.1) is called a **semigroup category**. The associativity constraint is also called an **associator** for $\otimes$

A **unit** for the semigroup category $\mathcal{C}=(\mathcal{C},\otimes,a)$ is an triple $(I,l,r)$, consisting of an object $I$ of $\mathcal{C}$ and two natural isomorphisms

$$l : I\otimes \_ \to \mathrm{Id}_{\mathcal{C}}, \quad r : \_ \otimes I \to \mathrm{Id}_{\mathcal{C}}$$

called the left and right unit constraints, such that

$$\begin{tikzcd}
(X\otimes I)\otimes Y \arrow[rr, "a"] \arrow[dr, "r_X\otimes Y"'] & & X\otimes(I\otimes Y) \arrow[dl, "X\otimes l_Y"] \\
 & X\otimes Y &
\end{tikzcd} \tag{1.2}$$

for all objects $X$ and $Y$. For simplicity, we will use the same notation for the unit object $I$ and the unit $(I,l,r)$.

**Definition 1.1.** A **monoidal category** is a quadruple $(\mathcal{C},\otimes,a,I)$ that satisfies (1.1) and (1.2). The monoidal category is **strict** if $(X\otimes Y)\otimes Z = X\otimes(Y\otimes Z)$ and $a_{X,Y,Z} : (X\otimes Y)\otimes Z \to X\otimes(Y\otimes Z)$ is the identity for every $X, Y, Z$.

In most of the examples treated here the associativity constraint will be the canonical one attached to the functor $\otimes$, and therefore we will refer to the monoidal category $\mathcal{C}=(\mathcal{C},a,I)$. Nevertheless, the same functor $\otimes$ may admit more than one associator, as we will see in Example 1.7.

Before proceeding, we should say some words about the motivation behind the pentagon axiom. Given objects $X_1,\dots,X_n$ of $\mathcal{C}$, at first the expression

$$X_1\otimes X_2\cdots\otimes X_n \tag{1.3}$$

does not make sense if $n\geq 3$. However, just as it is with semigroups, a generalized associativity for tensor products of objects holds : any two “bracketings” of this expression produce the same object up to isomorphism. The pentagon axiom (1.1) and the unit axiom (1.2) are there to ensure that such an isomorphism is unique if it is built up from concatenating and tensoring $a$, $l$, $r$, and the identities.

Examples of monoidal categories abound in nature, and, in fact, the same category might have more than one structure of monoidal category.

**Example 1.2.** The first main example is the category ${}_R\mathcal{M}$ of (left) modules over a commutative ring $R$ with the usual tensor product $\otimes_R$ over $R$. The associativity constraint is the canonical one, given by

$$\begin{array}{rccc} a_{L,M,N} : & (L\otimes M)\otimes N & \to & L\otimes(M\otimes N) \\ & (x\otimes y)\otimes z & \mapsto & x\otimes(y\otimes z) \end{array}$$

A unit object for this category is the left module $R$, and $l, r$ are as expected:

$$\begin{array}{rccc} l_M: & M\otimes_R R & \to & M \\ & m\otimes r & \mapsto & rm \end{array} \qquad \begin{array}{rccc} r_M: & R\otimes_R M & \to & M \\ & r\otimes m & \mapsto & rm \end{array}$$

In this example we can also see that the pentagon axiom is not automatically satisfied by any natural transformation. Indeed, the collection of maps

$$\begin{array}{rccc} \beta_{L,M,N}: & (L\otimes M)\otimes N & \to & L\otimes(M\otimes N) \\ & (x\otimes y)\otimes z & \mapsto & -x\otimes(y\otimes z) \end{array}$$

defines a natural isomorphism but the pentagon axiom does not hold if $R$ has characteristic different from 2.

**Example 1.3.** The same category ${}_R\mathcal{M}$ has another monoidal structure, this time with respect to the bifunctor $\times : {}_R\mathcal{M} \times {}_R\mathcal{M}$ that sends the pair of $R$-modules $(M,N)$ to their direct product $M\times N$.

In this case the associativity constraint is the canonical isomorphism

$$\begin{array}{rccc} a_{L,M,N}: & (L\times M)\times N & \to & L\times(M\times N) \\ & ((x,y),z) & \mapsto & (x,(y,z)) \end{array}$$

and the unit is the trivial module 0 with the maps

$$\begin{array}{rccc} l_M: & M\times 0 & \to & M \\ & (m,0) & \mapsto & m \end{array} \qquad \begin{array}{rccc} r_M: & 0\times M & \to & M \\ & (0,m) & \mapsto & m \end{array}$$

**Example 1.4.** The second monoidal structure on ${}_R\mathcal{M}$ is in fact a consequence of the categorical definition of product.

If $\mathcal{C}$ is a category which has finite products, let $\times$ denote the functor $\times : \mathcal{C}\times\mathcal{C} \to \mathcal{C}$ that takes $(X,Y)$ to the product of $X$ and $Y$, and let $I$ be the terminal object which is the product of the empty family. Then $(\mathcal{C},\times,I)$ is a monoidal category.

Similarly, if $\mathcal{C}$ is a category which has finite coproducts, then if $\coprod$ denotes the coproduct and $I$ is the initial object which is the coproduct of the empty family, then $(\mathcal{C},\coprod,I)$ is a monoidal category.

**Example 1.5.** Let $G$ be a group. A **$k$-linear representation** of $G$ is a $k$-vector space $V$ with a group homomorphism $\rho$ from $G$ into the group $GL(V)$ of invertible operators in $V$. One also says that "$V$ is a representation of $G$". If $V=(V,\rho_V)$ and $V'=(V',\rho_{V'})$ are representations of $G$, a morphism of representations is a $k$-linear map $f: V\to V'$ such that $f(\rho_V(r)v)=\rho_{V'}(r)f(v)$ for each $v$ in $V$ and $r$ in $G$.

The $k$-linear representations coincide with the left modules over the group algebra $kG$, which is the vector space with basis $G$ endowed with the multiplication induced by the product in $G$, i.e.,

$$(\sum a_g g)(\sum_g b_g g) = \sum_{g,h} a_g b_h gh.$$

This is a $k$-algebra with unit, the unit being (identified with) the neutral element of $G$.

If one has a representation $\rho: G\to GL(V)$, one gets a left $kG$-module by the formula $(\sum_g a_g g)\cdot v = \sum_g a_g\rho(g)(v)$. Conversely, if $V$ is a left $kG$-module then the restriction of the module structure to $G$ defines a $k$-linear representation of $G$. Is is also clear that morphisms of representations correspond to $kG$-linear maps and conversely. Therefore, we have an isomorphism between the categories $\operatorname{Rep} G$ of $k$-linear representations of $G$ and ${}_{kG}\mathcal{M}$ of left $kG$-modules.

Let $G$ be a group, $k$ a field, and let $\operatorname{Rep} G$ be its category of $k$-linear representations. If $V$ and $W$ are representations, then $V\otimes W$ is also a representation by

$$g\cdot(v\otimes w) = (gv)\otimes(gw).$$

The tensor product of morphisms of representations is also a morphism of representations.

If $U$ is another representation, then the canonical isomorphism

$$\begin{array}{rccc} a_{U,V,W}: & U\otimes(V\otimes W) & \to & (U\otimes V)\otimes W \\ & u\otimes(v\otimes w) & \mapsto & (u\otimes v)\otimes w \end{array}$$

is also an isomorphism of representations, is natural on $U, V, W$ and satisfies the pentagon axiom (since it does so in $\underline{\mathrm{Vect}_k}$).

The unit object of this category is the trivial representation $k$ where $r \cdot 1 = 1$ for every $r \in G$. The unit constraints $l_V : V \otimes k \to V$ and $r_V : k \otimes V \to V$ are the same as those for vector spaces.

**Example 1.6.** Consider now a noncommutative $k$-algebra $A$. In this case we may consider the category ${}_A\mathcal{M}_A$ of $A$-bimodules, with the tensor product over $A$. The natural transformation $a$ is defined as the corresponding one in ${}_R\mathcal{M}$, and the unit is also the object $A$ with the maps

$$\begin{array}{rccc} l_M: & M\otimes_A A & \to & M \\ & m\otimes a & \mapsto & ma \end{array} \qquad \begin{array}{rccc} r_M: & R\otimes_A A & \to & M \\ & a\otimes m & \mapsto & am \end{array}$$

This category has one important difference with respect to the previous examples: all those examples have are examples of **symmetric** monoidal categories, i.e., they are monoidal categories where there is a natural isomorphism

$$\sigma_{X,Y} : X \otimes Y \to Y \otimes X$$

such that $\sigma_{X,Y}^{-1} = \sigma_{Y,X}$ for every pair of objects $X, Y$ the category (plus some other conditions that will be explained in Chapter 2). In $({}_R\mathcal{M}, \otimes_R, a, R)$ this is the flip morphism $x\otimes y \mapsto y\otimes x$; in $({}_R\mathcal{M}, \times, a, 0)$ it is the flip map $(x, y) \mapsto (y, x)$. On the other hand, there is no such symmetry in ${}_A\mathcal{M}_A$ in general.

In the previous examples the associators are fairly trivial and the categories are usually considered strict. The next example shows that there are nontrivial associators.

**Example 1.7.** Let $G$ be a finite group, $k$ a field of characteristic zero. We will consider the category of $G$-graded vector spaces. This category has a canonical tensor product which is compatible with several different associativity constraints. In order to explain their nature, we will begin by making a digression on group cohomology.

Let $n$ be a nonnegative integer. An $n$-cochain on $G$ with coefficients in the group $k^*$ is a function from $G^n$ to $k^*$ (a 0-cochain is an element of $k^*$). The set $C^n(G, k^*)$ of $n$-cochains is an abelian group with respect to pointwise multiplication. These groups form a complex of abelian groups with respect to the morphisms

$$d^n : C^n(G, k^*) \to C^{n+1}(G, k^*)$$

defined by

$$d^n(f)(r_1, \dots, r_{n+1}) = f(r_2, \dots, r_{n+1}) \times \prod_{k=1}^{n} f(r_1, \dots, r_k r_{k+1}, \dots, r_{n+1})^{(-1)^k} \times f(r_1, \dots, r_n)^{(-1)^{n+1}}.$$

For instance,

$$\begin{aligned} d^2(f)(r_1, r_2, r_3) &= f(r_2, r_3) f(r_1, r_2 r_3) f(r_1 r_2, r_3)^{-1} f(r_1, r_2)^{-1}, \\ d^3(f)(r_1, r_2, r_3, r_4) &= f(r_2, r_3, r_4) f(r_1, r_2 r_3, r_4) f(r_1, r_2, r_3) f(r_1 r_2, r_3, r_4)^{-1} f(r_1, r_2, r_3 r_4)^{-1}. \end{aligned}$$

An $n$-**cocycle** with coefficients in the group $k^*$ is a function $f : G^n \to k^*$ such that $d^n(f) = 1$; an $n$-**coboundary** is an element of the image of $d^{n-1}$. The groups of $n$-cocycles and $n$-coboundaries are usually denoted by $Z^n(G, k^*)$ and $B^n(G, k^*)$ respectively. The $n$-th **cohomology group** of $G$ with coefficients in $k^*$ is the quotient group $H^n(G, k^*) = \dfrac{Z^n(G, k^*)}{B^n(G, k^*)}$.

2-cocycles and the second cohomology group $H^2(G, k^*)$ appear naturally in the study of projective representations of $G$ over $k$ and also in the classification of central extensions of $G$ by $k^*$ (see for instance [7, 12]). We will see now how 3-cocycles appear in the study of monoidal structures in the category of $G$-graded vector spaces.

Consider the category $\underline{\mathrm{Vect}}_k^G$ of $G$-graded $k$-vector spaces, i.e., vector spaces equipped with a direct sum decomposition where the components are indexed by $G$. If $V$ and $W$ are two $G$-graded vector spaces then the vector space $V \otimes W$ has a canonical $G$-grading given by

$$(V \otimes W)_r = \bigoplus_{\substack{s,t \in G \\ st=r}} V_s \otimes W_t$$

$\underline{\mathrm{Vect}}_k^G$ is a monoidal category with this tensor product, with the same associator as the one in $\underline{\mathrm{Vect}}_k$, and with the field $k$ graded by $k_e = k$, $k_r = 0$ for every $r \neq e$, where $e$ is the neutral element of $G$ (and same left and right unit constraints as in $\underline{\mathrm{Vect}}_k$). But $\underline{\mathrm{Vect}}_k^G$ may have several other noncanonical associators.

Given $r \in G$, let $k(r)$ be the $G$-grading of $k$ where the only nonzero component is the one of degree $r$. Note that

$$k(r) \otimes k(s) = k(rs).$$

Every object in $\underline{\mathrm{Vect}}_k^G$ is isomorphic to a direct sum of one-dimensional objects of the form $k(r)$. Since the tensor product commutes with direct sums, it is enough to study the possible associators restricted to the full subcategory $\mathcal{C}$ whose objects are the $k(r)$. Let $a$ be an associator for $(\underline{\mathrm{Vect}}_k^G, \otimes, k(e))$. Restricting $a$ to $\mathcal{C}$ and identifying $k(r)$ with $r$, $a$ can be identified with a map

$$a : G \times G \times G \to k^*$$

To require that $a$ satisfies the pentagon axiom is to ask that

$$a(r,s,t)a(r,st,v)a(s,t,v)a(r,s,tv)^{-1}a(rs,t,v)^{-1} = 1$$

for every $(r,s,t,v) \in G^4$, which means that $a$ is a 3-cocycle on $G$ with coefficients in $k^*$. Therefore every nontrivial 3-cocycle (i.e., $a \neq 1$) provides a non-canonical, non-strict monoidal structure on $\mathcal{C}$ (and on $\underline{\mathrm{Vect}}_k^G$).

## 1.2 The unit of a monoidal category

### 1.2.1 Some properties of the unit

**Proposition 1.8.** *Let $(\mathcal{C}, \otimes, a, I)$ be a monoidal category. The following diagrams commute:*

$$\begin{tikzcd}
(X \otimes Y) \otimes I \arrow[rr, "a"] \arrow[dr, "r"'] & & X \otimes (Y \otimes I) \arrow[dl, "X \otimes r"] \\
& X \otimes Y &
\end{tikzcd}
\qquad
\begin{tikzcd}
(I \otimes X) \otimes Y \arrow[rr, "a"] \arrow[dr, "l \otimes Y"'] & & I \otimes (X \otimes Y) \arrow[dl, "l"] \\
& X \otimes Y &
\end{tikzcd}$$

*Proof.* We will prove that the first diagram commutes.

Since the unit appears between two objects in (1.2), it is reasonable to try to prove first that this diagram tensored by $I$ on the right is commutative or, more generally, to prove that the diagram below is commutative for every object $Z$.

$$\begin{tikzcd}
((X \otimes Y) \otimes I) \otimes Z \arrow[rr, "a \otimes Z"] \arrow[dr, "r \otimes Z"'] & & (X \otimes (Y \otimes I)) \otimes Z \arrow[dl, "(X \otimes r) \otimes Z"] \\
& (X \otimes Y) \otimes Z &
\end{tikzcd} \tag{1.4}$$

The point is that if this diagram commutes for each object $Z$ then, using the naturality of $r$, we can also prove that the first diagram is commutative.

This can be done, for instance, via the next diagram where (1.4) appears as a triangle in the left corner (it is better to think of this diagram as a prism of triangular bases, one of them being (1.4) and the other one being the first diagram of the statement of this proposition).

$$\begin{tikzcd}
& ((X\otimes Y)\otimes I)\otimes I \arrow[r, "r"] \arrow[d, "a\otimes I"] \arrow[ddl, "r\otimes I"'] & (X\otimes Y)\otimes I \arrow[d, "a"] \arrow[ddr, "r"] & \\
& (X\otimes (Y\otimes I))\otimes I \arrow[r, "r"] \arrow[dl, "(X\otimes r)\otimes I"] & X\otimes (Y\otimes I) \arrow[dr, "X\otimes r"'] & \\
(X\otimes Y)\otimes I \arrow[rrr, "r"'] & & & X\otimes Y
\end{tikzcd}$$

In this diagram, the central square commutes because of the naturality of $r$ applied to the morphism $a_{X,Y,I} : (X \otimes Y) \otimes I \to X \otimes (Y \otimes I)$. The trapezium below also commutes, now using the natural transformation $r$ and the morphism $X \otimes r_Y : X \otimes (Y \otimes I) \to X \otimes Y$. And the outer trapezium commutes also due to the naturality of $r$, this time with respect to the morphism $r_{X\otimes Y}$. Since all arrows are isomorphisms, we can begin with $(X \otimes r_Y)a_{X,Y,I}$ and, moving it around the diagram we get

$$\begin{aligned}
(X \otimes r_Y)a_{X,Y,I} &= (X \otimes r_Y)\; r_{X\otimes(Y\otimes I)}\; (a_{X,Y,I} \otimes I)\; r^{-1}_{(X\otimes Y)\otimes I} \\
&= r_{X\otimes Y}\; ((X \otimes r_Y) \otimes I)(a_{X,Y,I} \otimes I)\; r^{-1}_{(X\otimes Y)\otimes I} \\
&= r_{X\otimes Y}\; (r_{(X\otimes Y)\otimes I} \otimes I)\; r^{-1}_{(X\otimes Y)\otimes I} \\
&= r_{X\otimes Y}.
\end{aligned}$$

Now to prove (1.4). The idea is to "expand" it to an instance of the pentagon axiom: the triangle (1.4) fits in the left upper corner of the next diagram.

$$\begin{tikzcd}
& ((X\otimes Y)\otimes I)\otimes Z \arrow[dl, "a_{X,Y,I}\otimes Z"'] \arrow[d, "l\otimes \mathrm{id}"'] \arrow[dr, "a_{X\otimes Y,I,Z}"] & \\
(X\otimes (Y\otimes I))\otimes Z \arrow[r, "(X\otimes l)\otimes Z"'] \arrow[dd, "a"] & (X\otimes Y)\otimes Z \arrow[d, "a"'] & (X\otimes Y)\otimes (I\otimes Z) \arrow[l, "(X\otimes Y)\otimes r"] \arrow[dd, "a"'] \\
& X\otimes (Y\otimes Z) & \\
X\otimes ((Y\otimes I)\otimes Z) \arrow[rr, "X\otimes a"'] \arrow[ur, "X\otimes (l\otimes Z)"] & & X\otimes (Y\otimes (I\otimes Z)) \arrow[ul, "X\otimes (Y\otimes r)"']
\end{tikzcd}$$

The diagram on the outside is the pentagon axiom applied to $(X, Y, I, Z)$. The bottom triangle comes from (1.2) tensored by $X$ on the left; the left and right quadrilaterals above it are commutative because if the naturality of $a$; and the right top triangle is the unit axiom (1.2) (note that $\mathrm{id}_{X\otimes Y} = \mathrm{id}_X \otimes \mathrm{id}_Y$, because $\otimes$ sends identity to identity, and the identity of $(X,Y) \in \mathcal{C} \times \mathcal{C}$ is $(\mathrm{id}_X, \mathrm{id}_Y)$ ). Once again, since all arrows are isomorphisms, we conclude that the top left triangle must also be commutative.

$\square$

**Proposition 1.9.** *In a monoidal category $\mathcal{C}$ one has*

$$l_{I\otimes X} = I \otimes l_X, \quad r_{X\otimes I} = r_X \otimes \mathit{unit}, \quad \text{and } l_1 = r_1.$$

*Proof.* By the naturality of $l$, applied to the morphism $l_X : I \otimes X \to X$, the diagram below commutes:

$$\begin{tikzcd}
I \otimes (I \otimes X) \arrow[r, "I \otimes l_X"] \arrow[d, "l_{I \otimes X}"] & I \otimes X \arrow[d, "l_X"] \\
I \otimes X \arrow[r, "l_X"] & X
\end{tikzcd}$$

i.e., $l_X l_{I \otimes X} = l_X(I \otimes l_X)$. But $l_X$ is an isomorphism, and therefore $l_{I \otimes X} = I \otimes l_X$. The proof of the second equation is analogous.

For the third equation, note first that Proposition 1.8 implies the commutativity of the diagram

$$\begin{tikzcd}
(I \otimes I) \otimes I \arrow[r, "a"] \arrow[d, "l_{I \otimes I}"] & I \otimes (I \otimes I) \arrow[dl, "l_{I \otimes I}"] \\
I \otimes I &
\end{tikzcd}$$

and using this diagram and the first equality of this proposition,

$$l_I \otimes I = l_{I \otimes I} a = (I \otimes l_I) a$$

From (1.2) it follows that

$$(I \otimes l_I) a = r_I \otimes I$$

and therefore $l_I \otimes I = r_I \otimes I$.

Since $r : \_ \otimes I \to \mathrm{Id}_{\mathcal{C}}$ is a natural isomorphism, we can conclude that $l_I = r_I$. In detail, given any morphism $f : X \to X'$, the diagram

$$\begin{tikzcd}
X \otimes I \arrow[r, "r_I"] \arrow[d, "f \otimes I"'] & X \arrow[d, "f"] \\
X' \otimes I \arrow[r, "r_I"'] & X'
\end{tikzcd}$$

commutes. Taking $X = X' = I$ and the morphisms $l_I : I \to I$ and $r_I : I \to I$, we get

$$r_I = r_I(r_I \otimes I) r_I^{-1} = r_I(l_I \otimes I) r_I^{-1} = l_I.$$

□

### 1.2.2 Units in semigroup categories

If $(\mathcal{C}, \otimes, I)$ is a monoidal category, to what extent does the semigroup category determine the unit $I = (I, l, r)$? In order to answer this question let us say, more generally, that any triple $(I, l, r)$ that satisfies (1.2) is a **unit** of the semigroup category $(\mathcal{C}, \otimes)$.

It is clear that if $(I, l, r)$ and $(I', l', r')$ are units of $\mathcal{C}$ then the unit objects $I$ and $I'$ are isomorphic, since we have the isomorphism

$$I \xrightarrow{l_{I'}^{-1}} I \otimes I' \xrightarrow{r'_I} I'$$

but what about the left and right unit constraints? In order to take them into account, let us define a **morphism of units** $\psi : (I, l, r) \to (I', l', r')$ to be a morphism $\psi : I \to I'$ that satisfies canonical compatibilities with the unit constraints:

$$\begin{tikzcd}
I \otimes X \arrow[r, "l_X"] \arrow[d, "\psi \otimes X"'] & X \\
I' \otimes X \arrow[ur, "l'_X"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
X & X \otimes I \arrow[l, "r_X"'] \arrow[d, "X \otimes \psi"] \\
& X \otimes I' \arrow[ul, "r'_X"]
\end{tikzcd}
\tag{1.5}$$

**Proposition 1.10.** *Any two units in $\mathcal{C}$ are isomorphic via a unique isomorphism of units.*

*Proof.* To prove the existence of such an isomorphism, take the isomorphism (of unit objects) $\psi =: r'_I l_{I'}^{-1} : I \to I'$ as above; we will show that this is a morphism of units.

Consider the diagram

$$\begin{tikzcd}
 & I \otimes X \arrow[r, "l_X"] & X \\
(I \otimes I') \otimes X \arrow[ur, "r'_I \otimes X"] \arrow[r, "a"] \arrow[dr, "l_{I'} \otimes X"'] & I \otimes (I' \otimes X) \arrow[u, "I \otimes l'_X"'] \arrow[r, "l_{I' \otimes X}"] \arrow[d, "l_{I' \otimes X}"'] & I' \otimes X \arrow[u, "l'_X"'] \arrow[d, "l'_X"] \\
 & I' \otimes X \arrow[r, "l'_X"'] & X
\end{tikzcd}$$

The bottom square is obviously commutative, and the top square is commutative by the naturality of $l$. On the left of the diagram, the top triangle commutes because of the unit axiom (1.2), and the bottom triangle commutes because of Proposition 1.8. Once more, since all arrows are isomorphisms, the outline of this diagram is commutative, and it follows that

$$\begin{aligned}
l'_X(\psi \otimes X) &= l'_X(r'_I l_{I'}^{-1}) \otimes X \\
&= l'_X(r'_I \otimes X)(l_{I'}^{-1} \otimes X) \\
&= l'_X l'_X (l'_X)^{-1} l_X \\
&= l_X .
\end{aligned}$$

An analogous reasoning shows that the other diagram of the definition of morphism of units also commutes.

For the uniqueness, assume that $\theta : I \to I'$ is another isomorphism of units, and take $X = I$ in the left diagram in (1.5). Since $l_I$ and $l_{I'}$ are isomorphisms, there is at most one arrow making the diagram commute, and therefore $\psi \otimes I = \theta \otimes I$. By the same argument used in the end of the proof of Proposition 1.9, since $r : \_ \otimes I \to \mathrm{Id}_{\mathcal{C}}$ is a natural isomorphism we conclude that $\psi = \theta$.

$\square$

Other results about units in semigroup categories can be found in [6, 9, 10, 11].

## 1.3 Strict Monoidal Categories

### 1.3.1 Monoidal Functors

**Definition 1.11.** Let $\mathcal{C} = (\mathcal{C}, \otimes, I_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \boxtimes, I_{\mathcal{D}})$ be two monoidal categories.

A **monoidal functor** $F = (F, \phi, \phi_0)$ consists of a functor $F : \mathcal{C} \to \mathcal{D}$, a natural transformation $\phi : F(\_) \boxtimes F(\_) \to F(\_ \otimes \_)$ and a morphism $\phi_0 : F(I_{\mathcal{C}}) \to I_{\mathcal{D}}$ such that for all objects $X, Y, Z$ in $\mathcal{C}$,

$$\begin{tikzcd}
(F(X) \boxtimes F(Y)) \boxtimes F(Z) \arrow[r, "a^{\mathcal{D}}_{F(X),F(Y),F(Z)}"] \arrow[d, "\phi_{X,Y} \boxtimes F(Z)"'] & F(X) \boxtimes (F(Y) \boxtimes F(Z)) \arrow[d, "F(X) \boxtimes \phi_{Y,Z}"] \\
(F(X \otimes Y)) \boxtimes F(Z) \arrow[d, "\phi"'] & F(X) \boxtimes (F(Y \otimes Z)) \arrow[d, "\phi"] \\
F((X \otimes Y) \otimes Z) \arrow[r, "F(a^{\mathcal{C}}_{X,Y,Z})"'] & F(X \otimes (Y \otimes Z))
\end{tikzcd}$$

and

$$
\begin{tikzcd}
I_{\mathcal{D}} \boxtimes F(X) \arrow[r, "l_{F(X)}"] \arrow[d, "\phi_0 \boxtimes F(X)"'] & F(X) \\
F(I_{\mathcal{C}}) \boxtimes F(X) \arrow[r, "\phi_0"'] & F(I_{\mathcal{C}} \otimes X) \arrow[u, "F(l_X)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
F(X) & F(X) \boxtimes I_{\mathcal{D}} \arrow[l, "r_{F(X)}"'] \arrow[d, "F(X)\boxtimes\phi_0"] \\
F(X \otimes I_{\mathcal{C}}) \arrow[u, "F(r_X)"] & F(X) \boxtimes F(I_{\mathcal{C}}) \arrow[l, "\phi_0"]
\end{tikzcd}
$$

We will say that $F = (F, \phi, \phi_0)$ is a **strict** monoidal functor if $\Phi$ and $\Phi_0$ are isomorphisms.

Recall that the monoidal category $\mathcal{C}$ itself is called **strict** if all isomorphisms

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)$$

are identities. In this section we will prove that every monoidal is equivalent to a strict one via a monoidal functor.

**Example 1.12.** Let $G$ be a finite group, $k$ a field, and $\mathrm{Rep}^f G$ the category of finite-dimensional $k$-linear representations of $G$. The forgetful functor $F : \mathrm{Rep}^f G \to \underline{\mathrm{Vect}}_k$ that takes the representation $(V, \rho)$ to its underlying vector space is a strict monoidal functor.

**Example 1.13.** On the other hand, let $A$ be a $k$-algebra and let $F : {}_A\mathcal{M}_A \to \underline{\mathrm{Vect}}_k$ be the corresponding forgetful functor. In this case the monoidal structure on $F$ is not strict: the natural transformation

$$
\begin{array}{rccc}
\phi : & M \otimes N & \to & M \otimes_A N \\
& m \otimes n & \mapsto & m \otimes_A n
\end{array}
$$

is the projection of $M \otimes$ onto $M \otimes_A N$, and the morphism $\phi_0$ is the injection

$$
\begin{array}{rccc}
\phi_0 : & k & \to & A \\
& \lambda & \mapsto & \lambda 1_A
\end{array}
$$

**Example 1.14.** Let $G$ be a finite group. We have seen in Example 1.7 that for each 3-cocycle $a \in Z^3(G, k^*)$ there is a monoidal structure on the full subcategory $\mathcal{C}$ of the category $G$-graded $k$-vector spaces whose objects are the simple $G$-graded spaces $k(r)$. Given $a \in Z^3(G, k^*)$, let $\mathcal{C}_a$ denote the associated monoidal category.

Let $a, a'$ be two cocycles. We are going to derive necessary conditions for the identity functor $\mathrm{Id}_{\mathcal{C}} : C_a \to C_{a'}$ to be a monoidal functor, i.e., conditions for the existence of a natural transformation

$$\phi_{X,Y} : \mathrm{Id}_{\mathcal{C}}(X) \otimes \mathrm{Id}_{\mathcal{C}}(Y) \to \mathrm{Id}_{\mathcal{C}}(X \otimes Y)$$

and a morphism

$$\phi_0 : k \to k$$

satisfying Definition 1.11.

Once again identifying the objects of $\mathcal{C}$ and $G$, we obtain a map

$$\phi : G \times G \to k^*$$

such that (by the first diagram of Def. 1.11 )

$$a(r,s,t)\phi(s,t)\phi(r,st) = \phi(r,s)\phi(rs,t)a'(r,s,t)$$

i.e.,

$$(a/a')(r,s,t) = \phi(r,s)\phi(rs,t)\phi(s,t)^{-1}\phi(r,st)^{-1} = (d^2\phi)(r,s,t). \tag{1.6}$$

Hence, if the identity functor has a monoidal structure then $a/a'$ is a coboundary. This is also sufficient when we consider only the semigroup structure of the category, but we still have to take the unit into account.

The map $\phi_0$ can be identified with multiplication by a nonzero element $\lambda$ of $k$. The diagrams that $\phi_0$ must satisfy are translated, in this case, in the equations

$$\begin{aligned} \phi(e,e) &= \phi(e,r) = \phi(r,e), \quad \forall r \in G, && (1.7)\\ \lambda\, \phi(e,e) &= 1 && (1.8) \end{aligned}$$

Hence $\phi_0$ is determined by $\phi$, which is nice, but (1.7) adds an extra constraint. It follows that $\mathrm{Id}_\mathcal{C} : \mathcal{C}_a \to \mathcal{C}_{a'}$ will have a monoidal structure if and only if $a$ and $a'$ are cohomologous by a coboundary $d^2\phi$ such that $\phi$ satisfies (1.7), and every such $\phi$ defines a monoidal structure on $\mathrm{Id}_\mathcal{C}$.

The classification of the categories $\mathcal{C}_a$ that are equivalent as semigroup categories can be found in [6, Ex.13.5,p.118], where it is shown that the distinct classes are in bijection with the group $H^3(G,k^*)/\operatorname{Aut}(G)$.

### 1.3.2 Equivalence with Strict Monoidal Categories

**Definition 1.15.** Let $\mathcal{C} = (\mathcal{C}, \otimes, I_\mathcal{C})$ and $\mathcal{D} = (\mathcal{D}, \boxtimes, I_\mathcal{D})$ be two monoidal categories, and let $(F,\phi,\phi_0),(F',\phi',\phi_0')$ be two monoidal functors from $\mathcal{C}$ to $\mathcal{D}$. A natural monoidal transformation $\eta : (F,\phi,\phi_0) \to (F',\phi',\phi_0')$ is a natural transformation $\eta : F \to F'$ such that the following diagrams commute for every pair $(X,Y)$ of objects in $\mathcal{C}$:

$$\begin{tikzcd} I_\mathcal{D} \arrow[r, "\phi_0"] \arrow[dr, "\phi_0'"'] & F(I_\mathcal{C}) \arrow[d, "\eta_I"] \\ & G(I_\mathcal{C}) \end{tikzcd} \qquad \begin{tikzcd} F(X) \boxtimes F(Y) \arrow[r, "\phi'_{X,Y}"] \arrow[d, "\eta_X \boxtimes \eta_Y"'] & F(X \otimes Y) \arrow[d, "F(\eta_X \otimes \eta_Y)"'] \\ G(X) \boxtimes G(Y) \arrow[r, "\phi'_{X,Y}"] & G(X \otimes Y) \end{tikzcd}$$

A natural monoidal isomorphism is a natural monoidal transformation which is a natural isomorphism.

**Definition 1.16.** A **monoidal equivalence** between two monoidal categories $\mathcal{C}$ and $\mathcal{D}$ is a monoidal functor such that there exist a monoidal functor $G : \mathcal{D} \to \mathcal{C}$ and natural monoidal isomorphisms $\eta : \mathrm{Id}_\mathcal{D} \to FG$ and $\theta : GF \to \mathrm{Id}_\mathcal{C}$.

Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be a monoidal category. We will simultaneously build a "covering" strict monoidal category $\tilde{\mathcal{C}}$ and a monoidal equivalence $F : \tilde{\mathcal{C}} \to \mathcal{C}$.

The **objects** of $\tilde{\mathcal{C}}$ will be finite sequences $(X_1, \ldots, X_k)$ of objects of $\mathcal{C}$; we define the length of $(X_1, \ldots, X_k)$ to be $k$. We will also consider the empty sequence $\varnothing$ (which will be the unit of this category), with length equal to zero.

Let $\mathcal{S}$ be the class of all such sequences, and let $*$ be the operation of concatenation of sequences. For two nonempty sequences, $*$ is defined by

$$(X_1, \ldots, X_k) * (Y_1, \ldots, Y_l) = (X_1, \ldots, X_k, Y_1, \ldots, Y_l)$$

and for the empty sequence we define $S*\varnothing = \varnothing*S = S$, for every $S$ in $\mathcal{S}$. It is clear that $(S_1*S_2)*S_3 = S_1*(S_2*S_3)$ for all sequences $S_1, S_2, S_3$ in $\mathcal{S}$.

The **objects** of our category $\tilde{\mathcal{C}}$ are going to be the elements of $\mathcal{S}$. In order to define morphisms between then, define recursively a function $F$ from $\mathcal{S}$ to the class of objects of $\mathcal{C}$ by

$$F(\varnothing) = I, F((X)) = X \text{ and } F(S * (X)) = F(S) \otimes X. \tag{1.9}$$

In this manner we assign to each sequence $S = (X_1, X_2, \ldots, X_k)$ of objects of $\mathcal{C}$ the object $((\cdots((X_1 \otimes X_2) \otimes X_3) \cdots) \otimes X_k$.

Given sequences $S_1$ and $S_2$ of objects of $\mathcal{C}$, we define the set of **morphisms** $\tilde{\mathcal{C}}(S_1, S_2)$ to be the set $\mathcal{C}(F(S_1), F(S_2))$. Since $\mathcal{C}$ is a category, it follows that $\tilde{\mathcal{C}}$ is a category.

**Proposition 1.17.** *Let $\mathcal{C}$ and $\tilde{\mathcal{C}}$ be as before. The functor $F : \tilde{\mathcal{C}} \to \mathcal{C}$ which is defined on objects by (1.9), and sends the morphism $f : S_1 \to S_2$ to the morphism $f : F(S_1) \to F(S_2)$ is an equivalence of categories.*

*Proof.* It is easy to see that $F$ is a functor. In order to show that $F$ is an equivalence we will prove that it has a semi-inverse, i.e., a functor $G : \mathcal{C} \to \tilde{\mathcal{C}}$ such that $FG$ and $GF$ are naturally isomorphic to $\mathrm{Id}_{\mathcal{C}}$ and $\mathrm{Id}_{\tilde{\mathcal{C}}}$ respectively.

Define $G : \mathcal{C} \to \tilde{\mathcal{C}}$ as the functor that sends an object $X \neq I_{\mathcal{C}}$ to the unitary sequence $(X) \in \mathcal{S}$, $I_{\mathcal{C}}$ to $\varnothing$, and sends the morphism $X \xrightarrow{f} Y$ to the corresponding morphism $(X) \xrightarrow{f} (Y)$ of unitary sequences.

It is clear that $FG = \mathrm{Id}_{\mathcal{C}}$. By the definition of morphisms in $\tilde{\mathcal{C}}$, given two sequences $S, S'$, a morphism $S \xrightarrow{f} S'$ is a morphism in $\mathcal{C}$ from $F(S)$ to $F(S')$. Still by the definition of morphisms in $\tilde{\mathcal{C}}$, the identity $F(S)$ is an element of $\tilde{\mathcal{C}}((F(S), S)$ since the morphisms from $(F(S)) = GF(S)$ to $S$ are, by definition, the morphisms from $F(GF(S)) = (FG)(F(S)) = F(S)$ to $F(S)$. Therefore the collection of morphisms $\{F(S) : GF(S) \to S; S \in \mathcal{S}\}$ defines a natural isomorphism from $GF$ to $\mathrm{Id}_{\tilde{\mathcal{C}}}$. ∎

The concatenation of sequences induces a monoidal structure on $\tilde{\mathcal{C}}$. This is already defined on the level of objects, and on morphisms the "tensor product" is obtained via a natural isomorphism

$$\phi : F(\_) \otimes F(\_) \to F(\_ * \_)$$

which will be defined for pairs $(S, S')$ recursively on the length of $S'$.

For the empty sequence we define

$$\phi_{\varnothing,S} = l_S, \quad \phi_{S,\varnothing} = r_S.$$

For $S' = (X)$, $X$ an object of $\mathcal{C}$, define

$$\phi_{S,(X)} = F(S) \otimes X$$

which makes sense because $F(S * (X)) = F(S) \otimes X$.

If $\phi_{S,S'}$ is defined and $X$ is an object of $\mathcal{C}$, define

$$\phi_{S,S'*(X)} = (\phi_{S,S'} \otimes X) \circ a^{-1}_{F(S),F(S'),X}, \tag{1.10}$$

i.e., $\phi_{S,S'*X}$ is defined in such a manner that the diagram below commutes (remember that $F(S' * (X)) = F(S') \otimes X$.

$$\begin{array}{ccc} F(S) \otimes F(S' * (X)) & \xrightarrow{\phi_{S,S'*(X)}} & F(S * S' * X) \\ \downarrow{\scriptstyle a^{-1}} & & \| \\ (F(S) \otimes F(S')) \otimes X & \xrightarrow[\phi_{S,S'} \otimes X]{} & F(S * S') \otimes X \end{array}$$

By induction, $\phi_{S,S'}$ is defined for every pair $(S, S')$ of sequences.

$\phi$ is natural on $S$ and, by induction, on $S'$, since it is built up from natural isomorphisms. Now we can define the tensor product of two morphisms $f : S \to T$, $f' : S' \to T'$ of $\tilde{\mathcal{C}}$ by

$$f * f' = \phi_{T,T'}(f \otimes f')\phi^{-1}_{S,S'}. \tag{1.11}$$

**Proposition 1.18.** *$\tilde{\mathcal{C}}$ is a strict monoidal category.*

Finally, we will see that $F : \tilde{\mathcal{C}} \to \mathcal{C}$ is a monoidal functor.

The monoidal structure is given by the natural isomorphism $\phi$ and by $\phi_0 = I$.

**Lemma 1.19.** *If $S, S', S''$ are objects of $\mathcal{C}$ then*

$$\phi_{S,S'*S''} \circ (S \otimes \phi_{S',S''}) \circ a_{F(S),F(S'),F(S'')} = \phi_{S*S',S''} \circ (\phi_{S,S'} \otimes S''). \tag{1.12}$$

*i.e., the diagram below commutes.*

$$\begin{tikzcd}
(F(S) \otimes F(S')) \otimes F(S'') \arrow[rr, "a"] \arrow[d, "\phi_{S,S'} \otimes F(S'')"'] & & F(S) \otimes (F(S') \otimes F(S'')) \arrow[d, "F(S) \otimes \phi_{S',S''}"] \\
(F(S * S')) \otimes F(S'') \arrow[d, "\phi_{S*S',S''}"'] & & F(S) \otimes (F(S' * S'')) \arrow[d, "\phi_{S,S'*S''}"] \\
F(S * S' * S'') \arrow[rr, equal] & & F(S * S' * S'')
\end{tikzcd}$$

*Proof.* The proof will be done by induction on the length of $S''$. First, if $S'' = \varnothing$ then

$$\begin{aligned}
&\phi_{S,S'} \circ (S \otimes \phi_{S',\varnothing}) \circ a_{F(S),F(S'),I} \\
&= \phi_{S,S'} \circ (F(S) \otimes r_{F(S')}) \circ a_{F(S),F(S'),I}
\end{aligned}$$

by the definition of $\phi_{S',\varnothing}$. By Proposition 1.8, the diagram below commutes

$$\begin{tikzcd}
(F(S) \otimes F(S')) \otimes I \arrow[rr, "a_{F(S),F(S'),I}"] \arrow[dr, "r_{(F(S)\otimes F(S'))}"'] & & F(S) \otimes (F(S') \otimes I) \arrow[dl, "F(S)\otimes r_{F(S')}"] \\
& F(S) \otimes F(S') &
\end{tikzcd}$$

and therefore

$$\phi_{S,S'} \circ (F(S) \otimes r_{F(S')}) \circ a_{F(S),F(S'),I} = \phi_{S,S'} \circ r_{(F(S)\otimes F(S'))}.$$

Since $r$ is a natural isomorphism, we also have the commutative diagram

$$\begin{tikzcd}
(F(S) \otimes F(S')) \otimes I \arrow[r, "r_{(F(S)\otimes F(S'))}"'] \arrow[d, "\phi"'] & F(S) \otimes F(S') \arrow[d, "\phi"] \\
F(S * S') \otimes I \arrow[r, "r_{F(S*S')}"'] & F(S * S')
\end{tikzcd}$$

and it follows that

$$\begin{aligned}
\phi_{S,S'} \circ r_{(F(S)\otimes F(S'))} &= r_{F(S*S')}(\phi_{S,S'} \otimes I) \\
&= \phi_{S*S',\varnothing} \circ (\phi_{S,S'} \otimes I).
\end{aligned}$$

thus showing that Equation(1.12) holds for the triple $(S, S', \varnothing)$.

Assume that (1.12) holds for $(S, S', S'')$, and let us show that it holds for $(S, S', S'' * (X))$, for any object $X$ in $\mathcal{C}$, i.e., we will show that

$$\begin{aligned}
&\phi_{S,S'*S''*(X)} \circ (S \otimes \phi_{S',S''*(X)}) \circ a_{F(S),F(S'),F(S''*(X))} \\
&= \phi_{S*S',S''*(X)} \circ (\phi_{S,S'} \otimes S'' * (X)).
\end{aligned} \tag{1.13}$$

The idea for the proof is to "move parentheses to the left" until we obtain the LHS of the induction hypothesis tensored by $X$. The diagram below contains the first part of the proof. Keeping the top and bottom vertices fixed, we're going to move from right to left.

$$\begin{tikzcd}[column sep=small]
& & (F(S)\otimes F(S'))\otimes F(S''*(X)) \arrow[dll, "a^{-1}"'] \arrow[drr, "a"] & & \\
((F(S)\otimes F(S'))\otimes F(S''))\otimes X \arrow[dd, "a\otimes X"'] & & & & F(S)\otimes(F(S')\otimes F(S''*(X))) \arrow[dll, "F(S)\otimes a^{-1}"'] \arrow[dd, "F(S)\otimes\phi"] \\
& & F(S)\otimes((F(S')\otimes F(S''))\otimes X) \arrow[dll, "a^{-1}"] \arrow[drr, "F(S)\otimes(\phi\otimes X)"] & & \\
(F(S)\otimes(F(S')\otimes F(S'')))\otimes X \arrow[drr, "(F(S)\otimes\phi)\otimes X"'] & & & & F(S)\otimes F(S'*S''*S*(X)) \arrow[dll, "a^{-1}"] \arrow[dd, "\phi"] \\
& & (F(S)\otimes F(S'*S''))\otimes X \arrow[drr, "\phi\otimes X"'] & & \\
& & & & F(S*S'*S''*(X))
\end{tikzcd}$$

To begin with, the recursive definition of $\phi_{S,S'*S''*(X)}$ yields

$$\begin{aligned}
&\phi_{S,S'*S''*(X)}\circ(S\otimes\phi_{S',S''*(X)})\circ a_{F(S),F(S'),F(S''*(X))}=\\
&=(\phi_{S,S'*S''}\otimes X)a^{-1}_{F(S),F(S'*S''),X}(S\otimes\phi_{S',S''}\otimes S)\\
&(S\otimes a^{-1}_{F(S'),F(S''),X})a_{F(S),F(S'),F(S''*(X))}
\end{aligned}$$

The naturality of $a : (\_\otimes\_)\otimes\_ \to \_\otimes(\_\otimes\_)$ implies the commutativity of the square in the next diagram:

$$\begin{tikzcd}
F(S)\otimes((F(S')\otimes F(S''))\otimes X) \arrow[r, "F(S)\otimes(\phi\otimes X)"] \arrow[d, "a^{-1}"'] & F(S)\otimes(F(S'*S'')\otimes X) \arrow[d, "a^{-1}"'] \\
(F(S)\otimes((F(S')\otimes F(S''))\otimes X \arrow[r, "(F(S)\otimes\phi)\otimes X"'] & (F(S)\otimes F(S'*S''))\otimes X
\end{tikzcd}$$

Using this diagram and then using the pentagon axiom applied to the quadruple $(F(S),F(S'),F(S''),X)$, we get

$$\begin{aligned}
&=(\phi_{S,S'*S''}\otimes X)a^{-1}_{F(S),F(S'*S''),X}(S\otimes\phi_{S',S''}\otimes S)\\
&(S\otimes a_{F(S'),F(S''),X})a_{F(S),F(S'),F(S''*(X))}=\\
&=(\phi_{S,S'*S''}\otimes X)(S\otimes\phi_{S',S''}\otimes S)a_{F(S),F(S')\otimes F(S''),X}(S\otimes a^{-1}_{F(S'),F(S''),X})a^{-1}_{F(S),F(S'),F(S'')\otimes X}\\
&=(\phi_{S,S'*S''}\otimes X)(S\otimes\phi_{S',S''}\otimes X)(a_{F(S),F(S'),F(S'')\otimes X}a^{-1}_{F(S)\otimes F(S'),F(S''),X}\\
&=([\phi_{S,S'*S''}(S\otimes\phi_{S',S''})a_{F(S),F(S'),F(S'')}]\otimes X)a^{-1}_{F(S)\otimes F(S'),F(S''),X}
\end{aligned}$$

and we are already at the left side of the diagram.

To finish the proof we will again move from right to left in the next diagram below, keeping the top left and bottom right vertices “fixed”. Note that the rightmost path from top to bottom is the leftmost path in the previous diagram.

$$\begin{tikzcd}[column sep=small]
(F(S)\otimes F(S'))\otimes F(S''*(X)) \arrow[r, "a^{-1}"] \arrow[d, "\phi\otimes(F(S'')\otimes X)"'] & ((F(S)\otimes F(S')\otimes F(S''))\otimes X \arrow[r, "a\otimes X"] \arrow[d, "(\phi\otimes F(S''))\otimes X"] & (F(S)\otimes(F(S')\otimes F(S''))\otimes X \arrow[d, "(F(S)\otimes\phi)\otimes X"] \\
F(S*S')\otimes F(S''*(X)) \arrow[r, "a^{-1}"] \arrow[dr, "\phi"'] & ((F(S*S')\otimes F(S''))\otimes X \arrow[d, "\phi\otimes X"] & (F(S)\otimes(F(S'*S''))\otimes X \arrow[d, "\phi\otimes X"] \\
& F(S*S'*S''*(X)) \arrow[r, equal] & F(S*S'*S''*(X))
\end{tikzcd}$$

By the induction hypothesis, we have

$$
\begin{aligned}
&(\phi_{S*S',S''}(\phi_{S,S'} \otimes S'' \otimes X))a^{-1}_{F(S)\otimes F(S'),F(S''),X} \\
&= (\phi_{S*S',S''} \otimes X)((\phi_{S,S'} \otimes S'' \otimes X))a^{-1}_{F(S)\otimes F(S'),F(S''),X}
\end{aligned}
$$

and then, by naturality of the associator and the definition of $\phi_{S*S',S''*(X)}$, it follows that

$$
\begin{aligned}
&(\phi_{S*S',S''} \otimes X)((\phi_{S,S'} \otimes S'' \otimes X))a^{-1}_{F(S)\otimes F(S'),F(S''),X} \\
&= (\phi_{S*S',S''} \otimes X)a^{-1}_{F(S*S'),F(S''),X}(\phi_{S,S'} \otimes S'' \otimes X) \\
&= \phi_{S*S',S''*(X)}(\phi_{S,S'} \otimes S'' * (X)).
\end{aligned}
$$

□

**Theorem 1.20.** *The categories $\tilde{\mathcal{C}}$ and $\mathcal{C}$ are monoidally equivalent via a strict monoidal functor.*

*Proof.* In fact, in Lemma 1.19 we've proved that $(F, \phi, I)$ is a strict monoidal functor from $\tilde{\mathcal{C}}$ to $\mathcal{C}$. It can be proved that $G$ is also a strict monoidal functor. By the proof of Proposition 1.17, $F : \tilde{\mathcal{C}} \to \mathcal{C}$ is an equivalence with quasi-inverse $G$. □

As a corollary, we can state an informal version of Mac Lane's Coherence Theorem: Given objects $X_1, \ldots, X_n$ in a monoidal category $\mathcal{C}$, any isomorphism between two bracketings of

$$X_1 \otimes X_2 \cdots \otimes X_n$$

which is obtained from concatenating and tensoring $a$, $l$, $r$, and the identities is the same.

In fact, this statement is obviously true when $\mathcal{C}$ is a strict monoidal category; since we have seen that any monoidal category is monoidally equivalent to a strict one, the claim holds in any monoidal category.

## 1.4 Algebras and Coalgebras in a Monoidal Category

The concept of monoidal category allows us to unify several algebraic structures as different instances of the same notion, that of an algebra object in a monoidal category.

**Definition 1.21.** Let $(C, \otimes, I)$ be a monoidal category. An **algebra object** in $\mathcal{C}$, or simply an **algebra** in $\mathcal{C}$, is a triple $(A, \mu, \eta)$ consisting of an object $A$, a morphism $\mu : A \otimes A \to A$ and a morphism $\eta : I \to A$ such that the following diagrams are commutative:

$$
\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "a"] \arrow[d, "\mu\otimes A"'] & A \otimes (A \otimes A) \arrow[r, "A\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[rr, "\mu"'] & & A
\end{tikzcd}
\tag{1.14}
$$

$$
\begin{tikzcd}
 & A \otimes A \arrow[dd, "\mu"'] & \\
I \otimes A \arrow[ur, "\eta\otimes I"] \arrow[dr, "l"'] & & A \otimes I \arrow[ul, "I\otimes\eta"'] \arrow[dl, "r"] \\
 & A &
\end{tikzcd}
\tag{1.15}
$$

The morphism $\mu$ is called the product of multiplication of the algebra object $A$, and $\eta$ is called the unit of $A$.

**Remark 1.22.** When a morphism $\mu : A \otimes A \to A$ satisfies (1.14) one says that $\mu$ is **associative**; the second diagram is known as the **unit axiom**. Usually the objects $(A \otimes A) \otimes A$ and $A \otimes (A \otimes A)$ are identified via the associator $a$ and (1.14) is rewritten as

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "A\otimes\mu"] \arrow[d, "\mu\otimes A"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd} \tag{1.16}$$

**Example 1.23.** In the category $\underline{\mathrm{Vect}}_k$ the algebra objects are exactly the (unital) $k$-algebras.

If $A = (A, \mu, I)$ is an algebra object in $\underline{\mathrm{Vect}}_k$ then $A$ is a $k$-vector space with a multiplication

$$(a, b) \mapsto a \otimes b \mapsto \mu(a \otimes b)$$

which is $k$-bilinear, and associativity follows from (1.14). If we define $1_A = \eta(1)$ then it follows from (1.15) that

$$(1_A a) = (a 1_A) = A$$

for every $a \in A$.

Conversely, if $A = (A, m, 1_A)$ is a unital algebra then the map $(a, b) \mapsto ab = m(a, b)$ is $k$-bilinear and therefore there exists a unique linear map $\mu : A \otimes A \to A$ such that

$$\mu(a \otimes b) = ab.$$

Since $m$ is associative, the first diagram (1.14) commutes. The unit map is given by $\eta : k \to A$, $1 \mapsto 1_A$. It follows that the second diagram (1.15) is also commutative.

**Example 1.24.** The unit object of a monoidal category $\mathcal{C}$ is an algebra object. Remember from Proposition 1.9 that $l_I = r_I$; define $\mu = l_I = r_I$ and $\eta = I$. Then, from the definition of unit (1.2),

$$\mu(I \otimes \mu)a = \mu(I \otimes r_I)a = \mu(l_I \otimes I) = \mu(\mu \otimes I),$$

which proves (1.14), and (1.15) is straightforward.

**Example 1.25.** Let $R$ be a ring and let $\mathcal{C}$ be the monoidal category $\mathcal{C} = ({}_R\mathcal{M}, \times, 0)$. Every $R$-module has a unique structure of algebra object in $\mathcal{C}$. In fact, if $A = (A, \mu, \eta)$ is an algebra in $\mathcal{C}$, the unit map $\eta$ is the zero map $0 \to A$ (since this is the only morphism from 0 to $A$ in $\mathcal{C}$). The diagram 1.15 says that

$$\begin{aligned} a &= l_A(0, a) = \mu(\eta, A)(0, a) = \mu(0, a), \\ a &= r_A(a, 0) = \mu(A, \eta)(a, 0) = \mu(a, 0), \end{aligned}$$

and since $\mu$ is $R$-linear we have

$$\mu(a, b) = \mu(a, 0) + \mu(0, b) = a + b.$$

Hence $\mu : A \times A \to A$ is just the sum operation of $A$.

Conversely, if $M$ is an $R$-module it is easy to check that $(M, +, 0 \to M)$ is an algebra in $\mathcal{C}$.

**Example 1.26.** Let $G$ be a group and consider the monoidal category $\underline{\mathrm{Vect}}_k^G$ of $G$-graded vector spaces. If $(A, \mu, \eta)$ is an algebra in $\underline{\mathrm{Vect}}_k^G$, let $A = \oplus_{r\in G} A_r$ be its $G$-grading.

The forgetful functor from $\underline{\mathrm{Vect}}_k^G$ to $\underline{\mathrm{Vect}}_k$ is clearly a strict monoidal functor; therefore, if $(A, \mu, \eta)$ is an algebra in $\underline{\mathrm{Vect}}_k^G$ then its image in $\underline{\mathrm{Vect}}_k$, which consists of the same triple but forgetting about the $G$-grading, is an algebra in $\underline{\mathrm{Vect}}_k$, i.e., $A$ is a $k$-algebra.

Morphisms in $\underline{\mathrm{Vect}}_k^G$ are $k$-linear maps that preserve degree; since $\mu : A \otimes A \to A$ is a morphism in $\underline{\mathrm{Vect}}_k^G$, it follows that for any $r, s$ in $G$ one has

$$A_r A_s = \mu(A_r \otimes A_s) \subset A_{rs}.$$

In other words, $A$ is a $G$-graded algebra. Conversely, it can be shown that every $G$-graded $k$-algebra corresponds to an algebra object in $\underline{\mathrm{Vect}}_k^G$.

**Example 1.27.** An algebra object in the monoidal category of Sets, $(\underline{\mathrm{Set}}, \times, \{*\})$, is the same thing as a monoid (a semigroup with neutral element).

**Definition 1.28.** Let $A = (A, \mu_A, \eta_A)$ and $B = (B, \mu_B, \eta_B)$ be algebra objects in the monoidal category $\mathcal{C}$. A **morphism** of algebra objects $f : A \to B$ is a morphism in $\mathcal{C}$ such that the following diagrams commute.

$$\begin{tikzcd} A \otimes A \arrow[r, "f \otimes f"] \arrow[d, "\mu_A"'] & B \otimes B \arrow[d, "\mu_B"] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] & B \\ k \arrow[u, "\eta_A"] \arrow[ur, "\eta_B"'] & \end{tikzcd}$$

**Example 1.29.** The concept of morphism of algebra objects in a monoidal category specializes to the usual definitions of morphisms of algebras, monoids, graded algebras, etc.

1. A morphism of algebras in $\underline{\mathrm{Vect}}_k$ is exactly a mophism of unital $k$-algebras. In fact, if $f : A \to B$ is a morphism of algebras in $\underline{\mathrm{Vect}}_k$ then

$$f(a)f(b) = \mu_B(f \otimes f)(a \otimes b) = f\mu_A(a \otimes b) = f(ab)$$

   and

$$f(1_A) = f\eta_A(1) = \eta_B(1) = 1_B.$$

2. By the same token, a morphism of algebra objects in $\underline{\mathrm{Set}}$ is the same as a morphism of monoids.

3. A morphism in $\underline{\mathrm{Vect}}_k^G$ is a linear map that preserves degree. Therefore, a morphism of algebra objects in $\underline{\mathrm{Vect}}_k^G$ is a morphism of $G$-graded algebras.

Dualizing Definition 1.21 we arrive at the definition of a coalgebra.

**Definition 1.30.** Let $(C, \otimes, I)$ be a monoidal category. A **coalgebra object** in $\mathcal{C}$, or simply a **coalgebra** in $\mathcal{C}$, is a triple $(C, \Delta, \epsilon)$ consisting of an object $C$, a morphism $\Delta : C \to C \otimes C$ and a morphism $\epsilon : C \to I$ such that the following diagrams are commutative:

$$\begin{tikzcd} C \arrow[rr, "\Delta"] \arrow[d, "\Delta"'] & & C \otimes C \arrow[d, "\Delta \otimes C"] \\ C \otimes C \arrow[r, "C \otimes \Delta"'] & C \otimes (C \otimes C) \arrow[r, "a^{-1}"'] & (C \otimes C) \otimes C \end{tikzcd} \tag{1.17}$$

$$\begin{tikzcd} & C \arrow[dl, "l^{-1}"'] \arrow[dr, "r^{-1}"] \arrow[dd, "\Delta"'] & \\ I \otimes C & & C \otimes I \\ & C \otimes C \arrow[ul, "\epsilon \otimes C"] \arrow[ur, "C \otimes \epsilon"'] & \end{tikzcd} \tag{1.18}$$

The morphism $\Delta$ is called the coproduct or comultiplication of the coalgebra object $C$, and $\epsilon$ is called the counit of $C$.

**Remark 1.31.** A morphism $\Delta : C \to C \otimes C$ is **coassociative** if it satisfies (1.17). The second diagram is called the **counit axiom**.

As in the previous case of algebra objects, one usually identifies $C \otimes (C \otimes C)$ with $(C \otimes C) \otimes C$ via $a^{-1}$ in the definition of coalgebra object. The coassociativity diagram is then rewritten as

$$\begin{tikzcd} C \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & C \otimes C \arrow[d, "\Delta\otimes C"] \\ C \otimes C \arrow[r, "C\otimes\Delta"'] & C \otimes C \otimes C \end{tikzcd} \tag{1.19}$$

**Example 1.32.** Let $V$ be a vector space and choose a basis $B$ of $V$. Define $\Delta : V \to V \otimes V$ and $\epsilon : B \to k^*$ by $\Delta(b) = b \otimes b$ and $\epsilon(b) = 1$ for every $b \in B$. Then it is easily seen that $(V, \Delta, \varepsilon)$ is a coalgebra in $\underline{\mathrm{Vect}}_k$.

A particular case of this construction is the coalgebra structure on the base field $k$ given by $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$.

**Example 1.33.** The unit object of a monoidal category has a canonical structure of coalgebra object: $\Delta = l_I^{-1} = r_I^{-1}$ (cf. Proposition 1.9) and $\epsilon = I$.

**Example 1.34.** In the monoidal category $(\underline{\mathrm{Set}}, \times, \{*\})$, every object has a unique structure of coalgebra. If $(C, \Delta, \epsilon)$ is a coalgebra in $\underline{\mathrm{Set}}$ then $\epsilon$ is the unique function $C \to \{*\}$, and $\Delta$ is also uniquely determined by 1.18 because of the universal property of the product $C \times C$. In fact, if $\Delta(x) = (x', x'')$, using the first half of
refeqn.axiom.counit.coalgebra we obtain

$$(*, x'') = (\epsilon, C)\Delta(x) = l^{-1}(x) = (*, x)$$

and therefore $x'' = x$. The other half implies that $x' = x$, hence $\Delta(x) = (x, x)$ for every $x \in C$.

Conversely, every set $X$ is a coalgebra with $\epsilon$ and $\Delta$ given by $\epsilon(x) = *$ and $\Delta(x) = (x, x)$.

**Remark 1.35.** The same arguments of the previous example show that if $\mathcal{C}$ is a monoidal category where the "tensor product" is a product, in the categorical sense, and the unit is a final object (the product of the empty family), then every object is a coalgebra object in a unique way.

**Example 1.36.** The **divided power coalgebra** is the vector space $D$ with basis $\{x_n \mid n \in \mathbb{N}\}$ with the linear maps

$$\Delta(x_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i}, \quad \epsilon(x_n) = \delta_{n,0}.$$

In fact, it is clear that (1.18) holds. For the coassociativity, note that $\Delta(x_n) = \sum_{k+l=n} x_k \otimes x_l$; rearranging the sum $\sum_{i+j+l=n} x_i \otimes x_j \otimes x_l$ we then obtain

$$\sum_{i+j+l=n} x_i \otimes x_j \otimes x_l = \sum_{k+l=n} \left( \sum_{i+j=k} x_i \otimes x_j \right) \otimes x_l = \sum_{k+l=n} \Delta(x_k) \otimes x_l = (\Delta \otimes D)\Delta(x_n)$$

and

$$\sum_{i+j+l=n} x_i \otimes x_j \otimes x_l = \sum_{k+l=n} x_k \otimes \left( \sum_{s+t=l} x_s \otimes x_t \right) = \sum_{k+l=n} x_k \otimes \Delta(x_l) = (D \otimes \Delta)\Delta(x_n)$$

and it follows that $\Delta$ is coassociative.

**Definition 1.37.** Let $C = (C, \Delta_C, \epsilon_C)$ and $D = (D, \Delta_D, \epsilon_D)$ be coalgebra objects in the monoidal category $(\mathcal{C}, \otimes, I)$. A morphism of coalgebra objects $f : C \to D$ is a morphism in $\mathcal{C}$ such that the diagrams below commute:

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\Delta_C"'] & D \arrow[d, "\Delta_D"] \\ C \otimes C \arrow[r, "f\otimes f"] & D \otimes D \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "f"] \arrow[d, "\epsilon_C"'] & D \arrow[dl, "\epsilon_D"'] \\ k & \end{tikzcd}$$

A motivation for considering coalgebras comes from the algebra of functions on a monoid.

**Example 1.38.** Let $k$ be a field and consider the contravariant functor from the category $\underline{\mathrm{Set}}$ to the category $\underline{\mathrm{Alg}}_k$ of $k$-algebras that sends a set $X$ to the algebra $k^X$ of $k$-valued functions on $X$, and sends a function $f : X \to Y$ to the algebra morphism $f^* : k^Y \to k^Y$, $g \in k^Y \mapsto gf \in k^X$.

If $S$ is a finite monoid then one obtains more structure on $k^S$. Let $\theta : k^* \to k$ be the canonical isomorphism $t \mapsto t(1)$; let $\{p_r; r \in S\}$ be the canonical basis of $S$, defined by $p_r(s) = \delta_{r,s}$, and let $\psi : k^S \otimes k^S \to k^{S\times S}$ be the linear isomorphism that sends $p_r \otimes p_s$ to the function $\psi(p_r \otimes p_s)(x, y) = p_r(x)p_s(y)$.

Using the presentation of $S$ as an algebra object $(S, \mu, \eta)$ in $\underline{\mathrm{Set}}$, consider the functions $\epsilon = t^{-1} \circ \eta^* : k^S \to k$ and $\Delta = \psi^{-1} \circ \mu^* : k^S \to k^S \otimes k^S$. Then the commutativity of the diagrams (1.14) and (1.15) and the fact that $X \mapsto k^X$ is a contravariant functor imply the commutativity of the diagrams (1.17) and (1.18) that define a coalgebra, thus showing that $(k^S, \Delta, \epsilon)$ is a coalgebra.

Calculating $\Delta$ and $\epsilon$ with respect to the basis $\{p_r; r \in S\}$, we obtain

$$\Delta(p_r) = \sum_{xy=r} p_x \otimes p_y, \quad \epsilon(p_r) = \delta_{r,e}$$

(where $e$ is the neutral element of the monoid $S$).

In this last example, the structures of algebra and coalgebra are *compatible* in the sense that $\Delta$ and $\epsilon$ are morphisms of algebras, and $\mu$ and $\eta$ are morphisms of coalgebras. This is a first example of a **bialgebra**, which will be the topic of the next Chapter.

We may also consider left and right modules over an algebra object. We will give the definition as if the category were strict, for it is clear where the associator must be placed, if necessary.

**Definition 1.39.** Let $\mathcal{C} = (\mathcal{C}, \otimes, I)$ be a monoidal category and let $A = (A, \mu, \eta)$ be an algebra object in $\mathcal{C}$. A **left module** over $A$ is a pair $(M, \alpha)$, where $M$ is a object of $\mathcal{C}$ and $\alpha : A \otimes M \to M$ is a morphism of $\mathcal{C}$ such that

$$\begin{tikzcd} A \otimes A \otimes M \arrow[r, "A\otimes\alpha"] \arrow[d, "\mu\otimes M"'] & A \otimes M \arrow[d, "\alpha"] \\ A \otimes M \arrow[r, "\alpha"'] & M \end{tikzcd} \qquad \begin{tikzcd} A \otimes M \arrow[r, "\alpha"] & M \\ I \otimes M \arrow[u, "\eta\otimes M"] \arrow[ur, "l_M"'] & \end{tikzcd}$$

Right $A$-modules are defined in an analogous manner.

**Definition 1.40.** Let $A$ be an algebra object in $\mathcal{C}$ and let $(M, \alpha_M)$, $(N, \alpha_N)$ be left $A$-modules. A morphism $f : M \to N$ in $\mathcal{C}$ is a morphism of $A$-modules if the diagram below commutes.

$$\begin{tikzcd} A \otimes M \arrow[r, "A\otimes f"] \arrow[d, "\alpha_M"'] & A \otimes N \arrow[d, "\alpha_N"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

As expected, if $A$ is a $k$-algebra then a left $A$-module in $\underline{\mathrm{Vect}_k}$ is the same thing as a usual left $A$-module. The next example is also simple but less obvious.

**Example 1.41.** Let $S = (S, \mu, \eta)$ be an algebra object in $\underline{\mathrm{Set}}$, i.e., a monoid; then a structure of left $S$-module on a set $X$ is the same as a left action of $S$ on $X$.

If we begin with a left $S$-module $X = (X, \alpha)$, then define a map $\Phi : S \to \mathrm{End}(X)$ by $\Phi(s)(x) = \alpha(s, x)$. Then

$$\Phi(st)(x) = \alpha(\mu(s,t), x) = \alpha(\mu, S)(s,t,x) = \alpha(S, \alpha)(s,t,x) = \alpha(s, \alpha(t,x)) = \Phi(s)(\Phi(t)(x))$$

and thus $\Phi$ is a left action. Conversely, if $X$ is an $S$-set via a morphism of monoids $\Phi : S \to \mathrm{End}(X)$ then $\alpha(s,x) = \Phi(s)(x)$ defines a structure of left $S$-module on $X$.

As before, we may consider the dual definitions.

**Definition 1.42.** Let $C = (C, \Delta, \epsilon)$ be a coalgebra object in a monoidal category $\mathcal{C}$. A **right $C$-comodule** is a pair $(M, \rho)$ where $M$ is an object in $\mathcal{C}$ and $\rho : M \to M \otimes C$ is a morphism in $\mathcal{C}$ such that the following diagrams commute.

$$\begin{tikzcd} M \arrow[r, "\rho"] \arrow[d, "\rho"'] & M \otimes C \arrow[d, "M\otimes\Delta"] \\ M \otimes C \arrow[r, "\rho\otimes C"] & M \otimes C \otimes C \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "l_M^{-1}"] \arrow[d, "\rho"'] & M \otimes I \\ M \otimes C \arrow[ur, "M\otimes\epsilon"'] & \end{tikzcd}$$

**Definition 1.43.** Let $C$ be a coalgebra object in $\mathcal{C}$ and let $(M, \rho_M)$, $(N, \rho_N)$ be two right $C$-comodules. A morphism $\rho : M \to N$ in $\mathcal{C}$ is a morphism of comodules if the diagram commutes:

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "\rho_M"] & N \arrow[d, "\rho_N"] \\ M \otimes C \arrow[r, "f\otimes C"'] & N \otimes C \end{tikzcd}$$

**Example 1.44.** Consider the $k$-vector space $k\mathbb{Z}$ generated by $\mathbb{Z}$, which is a coalgebra via

$$\Delta(n) = n \otimes n, \quad \varepsilon(n) = 1$$

for all $n \in \mathbb{Z}$ (as in Example 1.32). Then a right $k\mathbb{Z}$-comodule is the same as a $\mathbb{Z}$-graded vector space.

If $V$ is a vector space with a $\mathbb{Z}$-grading and $v = \sum_n v_n \in V$, define $\rho(v) = \sum_n v_n \otimes n$. In other words, this map sends a homogeneous element $v$ to the element $v \otimes \deg(v)$. It is easy to see that $\rho : V \to V \otimes k\mathbb{Z}$ is a right comodule over the coalgebra $k\mathbb{Z}$. Conversely, if $(V, \rho)$ is a right $k\mathbb{Z}$-comodule, the $\mathbb{Z}$-grading of $V$ comes from defining $V_n = \rho^{-1}(n)$: given a vector $v \in V$ one has

$$\rho(v) = \sum_m v_m \otimes m$$

and by the second diagram we get

$$v \otimes 1 = (M \otimes \epsilon)\rho(v) = \sum_m v_m \otimes \epsilon(m) = (\sum_m v_m) \otimes 1$$

and therefore $v = \sum_m v_m$. Now, using the first diagram in the definition of comodule we get

$$\sum_m v_m \otimes m \otimes m = (M \otimes \Delta)\rho(v) = (\rho \otimes M)\rho(v) = \sum_m \rho(v_m) \otimes m.$$

If the functionals $f_n : k\mathbb{Z} \to k$ are defined by $f_n(m) = \delta_{n,m}$, for each $m, n$ in $\mathbb{Z}$, then applying $M \otimes f_n$ to the previous equations we conclude that $\rho(v_m) = v_m \otimes m$ for each $v_m$. Therefore $v \in \sum_{v_m \neq 0} V_m$, and $V = \sum_m V_m$. This sum is direct: if

$$u_{m_1} + \cdots + u_{m_k} = 0$$

for $m_1, \ldots, m_k \in \mathbb{Z}$ then

$$0 = \rho(u_{m_1} + \cdots + u_{m_k}) = u_{m_1} \otimes m_1 + \cdots + u_{m_k} \otimes m_k$$

from which follows that $u_{m_1} = \cdots = u_{m_k} = 0$.

**Example 1.45.** If $G$ is a group, analogous arguments of the last example show that a right $kG$-comodule is the same as a $G$-graded vector space.

# Chapter 2

# Bialgebras and Hopf Algebras

## 2.1 Braided monoidal categories

As we have seen in the previous chapter, in any monoidal category there can be algebra objects and coalgebras objects. It happens even that some objects in a monoidal category have both structures, that is, it is at the same time an algebra and a coalgebra. In this chapter, we are going to study objects in which both structures exist in such a way that they are compatible. More specifically, suppose that an object $B$ in a monoidal category $(\mathcal{C}, \otimes, I)$ is both an algebra and a coalgebra object in this category. The there are four morphisms, $(\mu : B \otimes B \to B,\ \eta : I \to B, \Delta : B \to B \otimes B,\ \epsilon : B \to I)$, such that $(B, \mu, \eta)$ is an algebra an $(B, \Delta, \epsilon)$ is a coalgebra. In order to say that these two structures are compatible, one should expect at least that the comultiplication $\Delta : B \to B \otimes B$ and the counit $\epsilon : B \to I$ are morphisms of algebras, whereas the multiplication $\mu : B \otimes B \to B$ and the unit $\eta : I \to B$ are morphisms of coalgebras. The unit object $I_{\mathcal{C}}$ has naturally a structure of algebra, $(I, \mu = l_I = r_I, \eta = I)$, and a structure of coalgebra, $(I, \Delta = l_I^{-1} = r_I^{-1}, \epsilon = I)$. The main technical difficulty is to define an algebra structure on the tensor product of two algebras and a coalgebra structure on the tensor product of two coalgebras. This is not possible to do in any monoidal category, the category must also be braided.

**Definition 2.1.** A monoidal category $(\mathcal{C}, \otimes, I)$ is a braided monoidal category if there is a natural isomorphism $\sigma : \otimes \Rightarrow \otimes\tau$, where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip functor defined as $\tau(M, N) = (N, M)$. The naturality means that for every pair of objects $M, N \in \mathcal{C}$ there is an isomorphism $\sigma_{M,N} : M \otimes N \to N \otimes M$ such that the following square commutes for every pair of morphisms $f : M \to M'$ and $g : N \to N'$

$$\begin{tikzcd}
M \otimes N \arrow[r, "\sigma_{M,N}"] \arrow[d, "f\otimes g"'] & N \otimes M \arrow[d, "g\otimes f"] \\
M' \otimes N' \arrow[r, "\sigma_{M',N'}"] & N' \otimes M'
\end{tikzcd}$$

Moreover, the following commutativity axioms have to be satisfied

$$
\begin{tikzcd}
(M\otimes N)\otimes P \arrow[r, "a_{M,N,P}"] \arrow[d, "\sigma_{M,N}\otimes P"'] & M\otimes(N\otimes P) \arrow[d, "\sigma_{M,(N\otimes P)}"] \\
(N\otimes M)\otimes P \arrow[d, "a_{N,M,P}"'] & (N\otimes P)\otimes M \arrow[d, "a_{N,P,M}"] \\
N\otimes(M\otimes P) \arrow[r, "N\otimes\sigma_{M,P}"] & N\otimes(P\otimes M)
\end{tikzcd}
\qquad
\begin{tikzcd}
M\otimes(N\otimes P) \arrow[r, "a^{-1}_{M,N,P}"] \arrow[d, "M\otimes\sigma_{N,P}"'] & (M\otimes N)\otimes P \arrow[d, "\sigma_{(M\otimes N),P}"] \\
M\otimes(P\otimes N) \arrow[d, "a^{-1}_{M,P,N}"'] & P\otimes(M\otimes N) \arrow[d, "a^{-1}_{P,M,N}"] \\
(M\otimes P)\otimes N \arrow[r, "\sigma_{M,P}\otimes N"] & (P\otimes M)\otimes N
\end{tikzcd}
$$

A braided monoidal category $(\mathcal{C},\otimes,I,\sigma)$ is symmetric if for any pair of objects $M,N\in\mathcal{C}$ we have $\sigma_{N,M}\circ\sigma_{M,N}=M\otimes N$

As examples of braided monoidal categories, we have

**Example 2.2.** The monoidal category $(\underline{\mathrm{Vect}}_k,\otimes,k)$ of vector spaces over a field $k$ is naturally braided with the flip morphism $\tau_{\mathbb{V},\mathbb{W}}:\mathbb{V}\otimes\mathbb{W}\to\mathbb{W}\otimes\mathbb{V}$ given by $\tau(v\otimes w)=w\otimes v$. This category is symmetric.

**Example 2.3.** The monoidal category $(\underline{\mathrm{Set}},\times,\{*\})$ is braided with the flip map $\tau_{X,Y}:X\times Y\to Y\times X$, given by $\tau(x,y)=(y,x)$. This category is also symmetric.

**Example 2.4.** As an example of a non-symmetric braided monoidal category, let us consider the category of crossed $G$ sets. Given a group $G$, a crossed $G$ set is a triple $(X,\alpha,|\,|)$ in which $X$ is a set, $\alpha:G\times X\to X$ is a left action of $G$ on $X$ and $|\,|$ is a map $|\,|:X\to G$ satisfying

$$|\alpha_g(x)|=g|x|g^{-1}.$$

A morphism between the crossed $G$ sets $X=(X,\alpha,|\,|)$ and $Y=(Y,\beta,\|\,\|)$ is a function $f:X\to Y$ such that for every $g\in G$ and $x\in X$,

$$f(\alpha_g(x))=\beta_g(f(x)),\quad \|f(x)\|=|x|.$$

The category of crossed $G$ sets is denoted by $X(G)$, which is a monoidal category with tensor product between $X=(X,\alpha,|\,|)$ and $Y=(Y,\beta,\|\,\|)$ given by

$$X\otimes Y=(X\times Y,\alpha\times\beta,|\,|_2),$$

where $|(x,y)|_2=|x|\|y\|$, and $(\alpha\times\beta)_g(x,y)=(\alpha_g(x),\beta_g(y))$. The unit object is given by $I=(\{*\},\{*\},|\,|_0)$, with $|*|_0=e$. This category is strict, because the tensor product of an arbitrary number of objects can be built straightforwardly from the cartesian product among the underlying sets. The braiding in the category of crossed $G$ sets is given by the family of maps

$$\begin{array}{rccc}\sigma_{X,Y}: & X\otimes Y & \to & Y\otimes X\\ & (x,y) & \mapsto & (\beta_{|x|}(y),x)\end{array}$$

It is easy to see that $\sigma_{X,Y}$ is a morphism of crossed $G$ sets, indeed

$$\begin{aligned}\sigma_{X,Y}(\alpha\times\beta)_g(x,y) &= \sigma_{X,Y}(\alpha_g(x),\beta_g(y))=(\beta_{|\alpha_g(x)|}\beta_g(y),\alpha_g(x))\\ &= (\beta_{g|x|g^{-1}}\beta_g(y),\alpha_g(x))=(\beta_g\beta_{|x|}(y),\alpha_g(x))\\ &= (\beta\times\alpha)_g(\beta_{|x|}(y),x)=(\beta\times\alpha)_g\sigma_{X,Y}(x,y),\end{aligned}$$

and

$$|\sigma_{X,Y}(x,y)|_2=|(\beta_{|x|}(y),x)|_2=\|\beta_{|x|}(y)\||x|=|x|\|y\||x|^{-1}|x|=|x|\|y\|=|(x,y)|_2.$$

Let us verify only one of the commutativity axioms for $\sigma$, consider $X = (X, \alpha, ||_X)$, $Y = (Y, \beta, ||_Y)$ and $Z = (Z, \gamma, ||_Z)$, as the category is strict, we need to show only that $\sigma_{X,Y\otimes Z} = (Y \otimes \sigma_{X,Z})(\sigma_{X,Y} \otimes Z)$. On one hand we have

$$\sigma_{X,Y\otimes Z}(x, y, z) = ((\beta \times \gamma)_{|x|_X}(y, z), x) = (\beta_{|x|_X}(y), \gamma_{|x|_X}(z), x).$$

On the other hand,

$$(Y \otimes \sigma_{X,Z})(\sigma_{X,Y} \otimes Z)(x, y, z) = (Y \otimes \sigma_{X,Z})(\beta_{|x|_X}(y), x, z) = (\beta_{|x|_X}(y), \gamma_{|x|_X}(z), x),$$

and this concludes our proof. The other commutativity axiom is analogous. Therefore, the category $X(G)$ of crossed $G$ sets is braided monoidal. It is easy to see that it is not symmetric.

## 2.2 Bialgebras

In a braided monoidal category, the tensor product of two algebra objects is an algebra object and the tensor product of two coalgebra objects is a coalgebra.

**Theorem 2.5.** *Let $(\mathcal{C}, \otimes, I, \sigma)$ be a (strict) braided monoidal category.*

1. *If $A$ and $B$ are two algebra objects in $\mathcal{C}$ then their tensor product $A \otimes B$ is also an algebra object.*
2. *If $C$ and $D$ are two coalgebra objects in $\mathcal{C}$, then their tensor product is also a coalgebra object.*

*Proof.* (1) Define in $A \otimes B$ the multiplication and unit maps

$$\mu_{A\otimes B} = (\mu_A \otimes \mu_B)(A \otimes \sigma_{B,A} \otimes B), \quad \eta_{A\otimes B} = \eta_A \otimes \eta_B.$$

First, due to the naturality of the braiding we have the following relations

$$\sigma_{B,A}(\mu_B \otimes A) = (A \otimes \mu_B)\sigma_{B\otimes B,A}$$

and

$$\sigma_{B,A}(B \otimes \mu_A) = (\mu_A \otimes B)\sigma_{B,A\otimes A}.$$

Let us verify the associativity

$$\begin{aligned}
& \mu_{A\otimes B}(\mu_{A\otimes B} \otimes A \otimes B) = \mu_{A\otimes B}(\mu_A \otimes \mu_B \otimes A \otimes B)(A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \sigma_{B,A} \otimes B)(\mu_A \otimes \mu_B \otimes A \otimes B)(A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(\mu_A \otimes A \otimes \mu_B \otimes B)(A \otimes A \otimes \sigma_{B\otimes B,A} \otimes B)(A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \mu_A \otimes B \otimes \mu_B)(A \otimes A \otimes \sigma_{B,A} \otimes B \otimes B) \\
& (A \otimes A \otimes B \otimes \sigma_{B,A} \otimes B)(A \otimes \sigma_{B,A} \otimes B \otimes A \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \mu_A \otimes B \otimes \mu_B)(A \otimes A \otimes \sigma_{B,A} \otimes B \otimes B) \\
& (A \otimes \sigma_{B,A} \otimes A \otimes B \otimes B)(A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \mu_A \otimes B \otimes \mu_B)(A \otimes \sigma_{B,A\otimes A} \otimes B \otimes B)(A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \sigma_{B,A} \otimes B \otimes \mu_B)(A \otimes B \otimes \mu_A \otimes B \otimes B)(A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \sigma_{B,A} \otimes B)(A \otimes B \otimes \mu_A \otimes \mu_B)(A \otimes B \otimes A \otimes \sigma_{B,A} \otimes B) \\
= \ & \mu_{A\otimes B}(A \otimes B \otimes \mu_{A\otimes B})
\end{aligned}$$

For the axiom of unit we have

$$\begin{aligned}
& \mu_{A\otimes B}(A \otimes B \otimes \eta_{A\otimes B}) = (\mu_A \otimes \mu_B)(A \otimes \sigma_{B,A} \otimes B)(A \otimes B \otimes \eta_A \otimes \eta_B) \\
= \ & (\mu_A \otimes \mu_B)(A \otimes \eta_A \otimes B \otimes \eta_B) = ((\mu_A(A \otimes \eta_A)) \otimes (\mu_B(B \otimes \eta_B))) \\
= \ & A \otimes B
\end{aligned}$$

The other side is analogous. Therefore $(A \otimes B, \mu_{A\otimes B}, \eta_{A\otimes B})$ is an algebra. ◻

**Proposition 2.6.** *Let $(\mathcal{C}, \otimes, I, \sigma)$ be a (strict) braided monoidal category and let $(B, \mu, \eta, \Delta, \epsilon)$ be an object in $\mathcal{C}$ which is, simultaneously, an algebra object and a coalgebra object. Then the following affirmatives are equivalent:*

*(a) The maps $\Delta$ and $\epsilon$ are morphisms of algebras.*

*(b) The maps $\mu$ and $\eta$ are morphisms of coalgebras.*

*Proof.* Consider the following diagram,

$$\begin{tikzcd}
B \otimes B \arrow[r, "\mu"] \arrow[d, "\Delta\otimes\Delta"'] & B \arrow[r, "\Delta"] & B \otimes B \\
B \otimes B \otimes B \otimes B \arrow[rr, "B\otimes\sigma_{B,B}\otimes B"] & & B \otimes B \otimes B \otimes B \arrow[u, "\mu\otimes\mu"']
\end{tikzcd}$$

This diagram can be read in two alternative ways, either as

$$\Delta \circ \mu = \mu_{B\otimes B} \circ (\Delta \otimes \Delta), \tag{2.1}$$

meaning that $\Delta$ is multiplicative, or as

$$\Delta \circ \mu = (\mu \otimes \mu) \circ \Delta_{B\otimes B}, \tag{2.2}$$

meaning that $\mu$ is comultiplicative.

For the units and counits, let us recall that $\mu_I = l_I = r_I$ , $\Delta_I = l_I^{-1} = r_I^{-1}$ and $\eta_I = \epsilon_I = I$. We are assuming that the monoidal category is strict, this implies that $l_I$ and $r_I$ can be viewed as identities. Consider the following diagrams

$$\begin{tikzcd}
I \arrow[r, "\Delta_I"] \arrow[d, "\eta"'] & I \otimes I \arrow[d, "\eta\otimes\eta"] \\
B \arrow[r, "\Delta"] & B \otimes B
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
B \otimes B \arrow[r, "\mu"] \arrow[d, "\epsilon\otimes\epsilon"'] & A \arrow[d, "\epsilon"] \\
I \otimes I \arrow[r, "\mu_I"] & I
\end{tikzcd}$$

$$\begin{tikzcd}
I \arrow[rr, "I"] \arrow[dr, "\eta"'] & & I \\
& B \arrow[ur, "\epsilon"'] &
\end{tikzcd}$$

The first can be read either as the map $\Delta$ is unital or as the map $\eta$ is comultiplicative. The second interchanges the roles of the multiplication map $\mu$ as a counital map or the counit $\epsilon$ as a multiplicative map. The third puts in the same footing the unit and the counit, one preserving the other. Putting all these informations together, we obtain the stated equivalence. $\square$

**Definition 2.7.** A bialgebra in a monoidal category is an object $B$ which is at the same time an algebra object and a coalgebra object, and the comultiplication and counit maps are morphisms of algebras.

**Definition 2.8.** A bialgebra morphism between two bialgebras $A$ and $B$ in a braided monoidal category $\mathcal{C}$ is a morphism $f \in \mathcal{C}(A, B)$ which is at the same time a morphism of algebras and a morphism of coalgebras.

**Remark 2.9.** In the category of $k$ vector spaces, $\underline{\text{Vect}}_k$, we usually denote the multiplication map on an algebra $A$ as

$$\mu(\sum_{i=1}^{n} a_i \otimes b_i) = \sum_{i=1}^{n} a_i b_i$$

and the unit map as $\eta(\lambda) = \lambda 1_A$, for $\lambda \in k$. In the category of sets we denote the multiplication on an algebra object $A$ (a monoid) by $\mu(a,b) = ab$ and the unit $\eta(*) = 1$.

**Example 2.10.** In the category of sets, $(\underline{\text{Set}}, \times, \{*\}, \tau)$, every object is a coalgebra object with $\Delta$ being the diagonal map and $\epsilon$ being the unique morphism onto the singleton. The algebra objects are the monoids. Due to the fact that the braiding in $\underline{\text{Set}}$ is the flip map, then the monoid structure on the cartesian product of monoids is the direct product structure, that is, if $M$ and $N$, the set $M \times N$ is a monoid with multiplication $(x,y)(x',y') = (xx', yy')$ and unit $1 = (1,1)$. Therefore it is easy to see that every monoid is a bialgebra in $\underline{\text{Set}}$.

**Example 2.11.** In the category of crossed $G$ sets, $(X(G), \times, \{*\}, \sigma)$, the algebra objects are monoids $X$, such that the action of the group $G$ is given by automorphisms of $X$ and the map $|\,| : X \to G$ is also multiplicative with $|1_X| = e$. The coalgebra objects, with the comultiplication given by the diagonal map are the objects $(X, \alpha, |\,|)$ such that $|x| = e$ for all $x \in X$, this is because the map $\Delta$ must be a morphism of crossed $G$ sets between $X$ and $X \times X$, then $|x|^2 = |(x,x)|_2 = |x|$. In this case, $X$ is automatically a bialgebra, indeed, let us verify that $\Delta$ is multiplicative

$$\begin{aligned}
(\mu \times \mu)\Delta_{X \times X}(x,y) &= (\mu \times \mu)(X \times \sigma_{X,X} \times X)(\Delta \times \Delta)(x,y) \\
&= (\mu \times \mu)(X \times \sigma_{X,X} \times X)(x,x,y,y) \\
&= (\mu \times \mu)(x, \alpha_{|x|}(y), x, y) \\
&= (\mu \times \mu)(x, \alpha_e(y), x, y) \\
&= (\mu \times \mu)(x, y, x, y) \\
&= (xy, xy) \\
&= \Delta(xy) = \Delta\mu(x,y)
\end{aligned}$$

**Example 2.12.** In the category of vector spaces over a field $k$, $(\underline{\text{Vect}}_k, \otimes, k, \tau)$. There are several well known examples of bialgebras, some of them were already seen in these lecture notes as motivating examples:

1. The group algebra $kG$ of a group $G$, with multiplication induced by the group, the unit being the neutral element of the group, the comultiplication given by $\Delta(g) = g \otimes g$ and counit $\epsilon(g) = 1$ for all $g \in G$.

2. The algebra $R(G)$ of representative functions of group $G$. Recalling that a function$f : G \to k$ is called representative if there is a representation $\pi : G \to GL(V)$ on a finite dimensional $k$ vector space $V$, a vector $v \in V$ and a functional $\phi \in V^*$ such that $f(g) = \phi(\pi(g)(v))$ for all $g \in G$. The product of representative functions is done pointwise, then $R(G)$ is commutative, and the unit is the constant function $1(g) = 1$, which is representative, because of the trivial representation $\epsilon : G \to k^\times$ given by $\epsilon(g) = 1$, in this case, $v = \phi = 1 \in k$. The comultiplication on a representative function $f$ is a function $\Delta(f) : G \times G \to k$, defined by $\Delta(f)(g,h) = f(gh)$, for all $g, h \in G$, and the counit of $f$ is given by $\epsilon(f) = f(e)$, where $e$ is the neutral element of the group $G$. In order to show that $\Delta(f) \in R(G) \otimes R(G)$, consider the finite dimensional representation $\pi : G \to GL(V)$, the vector $v \in V$ and the functional $\phi \in V^*$ such that $f(g) = \phi(\pi(g)(v))$. Take

$\{e_i\}_{i=1}^n$ a basis of $V$ and $\{\varepsilon^i\}_{i=1}^n$ its dual basis in $V^*$, then we have

$$\begin{aligned}
\Delta(f)(g,h) &= f(gh) = \phi(\pi(gh)(v)) = \phi(\pi(g)(\pi(h)(v))) \\
&= \sum_{i=1}^n \phi(\pi(g)e_i\varepsilon^i(\pi(h)(v))) = \sum_{i=1}^n \phi(\pi(g)e_i)\varepsilon^i(\pi(h)(v)) \\
&= \sum_{i=1}^n f_i'(g)f_i''(h)
\end{aligned}$$

where the functions $f_i'$ and $f_i''$ are representative functions, given by $f_i'(g) = \phi(\pi(g)(e_i))$ and $f_i''(g) = \varepsilon^i(\pi(g)(v))$. Therefore $\Delta(f) = \sum_{i=1}^n f_i' \otimes f_i'' \in R(G) \otimes R(G)$. It is straightforward to show that the comultiplication and the counit are algebra morphisms.

3. The dual of the group algebra $(kG)^* = k^G$, for a finite group $G$, generated by the elements $\{p_g\}_{g\in G}$, with multiplication $p_g p_h = \delta_{g,h}p_g$, unit $1 = \sum_{g\in G} p_g$, comultiplication given by $\Delta(p_g) = \sum_{hk=g} p_h \otimes p_k$ and counit $\epsilon(p_g) = \delta_{g,e}$. It is easy to see that these maps define a structure of a bialgebra on $(kG)^*$.

4. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. The comultiplication in $\mathcal{U}(\mathfrak{g})$, calculated on an element $\xi = X_1.X_2 \dots X_n$, with $X_i \in \mathfrak{g}$ is given by

$$\Delta(\xi) = \xi \otimes 1 + 1 \otimes \xi + \sum_{\sigma \in S_{n,k}} X_{\sigma(1)} \dots X_{\sigma(k)} \otimes X_{\sigma(k+1)} \dots X_{\sigma(n)},$$

where $S_{n,k} \subseteq S_n$ is the set of $(n,k)$ shuffles, that is, permutations of $n$ elements such that $\sigma(1) < \cdots < \sigma(k)$ and $\sigma(k+1) < \cdots < \sigma(n)$. In particular, for $\xi = X \in \mathfrak{g}$, we have $\Delta(X) = X \otimes 1 + 1 \otimes X$. We put also $\Delta(1) = 1 \otimes 1$. The counit is given by $\epsilon(1) = 1$ and $\epsilon(\xi) = 0$ for $\xi = X_1 \dots X_n$. It is straightforward to show that the comultiplication and the counit are multiplicative.

5. The divided power bialgebra $D = \{x_n \mid n \in \mathbb{N}\}$ with algebra structure given by

$$x_m x_n = \binom{m+n}{m} x_{m+n}, \quad 1_D = x_0,$$

and coalgebra structure given by

$$\Delta(x_n) = \sum_{i=0}^n x_i \otimes x_{n-i}, \quad \epsilon(x_n) = \delta_{n,0}.$$

These structures define a bialgebra. Indeed, let us verify, for example that the comultiplication is multiplicative

$$\begin{aligned}
\Delta(x_m)\Delta(x_n) &= \left(\sum_{i=0}^m x_i \otimes x_{m-i}\right)\left(\sum_{j=0}^n x_j \otimes x_{n-j}\right) \\
&= \sum_{i,j} \binom{i+j}{i} x_{i+j} \otimes \binom{m+n-(i+j)}{m-i} x_{m+n-(i+j)} \\
&= \sum_{k=0}^{m+n} \left(\sum_{i=0}^m \binom{k}{i}\binom{m+n-k}{m-i}\right) x_k \otimes x_{m+n-k} \\
&= \sum_{k=0}^{m+n} \binom{m+n}{m} x_k \otimes x_{m+n-k} \\
&= \Delta\left(\binom{m+n}{m} x_{m+n}\right) \\
&= \Delta(x_m x_n)
\end{aligned}$$

## 2.3 Bialgebras and their category of modules

There is another useful characterization of bialgebras via their categories of modules. First let us study in detail the case of bialgebras in the category of vector spaces over a field $k$. Let $B$ be a $k$ algebra, its module category ${}_B\mathcal{M}$ is a subcategory of $\underline{\text{Vect}}_k$. We define the forgetful functor $U : {}_B\mathcal{M} \to \underline{\text{Vect}}_k$ as the identity functor on objects and morphisms, but only keeping their underlying structure, that is, every $B$ module is a vector space and every morphism of $B$ modules is a $k$ linear map. Then, we have the following result

**Theorem 2.13.** *Let $B$ be a $k$ algebra, then $B$ is a bialgebra over $k$ if, and only if its category of modules, ${}_B\mathcal{M}$ is monoidal and the forgetful functor $U : {}_B\mathcal{M} \to \underline{Vect}_k$ is a strict monoidal functor.*

*Proof.* By saying that the forgetful functor is a strict monoidal functor, we mean that the tensor product of $B$ modules is given by the ordinary tensor product over $k$ of their underlying vector spaces and that the unit object in the category ${}_B\mathcal{M}$ is nothing more than the base field $k$, which is the unit object of $\underline{\text{Vect}}_k$.

Suppose first that $B$ is a bialgebra. Then there are two algebra morphisms $\Delta : B \to B \otimes B$ and $\epsilon : B \to k$ satisfying the coassociativity and counit axioms. Let $M$ and $N$ be two left $B$ modules, define on the tensor product $M \otimes N$ the following $B$ module structure:

$$\begin{array}{rccc} \cdot : & B \otimes M \otimes N & \to & M \otimes N \\ & a \otimes m \otimes n & \mapsto & a \cdot (m \otimes n) \end{array}$$

given by $a \cdot (m \otimes n) = (a_{(1)} \cdot m) \otimes (a_{(2)} \cdot n)$. One can show that this is indeed a $B$ module structure on $M \otimes N$,

$$1_B \cdot (m \otimes n) = (1_B \cdot m) \otimes (1_B \cdot n) = m \otimes n,$$

and

$$\begin{aligned} a \cdot (b \cdot (m \otimes n)) &= a \cdot ((b_{(1)} \cdot m) \otimes (b_{(2)} \cdot n)) \\ &= (a_{(1)} \cdot (b_{(1)} \cdot m)) \otimes (a_{(2)} \cdot (b_{(2)} \cdot n)) \\ &= (a_{(1)} b_{(1)} \cdot m) \otimes (a_{(2)} b_{(2)} \cdot n) \\ &= ((ab)_{(1)} \cdot m) \otimes ((ab)_{(2)} \cdot n) \\ &= ab \cdot (m \otimes n). \end{aligned}$$

On the base field $k$, we have the following $B$ module structure:

$$\begin{array}{rccc} \cdot : & B \otimes k & \to & k \\ & a \otimes \lambda & \mapsto & a \cdot \lambda = \epsilon(a)\lambda \end{array}$$

Again, it is easy to see that this is a $B$ module structure, because

$$1_B \cdot \lambda = \epsilon(1_B)\lambda = 1.\lambda = \lambda,$$

and

$$a \cdot (b \cdot \lambda) = a \cdot (\epsilon(b)\lambda) = \epsilon(a)\epsilon(b)\lambda = \epsilon(ab)\lambda = ab \cdot \lambda.$$

The associator map and the left and right unit maps are $B$ module morphisms, indeed

$$\begin{aligned} a \cdot ((m \otimes n) \otimes p) &= (a_{(1)} \cdot (m \otimes n)) \otimes (a_{(2)} \cdot p) \\ &= (a_{(1)(1)} \cdot m) \otimes (a_{(1)(2)} \cdot n) \otimes (a_{(2)} \cdot p) \\ &= (a_{(1)} \cdot m) \otimes (a_{(2)(1)} \cdot n) \otimes (a_{(2)(2)} \cdot p) \\ &= (a_{(1)} \cdot m) \otimes (a_{(2)} \cdot (n \otimes p)) \\ &= a \cdot (m \otimes (n \otimes p)), \end{aligned}$$

and

$$\begin{aligned}
a \cdot l_B(\lambda \otimes m) &= a \cdot (\lambda m) = \lambda(a \cdot m) \\
&= \lambda(\epsilon(a_{(1)})a_{(2)} \cdot m) = \epsilon(a_{(1)})\lambda(a_{(2)} \cdot m) \\
&= l_B((a_{(1)} \cdot \lambda) \otimes (a_{(2)} \cdot m)) = l_B(a \cdot (\lambda \otimes m)),
\end{aligned}$$

and analogously for the right unit map $r_B$. Therefore, ${}_B\mathcal{M}$ is a monoidal category and the forgetful functor, which sees the underlying vector space structure, is monoidal and strict.

On the other hand, consider an algebra $B$ such that its module category ${}_B\mathcal{M}$ is monoidal and the forgetful functor is monoidal and strict. As $B$ is a left $B$ module by the left multiplication, then $B \otimes B \in {}_B\mathcal{M}$, define the linear map

$$\begin{aligned}
\Delta : \quad B &\to B \otimes B \\
a &\mapsto a \cdot (1_B \otimes 1_B)
\end{aligned}$$

One can easily prove that this map is a coassociative map. Indeed, denoting $a \cdot (1_B \otimes 1_B) = a_{(1)} \otimes a_{(2)}$ we have

$$\begin{aligned}
a_{(1)} \otimes a_{(2)(1)} \otimes a_{(2)(2)} &= a_{(1)} \otimes (a_{(2)} \cdot (1_B \otimes 1_B)) \\
&= a \cdot (1_B \otimes (1_B \otimes 1_B)) \\
&= a \cdot ((1_B \otimes 1_B) \otimes 1_B) \\
&= (a_{(1)} \cdot (1_B \otimes 1_B)) \otimes a_{(2)} \\
&= a_{(1)(1)} \otimes a_{(1)(2)} \otimes a_{(2)}.
\end{aligned}$$

In order to verify the other properties, first, note that for any $B$ modules $M$ and $N$ and elements $m \in M$ and $n \in N$ we can define the maps $\rho_m : B \to M$ and $\rho_n : B \to N$ given, respectively by $\rho_m(a) = a \cdot m$ and $\rho_n(a) = a \cdot n$, these maps are clearly morphisms of left $B$ modules. By the functoriality of the tensor product, we can see that, for any $m \in M$ and $n \in N$, the map $(\rho_m \otimes \rho_n) : B \otimes B \to M \otimes N$ given by $(\rho_m \otimes \rho_n)(a \otimes b) = a \cdot m \otimes b \cdot n$ is a morphism of $B$ modules as well, then we can write

$$\begin{aligned}
a \cdot (m \otimes n) &= a \cdot ((\rho_m \otimes \rho_n)(1_B \otimes 1_B)) = (\rho_m \otimes \rho_n)(a \cdot (1_B \otimes 1_B)) \\
&= (\rho_m \otimes \rho_n)(a_{(1)} \otimes a_{(2)}) = (a_{(1)} \cdot m) \otimes (a_{(2)} \cdot n).
\end{aligned}$$

With these facts, we can prove that $\Delta$ is an algebra morphism

$$\begin{aligned}
\Delta(a)\Delta(b) &= (a_{(1)} \otimes a_{(2)})(b_{(1)} \otimes b_{(2)}) = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \\
&= \rho_{b_{(1)}}(a_{(1)})\rho_{b_{(2)}}(a_{(2)}) = (\rho_{b_{(1)}} \otimes \rho_{b_{(2)}})(a_{(1)} \otimes a_{(2)}) \\
&= (\rho_{b_{(1)}} \otimes \rho_{b_{(2)}})(a \cdot (1_B \otimes 1_B)) = a \cdot ((\rho_{b_{(1)}} \otimes \rho_{b_{(2)}})(1_B \otimes 1_B)) \\
&= a \cdot (b_{(1)} \otimes b_{(2)}) = a \cdot (b \cdot (1_B \otimes 1_B)) \\
&= ab \cdot (1_B \otimes 1_B) = \Delta(ab).
\end{aligned}$$

As $k$ has a $B$ module structure, define the linear map

$$\begin{aligned}
\epsilon : \quad B &\to k \\
a &\mapsto a \cdot 1
\end{aligned}$$

This defines a counit in $B$, indeed

$$\begin{aligned}
(B \otimes \epsilon)\Delta(a) &= a_{(1)}\epsilon(a_{(2)}) = r_B(a_{(1)} \otimes \epsilon(a_{(2)})) \\
&= r_B(a_{(1)} \otimes (a_{(2)} \cdot 1)) = r_B((\rho_{1_B} \otimes \rho_1)(a_{(1)} \otimes a_{(2)})) \\
&= r_B((\rho_{1_B} \otimes \rho_1)(a \cdot (1_B \otimes 1_B))) = r_B(a \cdot (1_B \otimes 1)) \\
&= a \cdot r_B(1_B \otimes 1) = a \cdot 1_B = a1_B = a,
\end{aligned}$$

and a similar proof for $(\epsilon \otimes B)\Delta(a) = a$. Finally, the counit is also multiplicative,

$$\epsilon(ab) = ab \cdot 1 = a \cdot (b \cdot 1) = a \cdot (\epsilon(b)) = \epsilon(a)\epsilon(b).$$

Therefore, $B$ has a structure of a bialgebra over $k$. □

For the case of a general braided monoidal category we have a similar result, which we will state without proof in these notes.

**Theorem 2.14.** *Let $(\mathcal{C}, \otimes, I, \sigma)$ be a (strict) braided monoidal category and let $(B, \mu_B, \eta_B)$ be an algebra object in this category. Suppose that $I$ is a generator of $\mathcal{C}$ and that the functors $X \otimes \_$ and $\_ \otimes X$ preserve epimorphisms and coproducts for all $X \in \mathcal{C}$. Then there is a bijective correspondence between*

- *Monoidal structures on the module category ${}_B\mathcal{M}$ such that the forgetful functor $U : {}_B\mathcal{M} \to \mathcal{C}$ is a strict monoidal functor.*
- *Bialgebra structures $(B, \mu_B, \eta_B, \Delta_B, \epsilon_B)$ on $B$.*

Everything which has been done so far can be made if we consider the category of right $B$ modules instead of the category of left $B$ modules. There is yet another variation of the same result if we start with a coalgebra object $B$ on a braided monoidal category and demand that its category of left or right comodules should be a monoidal category the forgetful functor is strict monoidal. Then we have also a bialgebra

## 2.4 Hopf Algebras

In this section we will present all the constructions in the category $\underline{\text{Vect}}_k$, of vector spaces over a field $k$. Let $(C, \Delta, \epsilon)$ be a coalgebra and $(A, \mu, \eta)$ be an algebra. Then the vector space $\text{Hom}_k(C, A)$ of $k$ linear transformations from $C$ to $A$ has a structure of an associative algebra, namely the convolution algebra, with product given by

$$f * g = \mu \circ (f \otimes g) \circ \Delta$$

and unit given by

$$\mathbf{1} = \eta \circ \epsilon.$$

If $H$ is a bialgebra, then the vector space $\text{End}_k(H)$ is a convolution algebra, because $H$ is at the same time an algebra and a coalgebra.

**Definition 2.15.** A bialgebra $H$ is a Hopf algebra if the identity map is invertible in the convolution algebra $\text{End}_k(H)$.

Let $H$ be a Hopf algebra and let us denote by $S$ the inverse by convolution of the identity map. This map is called the antipode, and the fact that $S$ is the convolution inverse of the identity can be written explicitly as

$$\mu \circ (S \otimes H) \circ \Delta = \eta \circ \epsilon = \mu \circ (H \otimes S) \circ \Delta,$$

or in the Sweedler notation, for $h \in H$,

$$S(h_{(1)})h_{(2)} = \epsilon(h)1_H = h_{(1)}S(h_{(2)}).$$

**Example 2.16.** The examples of bialgebras, presented in the previous chapter are also examples of Hopf algebras:

1. The group algebra $kG$ of a group $G$. The antipode is given by $S(g) = g^{-1}$.

2. The algebra $R(G)$ of representative functions of group $G$. The antipode of a representative function $f$ is given by $S(f)(g) = f(g^{-1})$. This new function is representative because of the contragradient representation. More explicitly, given a finite dimensional representation $\pi : G \to GL(V)$, a vector $v \in V$ and a functional $\phi \in V^*$ such that $f(g) = \phi(\pi(g)(v))$, we consider the contragradient representation $\pi^* : G \to GL(V^*)$ defined as $(\pi^*(g)(\phi))(v) = \phi(\pi(g^{-1})(v))$, for any $\phi \in V^*$ and $v \in V$. Then we have

$$Sf(g) = f(g^{-1}) = \phi(\pi(g^{-1})(v)) = (\pi^*(g)(\phi))(v) = \mathrm{ev}_v(\pi^*(g)(\phi)).$$

3. The dual of the group algebra $(kG)^* = k^G$, for a finite group $G$, in this case, the antipode is given by $S(p_g) = p_{g^{-1}}$.

4. The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, in this algebra, the antipode is given by $S(1) = 1$, $S(X_1 \dots X_n) = (-1)^n X_n \dots X_1$.

5. The divided power bialgebra $D = \{x_n \,|\, n \in \mathbb{N}\}$, in this case, the antipode has to be derived recursively using the antipode axiom:

$$\sum_{i=0}^{n} S(x_i)x_{n-i} = \delta_{n,0} = \delta_{n,0}x_0.$$

Then we have $S(1) = S(x_0) = x_0$ and

$$S(x_n) = S(x_n)x_0 = -\sum_{i=0}^{n-1} S(x_i)x_{n-i}.$$

**Example 2.17.** In the category Set, we can also define Hopf algebra objects, which are nothing more than groups.

There are some important properties of the antipode which are useful in calculations.

**Theorem 2.18.** *Let $(H, \mu, \eta, \Delta, \epsilon, S)$ a Hopf algebra then*

*(a) $S(hk) = S(k)S(h)$, for all $h, k \in H$.*

*(b) $\Delta(S(h)) = (S \otimes S)\tau\Delta(h)$, for all $h \in H$, where $\tau$ is the flip morphism.*

*(c) $S(1_H) = 1_H$.*

*(d) $\epsilon(S(h)) = \epsilon(h)$, for all $h \in H$*

*Proof.* (a) Consider the vector space $\mathcal{A} = \mathrm{Hom}_k(H \otimes H, H)$ as $H \otimes H$ has a structure of coalgebra and $H$ is also an algebra, then $\mathcal{A}$ is a convolution algebra. Define two maps $F_1, F_2 : H \otimes H \to H$ given by

$$F_1(h \otimes k) = S(hk), \qquad F_2(h \otimes k) = S(k)S(h).$$

Let us show that both are the convolution inverse of the multiplication, on one hand we have

$$\begin{aligned} \mu * F_1(h \otimes k) &= \mu(h_{(1)} \otimes k_{(1)})F_1(h_{(2)} \otimes k_{(2)}) = h_{(1)}k_{(1)}S(h_{(2)}k_{(2)}) \\ &= (hk)_{(1)}S((hk)_{(2)}) = \epsilon(hk)1_H = \epsilon(h)\epsilon(k)1_H. \end{aligned}$$

On the other hand

$$\begin{aligned} F_2 * \mu(h \otimes k) &= F_2(h_{(1)} \otimes k_{(1)})\mu(h_{(2)} \otimes k_{(2)}) = S(k_{(1)})S(h_{(1)})h_{(2)}k_{(2)} \\ &= S(k_{(1)})\epsilon(h)k_{(2)} = \epsilon(h)S(k_{(1)})k_{(2)} \\ &= \epsilon(h)\epsilon(k)1_H. \end{aligned}$$

By the uniqueness of the inverse, we have that $F_1 = F_2$ and this implies that $S(hk) = S(k)S(h)$.

(b) Consider now the convolution algebra $\mathcal{B} = \mathrm{Hom}_k(H, H \otimes H)$ and two maps $G_1, G_2 : H \to H \otimes H$ given by

$$G_1(h) = \Delta(S(h)) = S(h)_{(1)} \otimes S(h)_{(2)}, \quad G_2(h) = (S \otimes S)\tau\Delta(h) = S(h_{(2)}) \otimes S(h_{(1)}).$$

Again, the idea is to show that both maps are the convolution inverse of the comultiplication, indeed, on one hand we have

$$\begin{aligned} G_1 * \Delta(h) &= G_1(h_{(1)})\Delta(h_{(2)}) = (S(h_{(1)})_{(1)} \otimes S(h_{(1)})_{(2)})(h_{(2)(1)} \otimes h_{(2)(2)}) \\ &= S(h_{(1)})_{(1)}h_{(2)(1)} \otimes S(h_{(1)})_{(2)}h_{(2)(2)} = (S(h_{(1)})h_{(2)})_{(1)} \otimes (S(h_{(1)})h_{(2)})_{(2)} \\ &= \Delta(S(h_{(1)})h_{(2)}) = \Delta(\epsilon(h)1_H) = \epsilon(h)(1_H \otimes 1_H). \end{aligned}$$

On the other hand

$$\begin{aligned} \Delta * G_2(h) &= \Delta(h_{(1)})G_2(h_{(2)}) = (h_{(1)} \otimes h_{(2)})(S(h_{(4)}) \otimes S(h_{(3)})) \\ &= h_{(1)}S(h_{(4)}) \otimes h_{(2)}S(h_{(3)}) = h_{(1)}S(h_{(3)}) \otimes \epsilon(h_{(2)})1_H \\ &= h_{(1)}S(h_{(2)}) \otimes 1_H = \epsilon(h)(1_H \otimes 1_H). \end{aligned}$$

Therefore $G_1 = G_2$ which implies the equality $S(h)_{(1)} \otimes S(h)_{(2)} = S(h_{(2)}) \otimes S(h_{(1)})$.

(c) Consider $h \in H$, then

$$\begin{aligned} S(1_H)h &= S(1_H)\epsilon(h_{(1)})h_{(2)} = S(1_H)S(h_{(1)})h_{(2)}h_{(3)} \\ &= S(h_{(1)})h_{(2)}h_{(3)} = \epsilon(h_{(1)})h_{(2)} = h. \end{aligned}$$

Analogously, one can show that $hS(1_H) = h$ for all $h \in H$, then by the uniqueness of the unit, we have $S(1_H) = 1_H$.

(d) Finally, consider $h \in H$, then

$$\begin{aligned} \epsilon(S(h_{(1)}))h_{(2)} &= \epsilon(S(h_{(1)}))\epsilon(h_{(2)})h_{(3)} = \epsilon(S(h_{(1)})h_{(2)})h_{(3)} \\ &= \epsilon(\epsilon(h_{(1)}1_H)h_{(2)} = \epsilon(h_{(1)})h_{(2)} = h. \end{aligned}$$

With similar calculations, one can prove that $h_{(1)}\epsilon(S(h_{(2)})) = h$, then by the uniqueness of the counit, we conclude that $\epsilon(S(h) = \epsilon(h)$ for all $h \in H$. □

**Remark 2.19.** Most of the results valid for bialgebras are automatically satisfied if these bialgebras are Hopf algebras. An example of this fact is that any morphism of bialgebras are automatically morphisms of Hopf algebras. More explicitly, if $H$ and $K$ are two Hopf algebras and $f : H \to K$ is a bialgebra morphism, then $f \circ S_H = S_K \circ f$. We leave to the reader the proof of this fact.

## 2.5 Representation theoretic characterizations of Hopf algebras

### 2.5.1 Hopf algebras and rigid monoidal categories

Hopf algebras can be also characterized by properties of their categories of modules. Given a Hopf algebra $(H, \mu, \eta, \Delta, \epsilon)$, and a finite dimensional $H$ module $M$, then it is easy to see that the dual vector space $M^* = \mathrm{Hom}_k(M, k)$ is also an $H$ module by the action

$$\begin{array}{rccc} \cdot : & H \otimes M^* & \to & M^* \\ & h \otimes \phi & \mapsto & h \cdot \phi \end{array}$$

given by

$$(h \cdot \phi)(m) = \phi(S(h) \cdot m)$$

**Proposition 2.20.** *Let $H$ be a $k$ Hopf algebra and $M$ be a finite dimensional $H$ module. Then the two maps, $ev : M^* \otimes M \to k$ and $coev : k \to M \otimes M^*$, given respectively as,*

$$ev(\phi \otimes m) = \phi(m) \quad \text{and} \quad coev(1) = \sum_{i=1}^{n} m_i \otimes m_i^*,$$

*where $\{m_i\}_{i=1}^n$ is a $k$ basis on $M$ and $\{m_i^*\}_{i=1}^n$ is its dual basis in $M^*$, are $H$ module morphisms.*

*Proof.* Consider $m \in M$, $\phi \in M^*$ and $h \in H$, then

$$\begin{aligned}
\mathrm{ev}(h \cdot (\phi \otimes m)) &= \mathrm{ev}((h_{(1)} \cdot \phi) \otimes (h_{(2)} \cdot m)) = (h_{(1)} \cdot \phi)(h_{(2)} \cdot m)) \\
&= \phi(S(h_{(1)}) \cdot (h_{(2)} \cdot m)) = \phi(S(h_{(1)})h_{(2)} \cdot m) \\
&= \phi(\epsilon(h)m) = \epsilon(h)\phi(m) = h \cdot \mathrm{ev}(\phi \otimes m),
\end{aligned}$$

which means that ev is an $H$ module morphism. For the coevalutation consider $h \in H$, then

$$h \cdot \mathrm{coev}(1) = h \cdot \sum_{i=1}^{n} m_i \otimes m_i^* = \sum_{i=1}^{n} (h_{(1)} \cdot m_i) \otimes (h_{(2)} \cdot m_i^*)$$

This element of $M \otimes M^*$ can be interpreted as a $k$ linear endomorphism in $M$, then applying on $m \in M$ we have

$$\begin{aligned}
(h \cdot \mathrm{coev}(1))(m) &= \sum_{i=1}^{n} (h_{(1)} \cdot m_i)(h_{(2)} \cdot m_i^*)(m) = \sum_{i=1}^{n} (h_{(1)} \cdot m_i) m_i^*(S(h_{(2)}) \cdot m) \\
&= h_{(1)} \cdot (S(h_{(2)}) \cdot m) = h_{(1)} S(h_{(2)}) \cdot m \\
&= \epsilon(h)m = \epsilon(h) \sum_{i=1}^{n} m_i m_i^*(m) \\
&= \epsilon(h)\mathrm{coev}(1)(m) = \mathrm{coev}(\epsilon(h)1)(m) \\
&= \mathrm{coev}(h \cdot 1)(m).
\end{aligned}$$

Then, coev is an $H$ module morphism. $\square$

Given a Hopf algebra $H$, the category of finite dimensional $H$ modules ${}_H\mathcal{M}^f \subseteq \underline{\mathrm{Vect}}_k^f$ is an example of a rigid monoidal category.

**Definition 2.21.** An object $M$ in a monoidal category $(\mathcal{C}, \otimes, I)$ is called left rigid if there is an object $M^* \in \mathcal{C}$ and two morphisms $\mathrm{ev} : M^* \otimes M \to I$ and $\mathrm{coev} : I \to M \otimes M^*$ such that the following diagrams commute

$$
\begin{tikzcd}
M \arrow[r, "M"] \arrow[d, "l_M^{-1}"'] & M \\
I \otimes M \arrow[d, "\mathrm{coev} \otimes M"'] & M \otimes I \arrow[u, "r_M"'] \\
(M \otimes M^*) \otimes M \arrow[r, "a"] & M \otimes (M^* \otimes M) \arrow[u, "M \otimes \mathrm{ev}"']
\end{tikzcd}
$$

$$
\begin{tikzcd}
M^* \arrow[r, "M^*"] \arrow[d, "r_{M^*}^{-1}"'] & M^* \\
M^* \otimes I \arrow[d, "M^* \otimes \mathrm{coev}"'] & I \otimes M^* \arrow[u, "l_{M^*}"'] \\
M^* \otimes (M \otimes M^*) \arrow[r, "a^{-1}"] & (M^* \otimes M) \otimes M^* \arrow[u, "\mathrm{ev} \otimes M^*"']
\end{tikzcd}
$$

A right rigid object can be defined in a symmetric way. An object is called rigid if it is both left and right rigid. A monoidal category $\mathcal{C}$ is said to be left (resp. right) rigid if every object in $\mathcal{C}$ is left (resp. right) rigid. Finally $\mathcal{C}$ is rigid if every object in $\mathcal{C}$ is rigid.

**Remark 2.22.** Note that if a monoidal category is braided, then an object is left rigid if, and only if it is right rigid. Then if a monoidal category is braided and left or right rigid, it is automatically rigid.

In fact there is a deeper connection between Hopf algebras and rigid monoidal categories, as stated by the Tannaka reconstruction theorem.

**Theorem 2.23.** *There is a bijective correspondence between the following objects:*

*(i) Hopf algebras over a field $k$.*

*(ii) Rigid monoidal categories $\mathcal{C}$ together with a strict monoidal functor $U : \mathcal{C} \to \underline{Vect}_k^f$, that preserves duals.*

*Under this correspondence, the category $\mathcal{C}$ is equivalent to the category ${}_H\mathcal{M}^f$ of finite dimensional $H$ modules and $U$ is the usual forgetful functor.*

### 2.5.2 The fundamental theorem for Hopf algebras

Given a $k$ bialgebra $H$, one can have $H$ modules and $H$ comodules as well. There are cases in which the same vector space can have both structures, of module and comodule. Then it is interesting to investigate when these structures of $H$ module and $H$ comodule are compatible, these objects are know as Hopf modules.

**Definition 2.24.** Given a $k$ bialgebra $H$, a right-right $H$ Hopf module is a vector space $M$ which is both a right $H$ module, with action

$$\begin{array}{rccc} \cdot: & M \otimes H & \to & H \\ & m \otimes h & \mapsto & m \cdot h \end{array}$$

and a right $H$ comodule, with coaction

$$\begin{array}{rccl} \rho: & M & \to & M \otimes H \\ & m & \mapsto & \rho(m) = m^{(0)} \otimes m^{(1)} \end{array}$$

satisfying the following compatibility condition:

$$\rho(m \cdot h) = m^{(0)} \cdot h_{(1)} \otimes m^{(1)} h_{(2)}.$$

A morphism between two right-right $H$ Hopf modules $M$ and $N$ is a linear map $f: M \to N$ which is at the same time a right $H$ module and a right $H$ comodule morphism. The category of right-right $H$ Hopf modules will be denoted by $\mathcal{M}_H^H$.

**Remark 2.25.** Note that the compatibility condition for right-right $H$ Hopf modules is equivalent to say that the coaction $\rho$ is a morphism of right $H$ modules, considering the right $H$ module structure on $M \otimes H$ as the standard structure on the tensor product, $(m \otimes h) \cdot k = m \cdot k_{(1)} \otimes hk_{(2)}$.

Obviously one can define left-left, left-right and right-left $H$ Hopf modules, whose categories are, respectively, denoted as, ${}_H^H\mathcal{M}$, ${}_H\mathcal{M}^H$ and ${}^H\mathcal{M}_H$. We are moving towards a characterization of Hopf algebras in terms of Hopf modules. For this purpose, we need to introduce another ingredient, the co-invariant sobcomodule.

**Definition 2.26.** Given a $k$ bialgebra $H$ and a right $H$ comodule $M$ the co-invariant subcomodule $M^{CoH}$ is the subspace $M^{CoH} = \mathrm{Ker}(\rho - M \otimes \eta_H)$, that is, it is the subspace

$$M^{CoH} = \{m \in M \mid \rho(m) = m \otimes 1_H\}.$$

**Lemma 2.27.** *There is an adjoint pair of functors*

$$\underline{Vect}_k \underset{(\_)^{CoH}}{\overset{\_\otimes H}{\rightleftarrows}} \mathcal{M}_H^H$$

*between the category of $k$ vector spaces and the category of right-right $H$ Hopf modules. Moreover the functor $\_ \otimes H$ is fully faithful.*

*Proof.* We leave to the reader the verification of the functoriality of $(\_)^{CoH}$. The image of the functor $\_\otimes H$ really is the category of right-right $H$ Hopf modules (Hopf modules, for short). Indeed, given a $k$ vector space $M$, the right $H$ module structure on $M \otimes H$ is by right multiplication, $(m \otimes h) \cdot k = m \otimes hk$. The right $H$ comodule structure on $M \otimes H$ is given by $\rho = M \otimes \Delta$. And given any linear transformation $f: M \to N$, the linear map $f \otimes H$ is clearly a morphism of Hopf modules between $M \otimes H$ and $N \otimes H$.

In order to prove that these two functors form an adjunction, one needs to construct a unit and a counit for them. Define, for each vector space $M$, the linear map

$$\begin{array}{rccl} \nu_M: & M & \to & (M \otimes H)^{CoH} \\ & m & \mapsto & m \otimes 1_H \end{array}$$

Also define, for each $N \in \mathcal{M}_H^H$ the linear map

$$\begin{array}{rccl} \zeta_N: & N^{CoH} \otimes H & \to & N \\ & m \otimes h & \mapsto & m \cdot h \end{array}$$

The linear map $\zeta_N$ is a morphism of Hopf modules. Indeed

$$\zeta_N((m\otimes h)\cdot k)=\zeta_M(m\otimes hk)=m\cdot hk=(m\cdot h)\cdot k=\zeta_M(m\otimes h)\cdot k.$$

and

$$\begin{aligned}\rho(\zeta_M(m\otimes h)) &= \rho(m\cdot h)=(m^{(0)}\cdot h_{(1)})\otimes m^{(1)}h_{(2)}=(m\cdot h_{(1)})\otimes h_{(2)}\\ &= \zeta_M(m\otimes h_{(1)})\otimes h_{(2)}=(\zeta_M\otimes H)\rho(m\otimes h).\end{aligned}$$

We leave to the reader the verification that $\nu$ and $\zeta$ are natural (see definition A.1 in Appendix A). In order to see that this pair of functors define an adjunction, that is, the natural transformation $\nu$ is a unit and the natural transformation $\zeta$ is a counit, we need to check whether these maps satisfy the commutative diagrams in the definition A.7, that is, for any $M\in \underline{\text{Vect}}_k$, one needs to prove that $\zeta_{M\otimes H}\circ(\nu\otimes H)=M\otimes H$ and for any $N\in\mathcal{M}_H^H$, one needs to prove that $(\zeta_N)^{CoH}\circ\nu_{N^{CoH}}=N^{CoH}$. for the first identity take an element $m\otimes h\in M\otimes H$ then

$$\zeta_{M\otimes H}\circ(\nu\otimes H)(m\otimes h)=\zeta_{M\otimes H}(m\otimes 1_H\otimes h(m\otimes 1_H)\cdot h=m\otimes h.$$

For the second identity, take $n\in N^{CoH}$, then

$$(\zeta_N)^{CoH}\circ\nu_{N^{CoH}}(n)=(\zeta_N)^{CoH}(n\otimes 1_H)=n\cdot 1_H=n.$$

Then, we have an adjunction.

The last claim of this lemma is that the functor $\_\otimes H$ is fully faithful, for this, according to the proposition A.12 we need only to check that the unit $\nu$ is a natural isomorphism, that is, for any vector space $N$, the linear map $\nu_N$ is an isomorphism. Define the linear map

$$\begin{array}{rccl}\widetilde{\nu}_M: & (M\otimes H)^{CoH} & \to & M\\ & m\otimes h & \mapsto & m\epsilon(h)\end{array}$$

Then, taking $m\in M$, we have

$$\widetilde{\nu}_M\circ\nu_M(m)=\widetilde{\nu}_M(m\otimes 1_H)=m\epsilon(1_H)=m,$$

and for an element $\sum_{i=1}^n m_i\otimes h_i\in(M\otimes H)^{CoH}$

$$\nu_M\circ\widetilde{\nu}_M(\sum_{i=1}^n m_i\otimes h_i)=\sum_{i=1}^n\nu_M(m_i\epsilon(h_i))=\sum_{i=1}^n m_i\otimes\epsilon(h_i)1_H.$$

At first sight these two maps doesn't seem to be mutually inverse, but note that if $\sum_{i=1}^n m_i\otimes h_i\in(M\otimes H)^{CoH}$, then

$$\sum_{i=1}^n m_i\otimes(h_i)_{(1)}\otimes(h_i)_{(2)}=\sum_{i=1}^n m_i\otimes h_i\otimes 1_H.$$

Applying $M\otimes\epsilon\otimes H$ on both sides of this identity we have

$$\sum_{i=1}^n m_i\otimes\epsilon((h_i)_{(1)})(h_i)_{(2)}=\sum_{i=1}^n m_i\otimes\epsilon(h_i)1_H.$$

Therefore

$$\sum_{i=1}^n m_i\otimes h_i=\sum_{i=1}^n m_i\otimes\epsilon(h_i)1_H,$$

which proves that $\widetilde{\nu}_M$ is the inverse of $\nu_M$. This concludes the proof. □

**Lemma 2.28.** *Given a $k$ bialgebra $H$, consider in $H\otimes H$ the left $H$module structure given by the left multiplication $h\cdot(k\otimes l)=hk\otimes l$, and the right $H$ comodule structure given by $\rho=H\otimes\Delta$. Then there is an anti isomorphism of algebras between the algebra ${}_H Hom^H(H\otimes H,H\otimes H)$, with the multiplication by composition, and the algebra $Hom_k(H,H)$ with multiplication by convolution.*

*Proof.* For any $F\in {}_H\mathrm{Hom}^H(H\otimes H,H\otimes H)$ define the linear map $\widehat{F}:H\to H$ by

$$\widehat{F}(h)=(H\otimes\epsilon)F(1\otimes h)$$

And for any $f\in\mathrm{Hom}_k(H,H)$ define the linear map $\widetilde{f}:H\otimes H\to H\otimes H$ given by

$$\widetilde{f}(h\otimes k)=hf(k_{(1)})\otimes k_{(2)}.$$

We leave to the reader the verification that $\widetilde{f}$ is indeed a morphism of left $H$ module and right$H$ comodule in $H\otimes H$. Then we can define two maps

$$\begin{array}{rccc}\widehat{(\,)}: & {}_H\mathrm{Hom}^H(H\otimes H,H\otimes H) & \to & \mathrm{Hom}_k(H,H)\\ & F & \mapsto & \widehat{F}\end{array}$$

and

$$\begin{array}{rccc}\widetilde{(\,)}: & \mathrm{Hom}_k(H,H) & \to & {}_H\mathrm{Hom}^H(H\otimes H,H\otimes H)\\ & f & \mapsto & \widetilde{f}\end{array}$$

We leave to the reader the detail that these maps are $k$ linear. To prove that they are mutually inverse, consider a linear map $f\in\mathrm{Hom}_k(H,H)$ and an element $h\in H$ then

$$\widehat{\widetilde{f}}(h)=(H\otimes\epsilon)\widetilde{f}(1_H\otimes h)=(H\otimes\epsilon)(f(h_{(1)})\otimes h_{(2)})=f(h_{(1)})\epsilon(h_{(2)})=f(h).$$

In its turn, consider a morphism $F\in {}_H\mathrm{Hom}^H(H\otimes H,H\otimes H)$ and an element $h\otimes k\in H\otimes H$, then

$$\begin{aligned}\widetilde{\widehat{F}}(h\otimes k) &= h\widehat{F}(k_{(1)})\otimes k_{(2)}=h((H\otimes\epsilon)F(1\otimes k_{(1)}))\otimes k_{(2)}\\ &= ((H\otimes\epsilon)(h\cdot F(1\otimes k_{(1)})))\otimes k_{(2)}=((H\otimes\epsilon)F(h\otimes k_{(1)}))\otimes k_{(2)}\\ &= (H\otimes\epsilon\otimes H)(F(h\otimes k_{(1)})\otimes k_{(2)})\\ &= (H\otimes\epsilon\otimes H)(F\otimes H)(H\otimes\Delta)(h\otimes k)\\ &= (H\otimes\epsilon\otimes H)(H\otimes\Delta)F(h\otimes k)=F(h\otimes k).\end{aligned}$$

Therefore, these maps are mutually inverse.

It remains only to verify that these maps are anti multiplicative, but as they are mutually inverse, one needs only to check one of them, let us verify for the map $\widetilde{\ }$, consider $f,g\in\mathrm{Hom}_k(H,H)$ and $h\otimes k\in H\otimes H$, then

$$\begin{aligned}\widetilde{f*g}(h\otimes k) &= h(f*g)(k_{(1)})\otimes k_{(2)}=hf(k_{(1)})g(k_{(2)})\otimes k_{(3)}\\ &= \widetilde{g}(hf(k_{(1)})\otimes k_{(2)})=\widetilde{g}\circ\widetilde{f}(h\otimes k).\end{aligned}$$

Therefore we have the anti-isomorphism. □

Now we have the needed ingredients to prove the fundamental theorem which characterizes a Hopf algebra by its category of Hopf modules.

**Theorem 2.29.** *(Fundamental Theorem of Hopf Algebras) Let $H$ be a bialgebra over $k$, then the following statements are equivalent:*

*(i) $H$ is a Hopf algebra.*

*(ii) The functor $(\_)^{CoH}$ is fully faithful.*

*(iii) The functor* $\_ \otimes H$ *is an equivalence of categories.*

*(iv) The canonical map* $can : H \otimes H \to H \otimes H$*, defined as* $can(h \otimes k) = hk_{(1)} \otimes k_{(2)}$*, is bijective.*

*Proof.* (i)⇒(ii) According to the proposition A.12 in Appendix A, in order to prove that the functor $(\_)^{CoH}$ is fully faithful, one needs only to show that the counit $\zeta$ defined in the lemma 2.27 is a natural isomorphism. Now we use the fact that $H$ is a Hopf algebra, then we use the antipode $S$ to produce a candidate of inverse of $\zeta$. Define, for any $M \in \mathcal{M}_H^H$, the linear map

$$\begin{array}{rccl} \widetilde{\zeta}_M : & M & \to & M \otimes H \\ & m & \mapsto & (m^{(0)} \cdot S(m^{(1)})) \otimes m^{(2)} \end{array}$$

The image of $\widetilde{\zeta}_M$ lies, in fact, in the subspace $M^{CoH} \otimes H$, indeed, applying $(\rho_M \otimes H)$ on $\widetilde{\zeta}_M(m)$ we have

$$\begin{aligned} (\rho_M \otimes H)(\widetilde{\zeta}_M(m)) &= (\rho_M \otimes H)((m^{(0)} \cdot S(m^{(1)})) \otimes m^{(2)}) \\ &= (m^{(0)} \cdot S(m^{(3)})) \otimes m^{(1)} S(m^{(2)}) \otimes m^{(4)} \\ &= (m^{(0)} \cdot S(m^{(2)})) \otimes \epsilon(m^{(1)}) 1_H \otimes m^{(3)} \\ &= (m^{(0)} \cdot S(m^{(1)})) \otimes 1_H \otimes m^{(2)}. \end{aligned}$$

Consider $m \in M$, then

$$\begin{aligned} \zeta_M \circ \widetilde{\zeta}_M(m) &= \zeta_M((m^{(0)} \cdot S(m^{(1)})) \otimes m^{(2)}) = (m^{(0)} \cdot S(m^{(1)})) \cdot m^{(2)} \\ &= m^{(0)} \cdot S(m^{(1)}) m^{(2)} = m^{(0)} \epsilon(m^{(1)}) = m. \end{aligned}$$

Now take $m \otimes h \in M^{CoH} \otimes H$, then

$$\begin{aligned} \widetilde{\zeta}_M \circ \zeta_M(m \otimes h) &= \widetilde{\zeta}_M(m \cdot h) = ((m \cdot h)^{(0)} \cdot S((m \cdot h)^{(1)})) \otimes (m \cdot h)^{(2)} \\ &= ((m^{(0)} \cdot h_{(1)}) \cdot S(m^{(1)} h_{(2)})) \otimes m^{(2)} h_{(3)} \\ &= ((m \cdot h_{(1)}) \cdot S(h_{(2)})) \otimes h_{(3)} = (m \cdot h_{(1)} S(h_{(2)})) \otimes h_{(3)} \\ &= m \otimes \epsilon(h_{(1)}) h_{(2)} = m \otimes h. \end{aligned}$$

Then $\zeta_M$ is an isomorphism, and this leads us to conclude that $(\_)^{CoH}$ is fully faithful.

(ii)⇒(iii) As the functor $\_ \otimes H$ is fully faithful as proved in lemma 2.27 and by hypothesis the functor $(\_)^{CoH}$ is fully faithful, the we have by the proposition A.12 that this is an equivalence of categories.

(iii)⇒(iv) Consider on the vector space $H \otimes H$ with right $H$ module structure given by $(h \otimes k) \cdot l = hl_{(1)} \otimes kl_{(2)}$ and right $H$ comodule structure $\rho = (H \otimes \Delta)$. It is straightforward to show that $H \otimes H$ with these structures is a Hopf module. We claim that the canonical morphism, can : $H \otimes H \to H \otimes H$ equals to $\zeta_{H \otimes H} \circ (\nu_H \otimes H)$, indeed consider an element $h \otimes k \in H \otimes H$ then

$$\zeta_{H \otimes H} \circ (\nu_H \otimes H)(h \otimes k) = \zeta_{H \otimes H}(h \otimes 1_H \otimes k) = (h \otimes 1_H) \cdot k = hk_{(1)} \otimes k_{(2)} = \mathrm{can}(h \otimes k).$$

As we have by hypothesis an equivalence of categories, this implies that both the unit $\nu_H$ and the counit $\zeta_{H \otimes H}$ are isomorphisms, therefore the map *can* is an isomorphism.

(iv)⇒(i) Note that *can* is a morphism of left $H$ module and of right $H$ comodule in $H \otimes H$ according to the structures defined in the lemma 2.28, also we have that the map $\widehat{(\ )}$ associates the canonical map *can* to the identity map in $H$, indeed

$$\widehat{\mathrm{can}}(h) = (H \otimes \epsilon)\mathrm{can}(1 \otimes h) = (H \otimes \epsilon)(h_{(1)} \otimes h_{(2)}) = h_{(1)} \epsilon(h_{(2)}) = h.$$

Then if *can* is bijective, it is invertible, this means that the identity map in $H$ is invertible by convolution, that is, there exists an antipode which makes $H$ into a Hopf algebra. ☐

**Remark 2.30.** The item (iv) in the fundamental theorem says that a Hopf algebra is a Hopf Galois extension of the base field $k$, that is $H$ is a Hopf Galois object in the category $\underline{\mathrm{Vect}}_k$. In fact, the fundamental theorem can be proved in any braided monoidal category in which there are equalizers. The equalizers are needed in order to define the coinvariant subcomodules.

### 2.5.3 Internal Homs and Hopf algebras

**Definition 2.31.** A monoidal category $(\mathcal{C}, \otimes, I)$ is said to be closed if for any object $M \in \mathcal{C}$ there exists a right adjoint $[M, \_]$ for the functor $\_ \otimes M$. These functors $[M, \_]$ are called internal homs in $\mathcal{C}$.

**Proposition 2.32.** *Let $(H, \mu, \eta, \Delta, \epsilon, S)$ be a $k$ Hopf algebra, then the category ${}_H\mathcal{M}$ is closed and the internal hom associated to $M \in {}_H\mathcal{M}$ is the functor $Hom_k(M, \_)$.*

*Proof.* Let $M, N \in {}_H\mathcal{M}$, then one can put a left $H$ module structure on $\mathrm{Hom}_k(M, N)$ by

$$(h \cdot f)(m) = h_{(1)} \cdot f(S(h_{(2)}) \cdot m).$$

Indeed, take $h, k \in H$, $f : M \to N$ and $m \in M$, then

$$\begin{aligned}
(h \cdot (k \cdot f))(m) &= h_{(1)} \cdot (k \cdot f)(S(h_{(2)}) \cdot m) = h_{(1)} \cdot (k_{(1)} \cdot f(S(k_{(2)}) \cdot (S(h_{(2)}) \cdot m))) \\
&= h_{(1)}k_{(1)} \cdot f(S(k_{(2)})S(h_{(2)}) \cdot m) = (hk)_{(1)} \cdot f(S((hk)_{(2)}) \cdot m) \\
&= (hk \cdot f)(m).
\end{aligned}$$

Consider $M, N, P \in {}_H\mathcal{M}$. By the Hom-tensor relation in the category of $k$ vector spaces, we have an isomorphism of $k$ vector spaces

$$\begin{array}{rcl}
\widehat{\ } : \mathrm{Hom}_k(M \otimes N, P) & \to & \mathrm{Hom}_k(M, \mathrm{Hom}_k(N, P)) \\
F & \mapsto & \widehat{F}
\end{array}$$

given by $(\widehat{F}(m))(n) = F(m \otimes n)$. We need only to check that this is a morphism of left $H$ modules, indeed, for $F \in \mathrm{Hom}_k(M \otimes N, P)$, $m \in M$, $n \in N$ and $h \in H$, we have

$$\begin{aligned}
((\widehat{h \cdot F})(m))(n) &= (h \cdot F)(m \otimes n) = h_{(1)} \cdot F(S(h_{(2)}) \cdot (m \otimes n)) \\
&= h_{(1)} \cdot F((S(h_{(3)}) \cdot m) \otimes (S(h_{(2)}) \cdot n)) \\
&= h_{(1)} \cdot ((\widehat{F}(S(h_{(3)}) \cdot m))(S(h_{(2)}) \cdot n)) \\
&= (h_{(1)} \cdot \widehat{F}(S(h_{(2)}) \cdot m))(n) = ((h \cdot \widehat{F})(m))(n).
\end{aligned}$$

This concludes the proof. □

**Definition 2.33.** Let $\mathcal{C}$ and $\mathcal{D}$ be two closed monoidal categories. Let $F : \mathcal{C} \to \mathcal{D}$ be a monoidal functor and for any $M, N \in \mathcal{C}$ let

$$\zeta_{M,N} : F(\mathcal{C}(M, N)) \to \mathcal{D}(F(M), F(N))$$

be the unique morphism such that the diagram

$$\begin{tikzcd}
F(\mathcal{C}(M,N)) \otimes F(M) \arrow[r, "\zeta_{M,N} \otimes F(M)"] \arrow[d, "\cong"'] & \mathcal{D}(F(M), F(N)) \otimes F(M) \arrow[d, "e_{F(M),F(N)}"'] \\
F(\mathcal{C}(M,N) \otimes M) \arrow[r, "F(E_{M,N})"'] & F(N)
\end{tikzcd}$$

commutes, where $e_{M,N} : \mathcal{C}(M, N) \otimes M \to N$ is given by the evaluation $e_{M,N}(f \otimes m) = f(m)$. We say that $F$ preserves internal homs if $\zeta_{M,N}$ is an isomorphism for all $M, N \in \mathcal{C}$.

If $H$ is a $k$ Hopf algebra, it is easy to see that the forgetful functor $U : {}_H\mathcal{M} \to \underline{\mathrm{Vect}_k}$ preserves internal homs. On the other hand, if we have a bialgebra $H$ such that the forgetful functor $U$ preserves internal homs, then one can prove that $H$ is a Hopf algebra.This last theorem we will state without proof and it shows another characterization of Hopf algebras by means of internal homs.

**Theorem 2.34.** *Let $H$ be a bialgebra, then the following statements are equivalent:*

1. *The category ${}_H\mathcal{M}$ is closed and the forgetful functor $U : {}_H\mathcal{M} \to \underline{Vect}_k$ preserves internal homs.*
2. *The canonical map $\overline{can} : H \otimes H \to H \otimes H$ given by $\overline{can}(h \otimes k) = h_{(1)} \otimes h_{(2)}k$ is bijective.*

The bijectivity of this left version of the canonical map introduced in the preceding section is also equivalent to the fact that $H$ is a Hopf algebra. Therefore one can see Hopf algebras as bialgebras whose category admit internal homs compatible with the internal homs in the category of vector space. This theorem also can be generalized for bialgebras in a braided monoidal category as well.

# Chapter 3

# Contructions over a Noncommutative Base

## 3.1 $R$-rings and $R$-corings

Throughout this chapter we will consider the category of bimodules over a not necessarily commutative, unital algebra $R$ over a commutative ring $k$, denoted by ${}_R\mathcal{M}_R$. This category is monoidal with tensor product over $R$ and which the unit object is also $R$. By the monoidality of $({}_R\mathcal{M}_R, \otimes_R, R)$ one can talk about its algebra objects and coalgebra objects, the $R$ rings and $R$ corings.

**Definition 3.1.** For an algebra $R$ over a commutative ring $k$, an $R$-ring is an algebra object in the monoidal category $({}_R\mathcal{M}_R, \otimes_R, R)$, that is, it is a triple $(A, \mu, \eta)$, such that $A$ is an $R$ bimodule and $\mu : A \otimes_R A \to A$ and $\eta : R \to A$ are $R$ bimodule maps satisfying the associativity and unit conditions.

A morphism of $R$ rings $f : (A, \mu, \eta) \to (A', \mu', \eta')$ is an $R$ bimodule map $f : A \to A'$ such that $\mu' \circ (f \otimes_R f) = f \circ \mu$ and $f \circ \eta = \eta'$.

A left $A$ module is a pair $(M, \cdot)$ where $M$ is an $R$ bimodule and $\cdot : A \otimes_R M \to M$ is an $R$ bimodule map satisfying the conditions $1_A \cdot m = m$, for all $m \in M$ and $a \cdot (b \cdot m) = ab \cdot m$ for all $a, b \in A$ and $m \in M$.

A morphism of left $A$ modules $f : (M, \cdot) \to (N, \bullet)$ is an $R$ bimodule morphism $f : M \to N$ such that $f(a \cdot m) = a \bullet f(m)$ for all $a \in A$ and $m \in M$.

If $(A, \mu, \eta)$ is an $R$ ring, then it is a $k$ algebra. The left/right multiplication of elements of $k$ by elements of $A$ is given by $k \cdot a = (k1_R) \cdot a = a \cdot (k1_R) = a \cdot k$, the multiplication $m : A \otimes_k A \to A$ can be obtained from the composition $\mu' = \mu \circ \Pi$ where $\Pi : A \otimes_k A \to A \otimes_R A$ is the canonical projection. The unit is given by the composition $\eta' = \eta \circ u$ where $u : k \to R$ is the unit of $R$ as a $k$ algebra. It is easy to see that the maps $\mu'$ and $\eta'$ are $k$ linear, then $(A, \mu', \eta')$ is a $k$ algebra. There is a basic characterization of $R$ rings considering its underlying $k$ algebra structure.

**Proposition 3.2.** *There is a bijective correspondence between $R$ rings $(A, \mu, \eta)$ and $k$ algebra morphisms $\eta : R \to A$.*

*Proof.* Consider $(A, \mu, \eta)$ an $R$ ring and the underlying $k$ algebra structure presented in the preceding paragraph. One needs only to show that $\mu : R \to A$ is a morphism of $k$ algebras. It is clear that this map is $k$ linear, in order to show that $\eta$ is multiplicative, take $r, s \in R$, then

$$\begin{aligned}
\eta(rs) &= \eta(r1_R s) = r \cdot \eta(1_R) \cdot s = r \cdot 1_A \cdot s \\
&= (r \cdot 1_A) \cdot s = ((r \cdot 1_A)1_A) \cdot s = (r \cdot 1_A)(1_A \cdot s) \\
&= (r \cdot \eta(1_R))(\eta(1_R) \cdot s) = \eta(r)\eta(s).
\end{aligned}$$

Conversely, suppose that $\eta : R \to A$ is a $k$ algebra morphism. In particular it is an $R$ bimodule morphism. We can endow $A$ and $A \otimes_k A$ with $R$ bimodule structures given, respectively by $r \cdot a \cdot s = \eta(r)a\eta(s)$ and $r \cdot (a \otimes b) \cdot s = \eta(r)a \otimes b\eta(s)$. One can easily show that the multiplication $m : A \otimes_k A \to A$ is an $R$ bimodule map. Then one needs only to construct a multiplication map $\mu : A \otimes_R A \to A$ which is an $R$ bimodule morphism. Using the morphism $\eta$ construct two maps $f, g : A \otimes_k R \otimes_k A \to A \otimes_k A$, given by

$$f(a \otimes r \otimes b) = a\eta(r) \otimes b, \quad g(a \otimes r \otimes b) = a \otimes \eta(r)b.$$

It is easy to see that these two maps are $R$ bimodule maps. The $R$ balanced tensor product is the coequalizer

$$\begin{tikzcd} A \otimes_k R \otimes_k A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \otimes_k A \arrow[r] & A \otimes_R A \end{tikzcd}$$

By the associativity of the multiplication $m : A \otimes_k A \to A$, one can obtain that

$$m \circ f(a \otimes r \otimes b) = m(a\eta(r) \otimes b) = (a\eta(r))b = a(\eta(r)b) = m(a \otimes \eta(r)b) = m \circ g(q \otimes r \otimes b).$$

Then, by the universal property of the coequalizer, there exists a unique $R$ bimodule map $\mu : A \otimes_R A \to A$ commuting the diagram

$$\begin{tikzcd} A \otimes_k R \otimes_k \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & A \otimes_k A \arrow[r] \arrow[dr, "m"] & A \otimes_R A \arrow[d, "\mu"] \\ & & A \end{tikzcd}$$

We leave to the reader the verification of the associativity of $\mu$. $\square$

**Proposition 3.3.** *Given an $R$-ring $(A, \mu, \eta)$ it is possible to lift the left regular $R$ module structure on $R$ to a left $A$ module structure if, and only if, there exists a $k$ module map $\chi : A \to R$ satisfying the following properties*

*(i)* $\chi(\eta(r)a) = r\chi(a)$*, for* $a \in A$ *and* $r \in R$*.*

*(ii)* $\chi(a(\eta \circ \chi)(b)) = \chi(ab)$*, for* $a, b \in A$*.*

*(iii)* $\chi(1_A) = 1_R$*.*

*The map $\chi$ obeying these properties is called a left character on the $R$ ring $(A, \mu, \eta)$.*

*Proof.* First consider $\chi : A \to R$ satisfying (i), (ii) and (iii). Define the left $A$ module structure on $R$ by $a \cdot r = \chi(a\eta(r))$, then we have

$$1_A \cdot r = \chi(1_A\eta(r)) = \chi(\eta(r)1_A) = r\chi(1_A) = r,$$

and

$$\begin{aligned} a \cdot (b \cdot r) &= a \cdot \chi(b\eta(r)) = \chi(a\eta(\chi(b\eta(r)))) \\ &= \chi(ab\eta(r)) = ab \cdot r. \end{aligned}$$

Therefore, $R$ is a left $A$ module.

On the other hand, if $R$ is a left $A$ module define $\chi : A \to R$ by $\chi(a) = a \cdot 1_R$. If this $A$ module structure lifts the usual left regular $R$ module structure on $R$, then $\eta(r) \cdot s = rs$. Then, we have

$$\chi(\eta(r)a) = \eta(r)a \cdot 1_R = \eta(r) \cdot (a \cdot 1_R) = \eta(r) \cdot \chi(a) = r\chi(a).$$

and

$$\begin{aligned} \chi(a\eta(\chi(b))) &= a\eta(\chi(b)) \cdot 1_R = a \cdot (\eta(\chi(b)) \cdot 1_R) \\ &= a \cdot \chi(b) = a \cdot (b \cdot 1_R) = ab \cdot 1_R = \chi(ab). \end{aligned}$$

Finally, $\chi(1_A) = 1_A \cdot 1_R = 1_R$. Therefore, the map $\chi$ is a left character of the $R$ ring $(A, \mu, \eta)$. $\square$

**Definition 3.4.** Given an $R$ ring $(A, \mu, \eta)$ with a left character $\chi : A \to R$ and a left $A$ module $M$ the submodule of invariants of $M$ with respect to the character $\chi$ is the $k$ submodule

$$M_\chi = \{m \in M \,|\, a \cdot m = \eta(\chi(a)) \cdot m, \ \forall a \in A\}.$$

**Proposition 3.5.** *(1) Given an $R$ ring $(A, \mu, \eta)$ with a left character $\chi : A \to R$ and a left $A$ module $M$, we have the isomorphism of $k$ bimodules*

$$\begin{array}{rccl} \phi : & M_\chi & \to & {}_AHom(R, M) \\ & m & \mapsto & \phi(m) \end{array}$$

*given by $\phi(m)(r) = \eta(r) \cdot m$.*
*(2) In particular, for the left $A$ module $R$, we have the subalgebra of invariants of $R$*

$$R_\chi = \{r \in R \,|\, \chi(a\eta(r)) = \chi(a)r \ \forall a \in A\}.$$

*Proof.* (1) First, one needs to check that $\phi(m) : R \to M$ is a left $A$ module morphism for any $m \in M_\chi$. Indeed, consider $a \in A$ and $r \in R$, then

$$\begin{aligned} \phi(m)(a \cdot r) &= \eta(a \cdot r) \cdot m = \eta(\chi(a\eta(r))) \cdot m \\ &= a\eta(r) \cdot m = a \cdot (\eta(r) \cdot m) = a \cdot \phi(m)(r). \end{aligned}$$

It is easy to see that $\phi$ is a morphism of $k$ bimodules. Let $m \in \mathrm{Ker}(\phi)$, then

$$0 = \phi(m)(1_R) = \eta(1_R) \cdot m = 1_A \cdot m = m,$$

then the map $\phi$ is a monomorphism. Finally consider $f \in {}_A\mathrm{Hom}(R, M)$, define $m = f(1_R)$ then

$$\phi(m)(r) = \eta(r) \cdot m = \eta(r) \cdot f(1_R) = f(\eta(r) \cdot 1_R) = f(r),$$

then $\phi$ is an epimorphism.

(2) Let $r \in R$, we have $\chi(a\eta(r)) = \chi(a)r$, if, and only if

$$a \cdot r = \chi(a)r = \eta(\chi(a)) \cdot r,$$

which means that $a \in R_\chi$ if and only if it satisfies the identity $\chi(a\eta(r)) = \chi(a)r$, for all $a \in A$. ☐

**Definition 3.6.** An $R$ ring $(A, \mu, \eta)$ is said to be a Galois $R$ ring if the canonical map

$$\begin{array}{rccl} \mathrm{can} : & A & \to & {}_{R_\chi}\mathrm{End}(R) \\ & a & \mapsto & (r \mapsto \chi(a\eta(r))) \end{array}$$

is bijective.

Dual to the notion of an $R$ ring, there is the notion of an $R$ coring.

**Definition 3.7.** For an algebra $R$ over a commutative ring $k$, and $R$-coring is a coalgebra object in the monoidal category $({}_R\mathcal{M}_R, \otimes_R, R)$, that is, it is a triple $(C, \Delta, \epsilon)$, such that $C$ is an $R$ bimodule and $\Delta : C \to C \otimes_R C$ and $\epsilon : C \to R$ are $R$ bimodule maps satisfying the coassociativity and counit conditions.

A morphism of $R$ corings $f : (C, \Delta, \epsilon) \to (C', \Delta', \epsilon')$ is an $R$ bimodule map $f : C \to C'$ such that $(f \otimes_R f) \circ \Delta = \Delta \circ f$ and $\epsilon' \circ f = \epsilon$.

A right $C$ comodule is a pair $(M, \rho)$, where $M$ is a right $R$ module and $\rho : M \to M \otimes_R C$ is a right $R$ module map satisfying $(\rho \otimes_R C) \circ \rho = (M \otimes_R \Delta) \circ \rho$ and $(M \otimes_R \epsilon) \circ \rho = M$.

A morphism of right $C$ comodules $f : (M, \rho) \to (N, \rho')$ is a morphism of right $R$ modules $f : M \to N$ such that $\rho' \circ f = (f \otimes_R C) \circ \rho$. The category of right $C$ comodules is denoted by $\mathcal{M}^C$.

In the theory of comodules, we use Sweedler notation for corings as well. Then for an $R$ coring $(C, \Delta, \epsilon)$ we write for any $c \in C$, $\Delta(c) = c_{(1)} \otimes_R c_{(2)}$. The counit axiom reads

$$c_{(1)} \cdot \epsilon(c_{(2)}) = c = \epsilon(c_{(1)}) \cdot c_{(2)}.$$

**Example 3.8.** The ring $R$ is itself an $R$ coring with the canonical isomorphisms $\Delta = l_R^{-1} : R \to R \otimes_R R$ and $\epsilon = R : R \to R$.

**Example 3.9.** Let $P$ be a right $R$ module which is finitely generated and projective. Let $S = \mathrm{End}_R(P)$, then $P$ is a left $S$ module. Let $P^* = \mathrm{Hom}_R(P, R)$ be the dual left $R$ module associated to $P$, it is easy to see that $P^*$ is a right $S$ module. Let$\{e_i\}_{i=1}^n \subseteq P$ and $\{\varepsilon^i\}_{i=1}^n \subseteq P^*$ be a dual basis for $P$. Then $C = P^* \otimes_S P$ is an $R$ coring, with comultiplication given by

$$\Delta(\phi \otimes_S p) = \sum_{i=1}^{n} \phi \otimes_S e_i \otimes_R \varepsilon^i \otimes_S p,$$

and counit

$$\epsilon(\phi \otimes_S p) = \phi(p).$$

We leave to the reader the verification of the axioms of coassociativity and counit.

**Example 3.10.** Let $\varphi : R \to S$ be an algebra extension, in particular, $S$ inherits the structure of an $R$ bimodules by $r \cdot a \cdot r' = \varphi(r) a \varphi(r')$. Then $C = S \otimes_R S$ has an $S$ coring defined by

$$\Delta(a \otimes_R b) = a \otimes_R 1_S \otimes_S 1_S \otimes_R b, \quad \epsilon(a \otimes_R b) = ab.$$

We leave to the reader the details for the verification of the axioms. This is the canonical coring or the Sweedler coring associated to the extension $\varphi$.

**Example 3.11.** An algebra extension $\varphi : R \to S$ is said to be Frobenius if there exists and element $e = \sum_{i=1}^n a_i \otimes_R b_i \in S \otimes_r S$ and an $R$ bimodules map $E : S \to R$ such that for any $a \in S$ we have

$$a = \sum_{i=1}^{n} E(a a_i) \cdot b_i = \sum_{i=1}^{n} a_i \cdot E(b_i a). \tag{3.1}$$

It is easy to show that for a Frobenius extension, the identity (3.1) is independent of the representative for the element $e$ in $S \otimes_R S$. It is also possible to show that for any $a \in S$ we have $ae = ea$, that is

$$\sum_{i=1}^{n} a a_i \otimes_R b_i = \sum_{i=1}^{n} a_i \otimes_R b_i a.$$

Given a Frobenius extension $\varphi : R \to S$ one can make $S$ into an $R$ coring with structure given by

$$\Delta(a) = ea = ae, \quad \epsilon(a) = E(a).$$

Again, the details are left to the reader.

**Proposition 3.12.** *A structure of (left or right) $R$ module of $R$ extends to a structure of an (left or right) comodule over an $R$ coring $(C, \Delta, \epsilon)$ if, and only if there exists an element $g \in C$ such that*

*(i)* $\Delta(g) = g \otimes_R g$.

*(ii)* $\epsilon(g) = 1_R$.

*Such an element satisfying the previous conditions is called a grouplike element in $C$.*

*Proof.* Let us do everything for the right case, the left case is analogous. First, suppose that there is a right $C$ comodule structure $\rho : R \to R \otimes_R C \cong C$. Define $g = \rho(1_R)$, then

$$\begin{aligned}\Delta(g) &= (R \otimes_R \Delta)(1_R \otimes_R g) = (R \otimes_R \Delta)\rho(1_R) \\ &= (\rho \otimes_R C)\rho(1_R) = (\rho \otimes_R C)(1_R \otimes_R g) = g \otimes_R g,\end{aligned}$$

and

$$\epsilon(g) = (R \otimes_R \epsilon)(1_R \otimes_R g) = (R \otimes_R \epsilon)\rho(1_R) = 1_R.$$

On the other hand, suppose that there exists a grouplike element $g \in C$ then define the map $\rho : R \to R \otimes_R C \cong C$ by $\rho(r) = g \cdot r$, it is a right $R$ module map. Moreover

$$\begin{aligned}(\rho \otimes_R C)\rho(r) &= (\rho \otimes_R C)(1_R \otimes_R (g \cdot r)) = (g \cdot 1_R) \otimes_R (g \cdot r) \\ &= (g \otimes_R g) \cdot r = \Delta(g) \cdot r = \Delta(g \cdot r) \\ &= (R \otimes_R \Delta)(1_R \otimes_R g \cdot r) = (R \otimes_R \Delta)\rho(r),\end{aligned}$$

and

$$(R \otimes_R \epsilon)\rho(r) = (R \otimes_R \epsilon)(1_R \otimes_R (g \cdot r)) = \epsilon(g \cdot r) = \epsilon(g)r = r.$$

Therefore, $\rho$ is a right $C$ comodule structure on $R$. $\square$

**Definition 3.13.** Let $(C, \Delta, \epsilon)$ be an $R$ coring with grouplike element $g \in C$ and $M \in \mathcal{M}^C$. The coinvariant comodulewith respect to $g$ is the $R$ submodule

$$M^g = \{m \in M \,|\, \rho(m) = m \otimes_R g\}.$$

**Proposition 3.14.** *Let $(C, \Delta, \epsilon)$ be an $R$ coring with grouplike element $g \in C$ and $M \in \mathcal{M}^C$. Then $M^g \cong Hom^C(R, M)$ as $k$ modules. In particular the coinvariant subcomodules of $R$, $B = R^g = \{r \in R \,|\, r \cdot g = g \cdot r\}$.*

*Proof.* Define the $k$ linear map

$$\begin{aligned}\phi : \quad M^g &\to \mathrm{Hom}_k(R, M) \\ m &\mapsto \phi(m)\end{aligned}$$

given by $\phi(m)(r) = m \cdot r$. First, note that for any $m \in M^g$, the map $\phi(m)$ is a morphism of right $C$ comodules. Indeed,

$$\begin{aligned}\rho_M(\phi(m)(r)) &= \rho_M(m \cdot r) = \rho_M(m) \cdot r = (m \otimes_R g) \cdot r \\ &= m \otimes_R (g \cdot r) = (m \cdot 1_R) \otimes_R (g \cdot r) \\ &= \phi(m)(1_R) \otimes_R (g \cdot r) = (\phi(m) \otimes_R C)(1_R \otimes_R (g \cdot r)) \\ &= (\phi(m) \otimes_R C)\rho_R(r).\end{aligned}$$

Then the image of the map $\phi$ is in $\mathrm{Hom}^C(R, M)$. For the inverse, define

$$\begin{aligned}\psi : \quad \mathrm{Hom}^C(R, M) &\to M \\ f &\mapsto \psi(f) = f(1_R)\end{aligned}$$

For any $f \in \mathrm{Hom}^C(R, M)$ we have that $f(1_R) \in M^g$, indeed

$$\begin{aligned}\rho_M(f(1_R)) &= (f \otimes_R C)\rho_R(1_R) = (f \otimes_R C)(1_R \otimes_R (g \cdot 1_R)) \\ &= f(1_R) \otimes_R g\end{aligned}$$

Then the image of the map $\psi$ is in $M^g$. They are mutually inverse:

$$\psi(\phi(m)) = \phi(m)(1_R) = m \cdot 1_R = m,$$

and

$$\phi(\psi(f))(r) = \psi(f) \cdot r = f(1_R) \cdot r = f(1_R r) = f(r).$$

Therefore, we have the isomorphism.

Now, for the right $C$ comodule $R$, take $r \in R^g$ then

$$1_R \otimes g \cdot r = \rho_R(r) = r \otimes_R g = 1_R \otimes_R r \cdot g$$

on the other hand, if $r \cdot g = g \cdot r$ then $\rho_R(r) = r \otimes_R g$. $\square$

**Proposition 3.15.** *Let $(C, \Delta, \epsilon)$ be an $R$ coring with grouplike element $g \in C$ and $M \in \mathcal{M}^C$. Then, denoting $M^g \in \mathcal{M}_{R^g}$. Moreover there is an adjunction*

$$\begin{tikzcd} \mathcal{M}_{R^g} \arrow[r, shift left, "\_\otimes_{R^g} R"] & \mathcal{M}^C \arrow[l, shift left, "(\_)^g"] \end{tikzcd}$$

*Considering the right $C$ comodule structure on $M \otimes_R R$ given by $(M \otimes_R \rho_R)$.*

*Proof.* Let $m \in M^g$ and $r \in r^G$, then

$$\begin{aligned} \rho_M(m \cdot r) &= \rho_M(m) \cdot r = (m \otimes_R g) \cdot r \\ &= m \otimes_R (g \cdot r) = m \otimes_R (r \cdot m) \\ &= (m \cdot r) \otimes_R g. \end{aligned}$$

Then $m \cdot r \in M^g$. It is not difficult to see that $(\_)^g$ is a functor. Then we need only to prove that this pair of functors form an adjunction by constructing the unit and the counit of this adjunction. Define, for each $M \in \mathcal{M}_{R^g}$ the map

$$\begin{array}{rccl} \nu_M : & M & \to & (M \otimes_R R)^g \\ & m & \mapsto & m \otimes_r 1_R \end{array}$$

and for each $N \in \mathcal{M}^C$ the map

$$\begin{array}{rccl} \zeta_N : & N^g \otimes_R R & \to & N \\ & n \otimes r & \mapsto & n \cdot r \end{array}$$

We leave to the reader the verification that both $\nu$ and $\zeta$ are natural transformations. Let us verify the commutativity of the diagrams in the definition A.7. Take first $M \in \mathcal{M}_{R^g}$, and elements $m \in M$ and $r \in R$, then

$$(\zeta_M \otimes_R R) \circ (\nu_M \otimes_R R)(m \otimes_R r) = (\zeta_M \otimes_R R)(m \otimes_R 1_R \otimes_R r) = m \otimes_R (1_R r) = m \otimes_R r.$$

Now, take $N \in \mathcal{M}^C$ and an element $n \in N$, then

$$(\zeta_N)^g \circ \nu_{N^g}(n) = (\zeta_N)^g(n \otimes_R 1_R) = n \cdot 1_R = n.$$

Therefore, we have an adjunction. $\square$

**Proposition 3.16.** *Let $R$ be a $k$ algebra and $(C, \Delta, \epsilon)$ be an $R$ coring then ${}^*C = {}_R Hom(C, R)$ is an $R$ ring with convolution product*

$$f * g(c) = g(c_{(1)} \cdot f(c_{(2)})),$$

*with unit given by $\epsilon$ and the morphism of $k$ algebras, $\theta : R \to {}^*C$, given by $\theta(r)(c) = \epsilon(c)r$.*

*Proof.* Let $f, g, h \in {}^*C$ and $c \in C$ then

$$\begin{aligned} ((f * g) * h)(c) &= h(c_{(1)} \cdot (f * g)(c_{(2)})) = h(c_{(1)} \cdot g(c_{(2)} \cdot f(c_{(3)})) \\ &= (g * h)(c_{(1)} \cdot f(c_{(2)}) = (f * (g * h))(c). \end{aligned}$$

For the unit, take $f \in {}^*C$ and $c \in C$, then

$$(f * \epsilon)(c) = \epsilon(c_{(1)} \cdot f(c_{(2)})) = \epsilon(c_{(1)})f(c_{(2)}) == f(\epsilon(c_{(1)}) \cdot c_{(2)}) = f(c).$$

In he same way, one can prove that $\epsilon * f = f$.

Finally, the map $\theta$ is clearly $k$ linear, it remains only to show that it is multiplicative, indeed, take $r, s \in R$ and $c \in C$ then

$$(\theta(r) * \theta(s))(c) = \theta(s)(c_{(1)} \cdot \theta(r)(c_{(2)})) = \epsilon(c_{(1)} \cdot \epsilon(c_{(2)})r)s = \epsilon(c)rs = \theta(rs)(c).$$

This concludes the proof. □

**Proposition 3.17.** *Let $(A, \mu\eta)$ be an $R$ ring which is finitely generated and projective as a right $R$ module. Then the right dual $A^* = Hom_R(A, R)$ is an $R$ coring with comultiplication given by*

$$\Delta(f) = \sum_{i=1}^{n} f(\_e_i) \otimes \varepsilon^i,$$

*where $\{e_i, \varepsilon^i\}_{i=1}^n$ is a dual basis for $A$, and coounit given by $\epsilon(f) = f(1_A)$. Moreover, the definition of the comultiplication is independent of the choice of dual basis in $A$.*

*Proof.* The $R$ bimodule structure on $A^*$ is given by $(r \cdot f \cdot s)(a) = rf(s \cdot a)$. In order to see that the definition of $\Delta$ is independent of choice of dual basis, take $\{e_i, \varepsilon^i\}_{i=1}^n$ and $\{a_j, \alpha^j\}_{i=1}^m$ two dual basis, then, for $f \in A^*$,

$$\begin{aligned}
\Delta(f) &= \sum_{i=1}^{n} f(_e_i) \otimes \varepsilon^i \\
&= \sum_{i=1}^{n} f\left(_\sum_{j=1}^{m} a_j \alpha^j(e_i)\right) \otimes \varepsilon^i \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} f(_a_j)\alpha^j(e_i) \otimes \varepsilon^i \\
&= \sum_{j=1}^{m} f(_a_j) \otimes \sum_{i=1}^{n} \alpha^j(e_i)\varepsilon^i \\
&= \sum_{j=1}^{m} f(_a_j) \otimes \alpha^j
\end{aligned}$$

For the coassociativity, we have

$$\begin{aligned}
(\Delta \otimes A^*)\Delta(f) &= \sum_{i=1}^{n} \Delta(f(_e_i)) \otimes \varepsilon^i \\
&= \sum_{i,j=1}^{n} f((_e_j)e_i)) \otimes \varepsilon^j \otimes \varepsilon^i \\
&= \sum_{i,j,k=1}^{n} f(_e_k\varepsilon^k(e_je_i)) \otimes \varepsilon^j \otimes \varepsilon^i \\
&= \sum_{i,k=1}^{n} f(_e_k) \otimes \sum_{j=1}^{n} \varepsilon^k(e_je_i)\varepsilon^j \otimes \varepsilon^i \\
&= \sum_{i,k=1}^{n} f(_e_k) \otimes \sum_{j=1}^{n} \varepsilon^k(_e_i) \otimes \varepsilon^i \\
&= (A^* \otimes \Delta)\Delta(f)
\end{aligned}$$

Finally, for the counit, we have

$$(\epsilon \otimes A^*)\Delta(f) = \sum_{i=1}^{n} \epsilon(f(\_e_i)) \cdot \varepsilon^i = \sum_{i=1}^{n} f(e_i) \cdot \varepsilon^i = f,$$

and

$$(A^* \otimes \epsilon)\Delta(f) = \sum_{i=1}^{n} f(\_e_i)\epsilon(\varepsilon^i) = f(\_\sum_{i=1}^{n} e_i\varepsilon^i(1_A)) = f.$$

Therefore, $(A^*, \Delta, \epsilon)$ is an $R$ coring. $\square$

There are some important duality results between $R$ rings and $R$ corings that we are going to state without proof.

**Theorem 3.18.** *Let $R$ be an algebra over a commutative ring $k$.*

1. *For an $R$ coring $(C, \Delta, \epsilon)$ which is a finitely generated projective left module over $R$, the second dual $({}^*C)^*$ is isomorphic to $C$ as an $R$ coring.*

2. *For an $R$ ring $(A, \mu, \eta)$ which is a finitely generated projective right module over $R$, the second dual ${}^*(A^*)$ is isomorphic to $A$ as an $R$ ring.*

3. *For an $R$ coring $(C, \Delta, \epsilon)$, any right $C$ comodule $(M, \rho)$ becomes a right ${}^*C$ module by the relation $m \cdot f = m^{(0)}f(m^{(1)})$, and this induces a faithful functor $\mathcal{M}^C \to \mathcal{M}_{{}^*C}$.*

4. *This functor defined above is an equivalence if, and only if, $C$ is finitely generated projective left $R$ module.*

## 3.2 Bialgebroids

As we have seen before, in order to define a bialgebra, one needs first to define an algebra structure on the tensor product of algebras. This can be obtained easily if the monoidal category where we are working is braided. In order to define the "bialgebra" objects in the category of $R$ bimodules, we need to overcome the inherent difficulty that ${}_R\mathcal{M}_R$ is not braided. One possible solution is to define an algebra structure on a subspace of the tensor space of two algebras such that the comultiplication can still be viewed as an algebra morphism.

Let $R$ be an algebra over a commutative ring $k$ and denote by $\overline{R} = R^{op}$ the opposite algebra associated to $R$, its elements will be denoted by $\bar{r}$, for each $r \in R$ and their product is given by $\bar{r}\bar{s} = \overline{sr}$. Let $R^e = R \otimes_k \overline{R}$ be the enveloping algebra of $R$. Note that an object $A \in {}_R\mathcal{M}_R$ is an $R^e$ algebra if and only if it is a $k$ algebra with an algebra map $s : R \to A$, named the source map, and an anti-algebra map $t : R \to A$, named the target map, such that for each $r, r' \in R$, $s(r)t(r') = t(r')s(r)$. Indeed, if $\phi : R^e \to A$ we can define $s(r) = \phi(r \otimes_k \overline{1_R})$ and $t(r) = \phi(1_R \otimes_k \bar{r})$, it is easy to see that their images commute in $A$. On the other hand, if we have $s$ and $t$, which commute, define $\phi : R^e \to A$ by $\phi(r \otimes_k \overline{r'}) = s(r)t(r')$. As an example of $R^e$ ring, one can see $A = {}_k\mathrm{End}(R)$ in this case, $s : R \to A$ is given by $s(r)(r') = rr'$ and $t(r)(r') = r'r$. The associativity in $R$ gives the commutativity of $s$ and $t$. In the sequel, the expression "let $(A, s, t)$ be an $R^e$ ring" will be understood simply as a $k$ algebra $A$ with the source and target maps above defined. Any $R^e$ ring can be endowed naturally with two different $R - R$ bimodule structures, the first, given by $r \cdot a \cdot r' = s(r)t(r')a$, is the left handed structure and the second, given by $r \cdot a \cdot r' = as(r')t(r)$ is the right handed structure. There will be situations when we are going to consider both structures on the same algebra, given by two distinct pairs of source and target maps, the left source and target $s$ and $t$ and the right source and target $\tilde{s}$ and $\tilde{t}$.

**Definition 3.19.** Let $M$ and $N$ be two $R^e - R^e$ bimodules. Define

$$\int_r {}_{\bar{r}}M \otimes_k {}_rN = M \otimes_k N / \langle \{\bar{r} \cdot m \otimes n - m \otimes rn \,|\, \forall r \in R\} \rangle,$$

that is, $\int_r {}_{\bar{r}}M \otimes_k {}_rN = M \otimes_R N$ considering the right $R$ module structure on $M$ given by its left $\overline{R}$ module structure. Define also the subspace

$$\int^r M_{\bar{r}} \otimes_k N = \{\sum_i m_i \otimes_k n_i \in M \otimes_k N \mid \sum_i m_i \bar{r} \otimes n_i = \sum_i m_i \otimes n_i r \,\forall r \in R\}.$$

The left Takeuchi product between $M$and $N$ is the $k$ module

$$M \times^l_R N = \int^s \int_r {}_{\bar{r}}M_{\bar{s}} \otimes_k {}_rN_s.$$

More explicitly

$$M \times^l_R N = \{\sum_i m_i \otimes_k n_i \in M \otimes_R N \mid \sum_i m_i \cdot \bar{r} \otimes n_i = \sum_i m_i \otimes n_i \cdot r \,\forall r \in R\}.$$

Analogously, considering the right handed $R$ bimodule structure given by $\tilde{s}$ and $\tilde{t}$ one can define the right Takeuchi product

$$M \times^r_R N = \{\sum_i m_i \otimes_k n_i \in M \otimes_R N \mid \sum_i r \cdot m_i \otimes n_i = \sum_i m_i \otimes \bar{r} \cdot n_i \,\forall r \in R\}.$$

In what follows, we will prove the results only for the left handed version of the Takeuchi product, leaving their right handed analogues to the reader. In contexts where there will be no ambiguity, we will denote the left Takeuchi product only by $M \times_R N$.

It is easy to see that $M \times_R N$ is an $R^e - R^e$ bimodule by

$$(r \otimes \bar{s}) \left( \sum_i m_i \otimes n_i \right) (t \otimes \bar{u}) = \sum_i r \cdot m_i \cdot t \otimes \bar{s} \cdot n_i \cdot \bar{u}$$

Moreover, the operation $\_ \times_R \_ : {}_{R^e}\mathcal{M}_{R^e} \times {}_{R^e}\mathcal{M}_{R^e} \to {}_{R^e}\mathcal{M}_{R^e}$ is a bifunctor.

**Theorem 3.20.** *Let $(A, s, t)$ and $(B, \tilde{s}, \tilde{t})$ be two $R^e$ rings, then the $R^e - R^e$ bimodule $A \times_R B$ is an $R^e$ ring with algebra map $\phi : R \otimes \overline{R} \to A \times_R B$ given by $\phi(r \otimes \overline{r'}) = s(r) \otimes \tilde{t}(r')$, multiplication*

$$\left( \sum_i a_i \otimes_R b_i \right) \left( \sum_j c_j \otimes_R d_j \right) = \sum_{i,j} a_i c_j \otimes_R b_i d_j$$

*and unit $1_A \otimes_R 1_B$.*

*Proof.* The first thing we need to prove is that the multiplication is well defined, that is, it is independent of the choice of the representatives of the elements in $A \times_R B$. Since the multiplication is clearly linear, it suffices to show that the result of the product is zero when one of the factors is zero.

First, assume that $\sum_i a_i \otimes_R b_i = 0$ by the characterization of the zero element, there exists finitely many elements $r_{ij} \in R$ and $x_j \in A$ such that $\sum_i \tilde{s}(r_{ij}) b_i = 0$ and $\sum_j t(r_{ij}) x_j = a_i$, then

$$\begin{aligned} \left( \sum_i a_i \otimes_R b_i \right) (c \otimes_R d) &= \sum_i a_i c \otimes_r b_i d = \sum_{i,j} t(r_{ij}) x_j c \otimes_R b_i d \\ &= \sum_j x_j c \otimes_R \sum_i \tilde{s}(r_{ij}) b_i d = 0 \end{aligned}$$

Note that we have made no use of the fact that $\sum_i a_i \otimes_R b_i \in A \times_R B$, this will be necessary only for the case when the second factor is zero. Indeed, let $\sum_j c_j \otimes_R d_j = 0$, then there are finitely many $r_{kj} \in R$

and $y_k \in A$ such that $\sum_j \tilde{s}(r_{kj})d_j = 0$ and $\sum_k t(r_{kj})y_k = c_j$. Consider an element $\sum_i a_i \otimes b_i \in A \times_R B$, then

$$\left(\sum_i a_i \otimes_R b_i\right)\left(\sum_j c_j \otimes_R d_j\right) = \sum_{i,j} a_i c_j \otimes b_i d_j$$

$$= \sum_{i,j} a_i \left(\sum_k t(r_{kj})y_k\right) \otimes_R b_i d_j = \sum_{i,j,k} (a_i t(r_{kj}) \otimes_R b_i)(y_k \otimes_R d_j)$$

$$= \sum_{i,j,k} (a_i \otimes_R b_i \tilde{s}(r_{kj}))(y_k \otimes_R d_j) = \sum_{i,k} a_i y_k \otimes_R b_i \left(\sum_j \tilde{s}(r_{kj}) d_j\right) = 0$$

Therefore, the multiplication is well defined. It is trivial to prove that the multiplication is associative and that $1_A \otimes_r 1_B$ is the unit in this associative algebra. Let us verify that $\phi$ is a morphism of algebras, take $r, s, t, u \in R$ then

$$\begin{aligned}
\phi((r \otimes \bar{s})(t \otimes \bar{u}) &= \phi(rt \otimes \bar{us}) = s(rt) \otimes_R \tilde{t}(us) \\
&= s(r)s(t) \otimes_R \tilde{t}(s)\tilde{t}(u) = (s(r) \otimes_R \tilde{t}(s))(s(t) \otimes_R \tilde{t}(u)) \\
&= \phi(r \otimes \bar{s})\phi(t \otimes \bar{u})
\end{aligned}$$

Then, $A \times_R B$ is an $R^e$ ring. □

**Definition 3.21.** Given an algebra $R$ over a commutative ring $k$, we define a left $R$ bialgebroid as a quintuple $(B, s, t, \underline{\Delta}, \underline{\epsilon})$ such that

1. $(B, s, t)$ is an $R^e$ ring, and the $R$ bimodule structure on $B$ is given by $r \cdot a \cdot r' = s(r)t(r')a$.
2. $(B, \underline{\Delta}, \underline{\epsilon})$ is an $R$ coring.
3. $\mathrm{Im}(\underline{\Delta}) \subseteq B \times_R B$ and the co-restriction $\underline{\Delta} : B \to B \times_R B$ is an algebra map.
4. $\underline{\epsilon}(1_B) = 1_R$ and for all $a, b \in B$

$$\underline{\epsilon}(ab) = \underline{\epsilon}(as(\underline{\epsilon}(b))) = \underline{\epsilon}(at(\underline{\epsilon}(b))).$$

Analogously, one can define a right $R$ bialgebroid.

**Definition 3.22.** Given an algebra $R$ over a commutative ring $k$, we define a right $R$ bialgebroid as a quintuple $(B, s, t, \underline{\Delta}, \underline{\epsilon})$ such that

1. $(B, s, t)$ is an $R^e$ ring, and the $R$ bimodule structure on $B$ is given by $r \cdot a \cdot r' = as(r')t(r)$
2. $(B, \underline{\Delta}, \underline{\epsilon})$ is an $R$ coring.
3. $\mathrm{Im}(\underline{\Delta}) \subseteq B \times_R^r B$ and the co-restriction $\underline{\Delta} : B \to B \times_R^r B$ is an algebra map.
4. $\underline{\epsilon}(1_B) = 1_R$ and for all $a, b \in B$

$$\underline{\epsilon}(ab) = \underline{\epsilon}(s(\underline{\epsilon}(a))b) = \underline{\epsilon}(t(\underline{\epsilon}(a))b).$$

**Example 3.23.** Bialgebras over a commutative ring $k$ are bialgebroids in the category of $k$ bimodules (which is, in fact, the category of left or right $k$ modules).

**Example 3.24.** Weak bialgebras are examples of bialgebroids [14].

**Example 3.25.** For any algebra $R$ over a commutative ring $k$, the algebra $R \otimes R^{op}$, it is a right $R$ bialgebroid, with source and target maps given by the inclusions

$$s(r) = r \otimes \overline{1_R}, \qquad t(r) = 1_R \otimes \bar{r}.$$

The comultiplication and the counit are given, respectively, by

$$\underline{\Delta}(r \otimes \bar{s}) = (1_R \otimes \bar{s}) \otimes_R (r \otimes \overline{1_R}), \quad \underline{\epsilon}(r \otimes \bar{s})) = sr.$$

The opposite co-opposite of this construction leads to a left $R^{op}$ bialgebroid on $R^{op} \otimes R \cong R \otimes R^{op}$.

**Example 3.26.** Let $H$ be a commutative $k$ bialgebra (Hopf algebra) and $A$ be a commutative right $H$ comodule algebra, then $A \otimes H$, with the standard algebra structure on the tensor product is an $A$ bialgebroid. The source and target maps are given, respectively, by $s(a) = a \otimes_H$ and $t(a) = \rho(a) = a^{(0)} \otimes a^{(1)}$. As $H$ and $A$ are commutative, it is obvious that the images of $s$ and $t$ commute and that $s$ and $t$ are algebra maps, and then $t$ is an anti-algebra map. Again, because of the commutativity, it is easy to see that the Takeuchi product $(A \otimes H) \times_A (A \otimes H)$ is simply the $A$ balanced tensor product $(A \otimes H) \otimes_A (A \otimes H)$. The comultiplication and the counit are, respectively, given by

$$\underline{\Delta}(a \otimes h) = a \otimes h_{(1)} \otimes_A 1_A \otimes h_{(2)}, \quad \underline{\epsilon}(a \otimes h) = a\epsilon(h).$$

This defines a left $A$ bialgebroid structure, we leave to the reader the verification of the details. For the right bialgebroid structure, we take simply $\tilde{s} = t$ and $\tilde{t} = s$, with the same comultiplication and the same counit, the right Takeuchi product also coincides with the tensor product over $A$. The detailed proof of this bialgebroid (Hopf algebroid) structure is given in [1], there everything is done for the case of partial coactions, but every coaction is a partial coaction, then it is easy to translate for this simple case.

**Example 3.27.** Let $H$ be a cocommutative $k$ Hopf algebra and $A$ be a commutative $H$ module algebra, then the smash product $A\#H$, which is a $k$ algebra, isomorphic to $A \otimes H$ as $k$ module, with multiplication given by $(a \otimes h)(b \otimes k) = a(h_{(1)} \cdot a) \otimes h_{(2)}k$ and unit $1_a \otimes 1_H$, is an $A$ bialgebroid. The source and target map, for both, left and right structures are the same:

$$s(a) = t(a) = \tilde{s}(a) = \tilde{t}(a) = a \otimes 1_H, \quad \forall a \in A.$$

But the $A$ bimodule structure defined by them are sensibly different. For the left structure we have

$$a \triangleright (b \otimes h) \triangleleft c = s(a)t(c)(b \otimes h)(a \otimes 1_H)(c \otimes 1_H)(b \otimes h) = abc \otimes h.$$

On the other hand, for the right structure we have

$$a \blacktriangleright (b \otimes h) \blacktriangleleft c = (b \otimes h)\tilde{s}(c)\tilde{t}(a) = (b \otimes h)(a \otimes 1_H)(c \otimes 1_H) = a(h_{(1)} \cdot ac) \otimes h_{(2)}.$$

With these left and right handed $A$ bimodule structures above defined, one can characterize the left Takeuchi product by

$$\underline{A\#H} \times^l_A \underline{A\#H} = \left\{ \sum x_i \otimes^l_A y_i \in \underline{A\#H} \otimes^l_A \underline{A\#H} | \sum x_i \otimes^l_A y_i = \sum a \blacktriangleright x_i \otimes^l_A y_i \blacktriangleleft a \, , \, \forall a \in A \right\},$$

and the right Takeuchi product by

$$\underline{A\#H} \times^r_A \underline{A\#H} = \left\{ \sum x_i \otimes^r_A y_i \in \underline{A\#H} \otimes^r_A \underline{A\#H} | \sum a \triangleright x_i \otimes^r_A y_i = \sum x_i \otimes^r_A y_i \triangleleft a \, , \, \forall a \in A \right\},$$

Where the tensor product $(A\#H) \otimes^l_A (A\#H)$ is balanced by the left structure, $(a \otimes h) \triangleleft b \otimes (c \otimes k) = (a \otimes h) \otimes b \triangleright (c \otimes d)$, while the tensor product $(A\#H) \otimes^r_A (A\#H)$ is balanced by the right structure, $(a \otimes h) \blacktriangleleft b \otimes (c \otimes k) = (a \otimes h) \otimes b \blacktriangleright (c \otimes d)$.

The comultiplication and the counit for the left $A$ bialgebroid structure are given by

$$\underline{\Delta}_l(a \otimes h) = a \otimes h_{(1)} \otimes^l_A 1_A \otimes h_{(2)}, \quad \underline{\epsilon}_l(a \otimes h) = a\epsilon(h).$$

The comultiplication and the counit for the right $A$ bialgebroid structure are given by

$$\underline{\Delta}_r(a \otimes h) = a \otimes h_{(1)} \otimes^r_A 1_A \otimes h_{(2)}, \quad \underline{\epsilon}_r(a \otimes h) = S(h) \cdot a.$$

The details of the proof for the case of partial actions can be found in [1], this simple case is obtained considering the partial action as a global action.

### 3.2.1 Equivalent formulations for bialgebroids

The concept of bialgebroid arose in several contexts, in [3] it was shown that these definitions of bialgebroids are in fact equivalent. There is a formulation by the anchor map [16] and the formulation by $\times_R$ bialgebras [15]. But the unifying point of view is exactly the fact that bialgebroids behave like bialgebras, namely, given an $R$ bialgebroid $B$ its module category, ${}_B\mathcal{M}$ is monoidal and the forgetful functor $U : {}_B\mathcal{M} \to {}_R\mathcal{M}_R$ is strict [13].

**Definition 3.28.** Let $(H, s, t)$ be an $R^e$ ring, the data $(H, s, t, \underline{\Delta}, \nu)$ is called an $R$ bialgebroid with an anchor $\mu$ if

(BA1) $\underline{\Delta} : H \to H \otimes_R H$ is a coassociative $R$ bimodule map.

(BA2) $\mathrm{Im}(\underline{\Delta}) \subseteq H \times^l_R H$ and its co restriction $\underline{\Delta} : H \to H \times^l_R H$ is an algebra map.

(BA3) The map $\nu : H \to \mathrm{End}_k(R)$, called the anchor map, is an algebra and an $R$ bimodule map, that is, $\nu(s(r)h)(r') = r\nu(h)(r')$ and $\nu(t(r)h)(r') = \nu(h)(r')r$, such that

(A1) $s(h_{(1)} \triangleright r)h_{(2)} = hs(r)$,

(A2) $t(h_{(2)} \triangleright r)h_{(1)} = ht(r)$,

where $h \triangleright r = \nu(h)(r)$, for all $h \in H$ and $r \in R$.

The original formulation of bialgebroids makes a more direct use of the $\times_R$ product [15]. The main problem is to overcome the fact that the product $\times_R$ is not associative. For $M$, $N$ and $P$ in ${}_{R^e}\mathcal{M}_{R^e}$, define

$$M \times_R N \times_R P = \int^{s,u} \int_{r,t} {}_{\bar{r}}M_{\bar{s}} \otimes {}_{r,\bar{t}}N_{s,\bar{u}} \otimes {}_t P_u.$$

There exist two canonical maps,

$$\alpha : (M \times_R N) \times_R P \to M \times_R N \times_R P, \qquad \alpha' : M \times_R (N \times_R P) \to M \times_R N \times_R P,$$

which are identities on elements. These maps are not isomorphism in general, even the maps being identities on elements, two different elements can be mapped into representatives of the same class in $M \times_R N \times_R P$. One can define also two maps,

$$\begin{array}{rccc} \theta : & M \times_R \mathrm{End}_k(R) & \to & M \\ & \sum_i m_i \otimes f_i & \mapsto & \sum_i \overline{f_i(1_R)} \cdot m_i \end{array}$$

and

$$\begin{array}{rccc} \theta' : & \mathrm{End}_k(R) \times_R M & \to & M \\ & \sum_i f_i \otimes m_i & \mapsto & \sum_i f_i(1_R) \cdot m_i \end{array}$$

**Definition 3.29.** A triple $(L, \Delta, \nu)$ is a $\times_R$ coalgebra if $L$ is an $R^e - R^e$ bimodule and

$$\Delta : L \to L \times_R L, \qquad \nu : L \to \mathrm{End}_k(R)$$

are $R^e$ bimodule maps such that

$$\alpha \circ (\Delta \times_R L) \circ \Delta = \alpha' \circ (L \times_R \Delta) \circ \Delta, \qquad \theta \circ (L \times_R \nu) \circ \Delta = L = \theta' \circ (\nu \times_R L) \circ \Delta.$$

The map $\Delta$ is called comultiplications and the map $\nu$ is called counit of the $\times_R$ coalgebra.

**Proposition 3.30.** *Let $i : L \times_R L \to L \otimes_R L$ be the canonical inclusion, and $\Delta : L \to L \times_R L$ and $\nu : L \to \mathrm{End}_k(R)$ two $R^e$ bimodule maps. Then $(L, \Delta, \nu)$ is a $\times_R$ coalgebra if, and only if, $(L, \underline{\Delta}, \underline{\epsilon})$ where $\underline{\Delta} = i \circ \Delta$ and $\underline{\epsilon}(l) = \nu(l)(1_R)$, is an $R$ coring.*

*Proof.* If $(L, \Delta, \nu)$ is a $\times_R$ algebra, then $\underline{\Delta}$ is automatically an $R$ bimodule map, let us show that $\underline{\epsilon}$ is also an $R$ bimodule map,

$$\underline{\epsilon}(r \cdot l \cdot s) = \nu((r \otimes \overline{1_R}) \cdot l \cdot (s \otimes \overline{1_R}))(1_R) = ((r \otimes \overline{1_R}) \cdot \nu(l) \cdot (s \otimes \overline{1_R}))(1_R) = r\nu(l)(1_R)s = r\underline{\epsilon}(l)s.$$

For the coassociativity, define $i^2 : L \times_R L \times_R L \to L \otimes_R L \otimes_R L$ the canonical inclusion, then, we have

$$\begin{aligned}
(\underline{\Delta} \otimes_R L) \circ \underline{\Delta} &= ((i \circ \Delta) \otimes_R L) \circ i \circ \Delta = (i \otimes_R L)(\Delta \otimes_R L) \circ i \circ \Delta \\
&= i^2 \circ \alpha \circ (\Delta \times_R L) \circ \Delta = i^2 \circ \alpha'(L \times_R \Delta) \circ \Delta \\
&= (L \otimes_R i)(L \otimes_R \Delta) \circ i \circ \Delta = (L \otimes_R (i \circ \Delta)) \circ i \circ \Delta \\
&= (L \otimes_R \underline{\Delta}) \circ \underline{\Delta}.
\end{aligned}$$

For the counit, take $l \in L$, then

$$\begin{aligned}
(\underline{\epsilon} \otimes_R L) \circ \underline{\Delta}(l) &= \underline{\epsilon}(l_{(1)}) \cdot l_{(2)} = \nu(l_{(1)})(1_R) \cdot l_{(2)} \\
&= \theta'(\nu(l_{(1)}) \otimes_R l_{(2)}) = \theta' \circ (\nu \times_R L)\Delta(l) = l,
\end{aligned}$$

and

$$\begin{aligned}
(L \otimes_R \underline{\epsilon}) \circ \underline{\Delta} &= l_{(1)} \cdot \underline{\epsilon}(l_{(2)}) = \overline{\underline{\epsilon}(l_{(2)})} \cdot l_{(1)} \\
&= \overline{\nu(l_{(2)})(1_R)} \cdot l_{(1)} = \theta(l_{(1)} \otimes_R \nu(l_{(2)})) \\
&= \theta \circ (L \times \nu) \circ \Delta(l) = l.
\end{aligned}$$

Similarly, if we assume the coassociativity of $\underline{\Delta}$ and the counit axiom for $\underline{\epsilon}$ then we can prove the coassociativity for $\Delta$ and the counit axiom for $\nu$. $\square$

**Definition 3.31.** Let $(H, s, t)$ be an $R^e$ ring, the data $(H, \Delta, \nu)$ is called a $\times_R$ bialgebra if it is a $\times_R$ coalgebra such that the comultiplication $\Delta$ and the counit $\nu$ are algebra maps.

**Theorem 3.32.** *Let $(H, s, t)$ be an $R^e$ ring. Then the following data are equivalent.*

*(1) A left $R$ bialgebroid structure $(H, s, t, \underline{\Delta}, \underline{\epsilon})$.*

*(2) An $R$ bialgebroid with anchor structure $(H, s, t, \underline{\Delta}, \nu)$.*

*(3) A $\times_R$ bialgebra structure $(H, s, t, \Delta, \nu)$.*

*(4) A monoidal structure on ${}_H\mathcal{M}$ such that the forgetful functor $U : {}_H\mathcal{M} \to {}_R\mathcal{M}_R$ is strict monoidal.*

*Proof.* The equivalence (3)⇔ (4) is proven in ( [13] Theorem 5.1)

(1)⇒ (2) Assume that $(H, s, t, \underline{\Delta}, \underline{\epsilon})$ is a left $R$ bialgebroid and define $\nu : H \to \mathrm{End}_k(R)$ by $\nu(h)(r) = \underline{\epsilon}(hs(r)) = \underline{\epsilon}(ht(r))$, then $\nu$ is a morphism of algebras, indeed

$$\begin{aligned}
\nu(hk)(r) &= \underline{\epsilon}(hks(r)) = \underline{\epsilon}(hs(\underline{\epsilon}(ks(r)))) \\
&= \underline{\epsilon}(hs(\nu(k)(r))) = \nu(h)(\nu(k)(r)).
\end{aligned}$$

And $\nu$ is also a morphism of $R$ bimodules:

$$\begin{aligned}
\nu(r \cdot h \cdot r')(r'') &= \nu(s(r)t(r')h)(r'') = \underline{\epsilon}(s(r)t(r')hs(r'')) \\
&= r\underline{\epsilon}(t(r')(hs(r''))) = r\underline{\epsilon}(hs(r''))r' \\
&= r\nu(h)(r'')r' = (r \cdot \nu(h) \cdot r')(r'').
\end{aligned}$$

Let us verify the axioms (A1) and (A2) for the anchor map, first we need to note that, for any $h \in H$ and $r \in R$ we have

$$\begin{aligned}
\underline{\Delta}(hs(r)) &= \underline{\Delta}(h)\underline{\Delta}(s(r)) = \underline{\Delta}(h)\underline{\Delta}(s \cdot 1_H) \\
&= (h_{(1)} \otimes_R h_{(2)})(s \cdot 1_H \otimes_R 1_H) \\
&= (h_{(1)} \otimes_R h_{(2)})(s(r) \otimes 1_H) \\
&= h_{(1)}s(r) \otimes_R h_{(2)}
\end{aligned}$$

and similarly $\underline{\Delta}(ht(r)) = h_{(1)} \otimes_R h_{(2)}t(r)$. Then,

$$s(\nu(h_{(1)})(r))h_{(2)} = s(\underline{\epsilon}(h_{(1)}s(r)))h_{(2)}(\underline{\epsilon} \otimes_R H)(h_{(1)}s(r) \otimes h_{(2)} = (\underline{\epsilon} \otimes_R H)\underline{\Delta}(hs(r)) = hs(r),$$

and

$$t(\nu(h_{(2)})(r))h_{(1)} = t(\underline{\epsilon}(h_{(2)}t(r)))h_{(1)}(H \otimes_R \underline{\epsilon})(h_{(1)} \otimes h_{(2)}t(r) = (H \otimes_R \underline{\epsilon})\underline{\Delta}(ht(r)) = ht(r).$$

Therefore $(H, s, t, \underline{\Delta}, \nu)$ is an $R$ bialgebroid with anchor.

(2)$\Rightarrow$ (1) Assume that $(H, s, t, \underline{\Delta}, \nu)$ is an $R$ bialgebroid with anchor, define $\underline{\epsilon} : H \to R$ as $\underline{\epsilon}(h) = \nu(h)(1_R)$, as $\nu$ is an $R$ bimodule morphism, it is easy to see that $\underline{\epsilon}$ is also an $R$ bimodule map. Let us verify the counit axioms, take $h \in H$ then

$$\begin{aligned} (\underline{\epsilon} \otimes_R H)\underline{\Delta}(h) &= \underline{\epsilon}(h_{(1)}) \cdot h_{(2)} = s(\underline{\epsilon}(h_{(1)}))h_{(2)} \\ &= s(\nu(h_{(1)})(1_R))h_{(2)} = hs(1_R) = h, \end{aligned}$$

and

$$\begin{aligned} (H \otimes_R \underline{\epsilon})\underline{\Delta}(h) &= h_{(1)} \cdot \underline{\epsilon}(h_{(2)}) = t(\underline{\epsilon}(h_{(2)}))h_{(1)} \\ &= t(\nu(h_{(2)})(1_R))h_{(1)} = ht(1_R) = h. \end{aligned}$$

Now, consider $h, k \in H$ then,

$$\begin{aligned} \underline{\epsilon}(hs(\underline{\epsilon}(k))) &= \nu(hs(\underline{\epsilon}(k)))(1_R) = \nu(h)(\nu(s(\underline{\epsilon}(k)))(1_R)) \\ &= \nu(h)(\nu(\underline{\epsilon}(k) \cdot 1_H)(1_R)) = \nu(h)((\underline{\epsilon}(k) \cdot \nu(1_H))(1_R)) \\ &= \nu(h)(\underline{\epsilon}(k)\nu(1_H)(1_R)) = \nu(h)(\underline{\epsilon}(k)) = \nu(h)(\nu(k)(1_R)) \\ &= \nu(hk)(1_R) = \underline{\epsilon}(hk). \end{aligned}$$

and by a similar calculation we get $\underline{\epsilon}(ht(\underline{\epsilon}(k))) = \underline{\epsilon}(hk)$. Therefore $(H, s, t, \underline{\Delta}, \underline{\epsilon})$ is a left $R$ bialgebroid.

(1)$\Rightarrow$(3) Assume that $(H, s, t, \underline{\Delta}, \underline{\epsilon})$ is a left $R$ bialgebroid and define $\Delta : H \to H \times_R H$ as the co restriction of $\underline{\Delta}$ and define $\nu : H \to \mathrm{End}_k(R)$ by $\nu(h)(r) = \underline{\epsilon}(hs(r)) = \underline{\epsilon}(ht(r))$, as before. The coassociativity of $\Delta$ is equivalent to the coassociativity of $\underline{\Delta}$ as we proved in the proposition 3.30. We leave to the reader to check that $\Delta$ and $\nu$ are algebra morphisms and $R^e$ bimodule maps. We are only going to check the counit axioms. Consider an element $h \in H$, then we have

$$\begin{aligned} \theta \circ (H \times_R \nu) \circ \Delta(h) &= \theta(h_{(1)} \otimes_R \nu(h_{(2)})) = t(\nu(h_{(2)})(1_R))h_{(1)} \\ &= t(\underline{\epsilon}(h_{(2)}s(1_R)))h_{(1)} = t(\underline{\epsilon}(h_{(2)}))h_{(1)} \\ &= (H \otimes_R \underline{\epsilon})\underline{\Delta}(h) = h, \end{aligned}$$

and

$$\begin{aligned} \theta' \circ (nu \times_R H) \circ \Delta(h) &= \theta'(\nu(h_{(1)}) \otimes_R h_{(2)}) = s(\nu(h_{(1)})(1_R))h_{(2)} \\ &= s(\underline{\epsilon}(h_{(1)}s(1_R)))h_{(2)} = s(\underline{\epsilon}(h_{(1)}))h_{(2)} \\ &= (\underline{\epsilon} \otimes_R H)\underline{\Delta}(h) = h. \end{aligned}$$

Therefore, $(H, s, t, \Delta, \nu)$ is a $\times_R$ bialgebra.

(3)$\Rightarrow$(1) Assume that $(H, s, t, \Delta, \nu)$ is a $\times_R$ bialgebra and define $\underline{\Delta} : H \to H \otimes_R H$ as $\underline{\Delta} = i \circ \Delta$, where $i : H \times_R H \to H \otimes_R H$ is the canonical inclusion. Define also the counit $\underline{\epsilon} : H \to R$ as $\underline{\epsilon}(h) = \nu(h)(1_R)$ as in the proposition 3.30. Again if $\Delta$ and $\nu$ are algebra morphisms and $R^e$ bimodule morphisms, then $\underline{\Delta}$ is an algebra and $R$ bimodule map, and $\underline{\epsilon}$ is an $R$ bimodule map. We need only to check that $\underline{\epsilon}(hs(\underline{\epsilon}(k))) = \underline{\epsilon}(ht(\underline{\epsilon}(k))) = \underline{\epsilon}(hk)$, but the proof is exactly the same as done in (2)$\Rightarrow$(1). Therefore $(H, s, t, \underline{\Delta}, \underline{\epsilon})$ is a left $R$ bialgebroid. $\square$

## 3.3 Hopf algebroids

The notion of a Hopf algebroid is the generalization of the notion of a Hopf algebra when we consider a noncommutative base ring $R$. As for bialgebroids, there are several different definitions of Hopf algebroids, but unlike in the bialgebroid case, the formulations for Hopf algebroids are not equivalent. In the sequel, we will follow the reference [2] to define Hopf algebroids.

**Definition 3.33.** Given two algebras $R$ and $L$ over a commutative ring $k$, a Hopf algebroid over the base algebras $R$ and $L$ is a triple $(H_L, H_R, \mathcal{S})$. Here, $(H_L, s_L, t_L, \underline{\Delta}_L, \underline{\epsilon}_L)$ is a left $L$ bialgebroid and $(H_R, s_R, t_R, \underline{\Delta}_R, \underline{\epsilon}_R)$ is a right $R$ bialgebroid, such that their underlying $k$ algebra $H$ is the same. The antipode $\mathcal{S}$ is a $k$ module map satisfying the following axioms:

(i) $s_L \circ \underline{\epsilon}_L \circ t_R = t_R,\ \ t_L \circ \underline{\epsilon}_L \circ s_R = s_R,\ \ s_R \circ \underline{\epsilon}_R \circ t_L = t_L,$ and $t_R \circ \underline{\epsilon}_R \circ s_L = s_L$.

(ii) $(\underline{\Delta}_L \otimes_R H) \circ \underline{\Delta}_R = (H \otimes_L \underline{\Delta}_R) \circ \underline{\Delta}_L$ and $(\underline{\Delta}_R \otimes_L H) \circ \underline{\Delta}_L = (H \otimes_R \underline{\Delta}_L) \circ \underline{\Delta}_R$.

(iii) For $l \in L$, $r \in R$ and $h \in H$, $\mathcal{S}(t_L(l)ht_R(r)) = s_R(r)\mathcal{S}(h)s_L(l)$.

(iv) $\mu_L \circ (\mathcal{S} \otimes_L H) \circ \underline{\Delta}_L = s_R \circ \underline{\epsilon}_R$ and $\mu_R \circ (H \otimes_R \mathcal{S}) \circ \underline{\Delta}_R = s_L \circ \underline{\epsilon}_L$.

**Remark 3.34.** This long definition requires some observations.

1. By the axioms of left/right bialgebroids, one can see that the maps $s_L \circ \underline{\epsilon}_L$, $t_L \circ \underline{\epsilon}_L$, $s_R \circ \underline{\epsilon}_R$ and $t_R \circ \underline{\epsilon}_R$ are idempotent maps in $\mathrm{End}_k(H)$. the axiom (i) says simply that the ranges of $s_L$ and $t_R$, also the ranges of $s_R$ and $t_L$ are coinciding subalgebras of $H$. This implies, for example, that $\underline{\Delta}_L$ is not only an $L$ bimodule map, but also an $R$ bimodule map. The same for $\underline{\Delta}_R$, it is an $R$ bimodule map.

2. It follows basically from the axiom (i) that the base algebras $L$ and $R$ are anti-isomorphic, this isomorphism is performed by $\underline{\epsilon}_L \circ s_R : R^{op} \to L$ and $\underline{\epsilon}_r \circ t_L : L \to R^{op}$.

3. The $k$ module $H$ underlying the left/right bialgebroid is a left and right comodule via the coproduct, that is, $H$ is a left and right $H_L$ comodule via the coproduct $\underline{\Delta}_L$ and a left and right $H_R$ comodule via the coproduct $\underline{\Delta}_R$. The axiom (ii) expresses that these regular coactions commute, that is, $H$ is an $H_L - H_R$ bicomodule and an $H_R - H_L$ bicomodule.

   Alternatively, the first identity in the axiom (ii) says that the comultiplication $\underline{\Delta}_L$ is a right $H_R$ comodule map. Symmetrically, the same identity can be viewed as a left $H_L$ comodule map property of $\underline{\Delta}_R$. The second identity says that $\underline{\Delta}_R$ is right $H_L$ colinear and $\underline{\Delta}_L$ is left $H_R$ colinear.

4. Axiom (iii) says that the antipode $\mathcal{S}$ is an $R - L$ bimodule map.

5. The axiom (iv) is the analogous to the axiom of antipode for Hopf algebras, but the notion of convolution products with two different base algebras is far more complicated and we are not going to discuss it here.

**Example 3.35.** Hopf algebras over a commutative ring $k$ are Hopf algebroids over $L = R = k$, with the left and right bialgebroid structure being the same and the antipode being the original antipode $S : H \to H$.

**Example 3.36.** Not every Hopf algebroid over a commutative ring $k$ is a Hopf algebra. Let $H$ be a Hopf algebra over $k$, one can consider the left bialgebroid structure on $H$ by its standard bialgebra structure $(H, \underline{\Delta}_L = \Delta, \underline{\epsilon}_L = \epsilon)$. Let $\chi : H \to k$ be a character, that is, a $k$ algebra map between $H$ and $k$. Then there is a second bialgebroid structure on $H$ with the twisted coproduct $\underline{\Delta}_R : H \to H \otimes H$, given by $\underline{\Delta}_R(h) = h_{(1)} \otimes \chi(S(h_{(2)}))h_{(3)}$ and the counit $\underline{\epsilon}_R = \chi$. Then we have a new antipode $\mathcal{S}$ defined by $\mathcal{S}(h) = \chi(h_{(1)})S(h_{(2)})$. This construction gives rise to a new Hopf algebroid structure on $H$ which is not a Hopf algebra.

**Example 3.37.** Weak Hopf algebras are Hopf algebroids [14], in particular, given a finite groupoid $\mathcal{G}$, its groupoid algebra $k\mathcal{G}$ is a Hopf algebroid.

**Example 3.38.** Given any algebra $R$ over a commutative ring $k$, the algebra $R\otimes R^{op}$ has a structure of left $R^{op}$ bialgebroid and right $R$ bialgebroid. One can put an structure of Hopf algebroid on $R\otimes R^{op}$ by defining the antipode as $\mathcal{S}(r\otimes \bar{s}) = s\otimes \bar{r}$.

**Example 3.39.** Let $\mathcal{G}$ be a finite groupoid and $k$ a commutative ring. Define $H$ be the algebra $k$ valued functions on $\mathcal{G}$ and $R = L$ the algebra of $k$ valued functions on the space $\mathcal{G}^{(0)}$ of the units of the groupoid $\mathcal{G}$. Define $s_L = t_R : R \to H$ by the formula $(s_L(f))(g) = f(t(g))$, and $t_L = s_R : R \to H$ by the formula $(t_L(f))(g) = f(s(g))$, where $s, t : \mathcal{G} \to \mathcal{G}^{(0)}$ are the source and target maps on the groupoid. The left and right Takeuchi product, as the algebras are commutative, coincide with the algebra of $k$ valued functions on the space

$$\mathcal{G}^{(2)} = \{(g,h) \in \mathcal{G}\times\mathcal{G} \,|\, s(g) = t(h)\},$$

that is, the set of pairs $(g,h)$ such that there exists the element $gh$. The left/right comultiplication is defined by

$$\underline{\Delta}(f)(g,h) = \begin{cases} f(gh) & \text{if} \quad (g,h)\in\mathcal{G}^{(2)} \\ 0 & \qquad\text{otherwise} \end{cases}$$

and the counit for both, left and right structures is simply the restriction to $\mathcal{G}^{(0)}$. The antipode is given by the inversion in the groupoid, that is $(\mathcal{S}(f))(g) = f(g^{-1})$. This gives to $H$ a structure of an $R$ Hopf algebroid.

**Example 3.40.** Given a commutative Hopf algebra $H$ and a commutative right $H$ comodule algebra $A$, we saw that $A\otimes H$ has two bialgebroid structures. These bialgebroid structures can be seen easily that satisfy the axioms (i) and (ii). The antipode in $A\otimes H$ is given by $\mathcal{S}(a\otimes h) = a^{(0)}\otimes a^{(1)}S(h)$. The details can be found in [1] for the case of partial coactions, but the ideas are easily translated to the global case.

**Example 3.41.** Given a cocommutative Hopf algebra $H$ and a commutative left $H$ module algebra $A$, the smash product $A\#H$ has two bialgebroid structures satisfying the compatibility axioms (i) and (ii). One can define the antipode for $A\#H$ by $\mathcal{S}(a\otimes H) = (S(h_{(2)})\cdot a)\otimes S(h_{(1)})$. The details can be found in [1] for the case of partial actions, but the ideas are easily translated to the global case.

**Example 3.42.** (The algebraic quantum torus) consider the $k$ algebra $\mathbb{T}_q$ generated by two invertible elements $U$ and $V$, satisfying the commutation relation $UV = qVU$, where $q\in k$ in an invertible element. The algebra $\mathbb{T}_q$ is endowed with a right bialgebroid structure over the commutative subalgebra $R = k[U]$. The source and target maps are given by the inclusion map $s_R = t_R = i : R \to \mathbb{T}_q$. The right comultiplication is given by $\underline{\Delta}_R(V^mU^n) = V^mU^n\otimes_R V^m$ and the right counit is given by $\underline{\epsilon}_R(V^mU^n) = U^n$. Symmetrically, there is a left $R$ bialgebroid structure with left comultiplication given by $\underline{\Delta}_L(U^nV^m) = U^nV^m\otimes V^m$ and left counit given by $\underline{\epsilon}_L(U^nV^m) = U^n$. Finally, the antipode for the Hopf algebroid structure is given by $\mathcal{S}(U^nV^m) = V^{-m}U^n$. We leave to the reader the verification of the details.

As we said before, there are some alternative constructions for bialgebroids. One of the formulations is due to Peter Schauenburg, which introduces the concept of a $\times_R$ Hopf algebra.

**Definition 3.43.** Let $H$ be a left bialgebroid over an algebra $L$. Consider the left regular $H$ comodule, whose coinvariant subalgebra is $t(L^{op})$ The algebra $H$ is said to be a $\times_L$ Hopf algebra provided that the algebra extension $t : L^{op} \to H$ is a left $H$ Galois extension, that is, the canonical map, $\mathrm{can} : H\otimes_{L^{op}} H \to H\otimes_L H$, given by $\mathrm{can}(h\otimes_{L^{op}} k) = h_{(1)}\otimes_L h_{(2)}k$, is bijective.

Any Hopf algebroid $(H_L, H_R, \mathcal{S})$ is a $\times_L$ Hopf algebra (in this case $L^{op}\cong R$. Indeed, the canonical map has the inverse $\mathrm{can}^{-1}(h\otimes_L k) = h_{[1]}\otimes_R \mathcal{S}(h_{[2]})k$, where $\underline{\Delta}_R(h) = h_{[1]}\otimes h_{[2]}$. Although it is believed that not every $\times_L$ Hopf algebra is the constituent left bialgebroid of a Hopf algebroid, no couterexamples are known so far. The concept of a $\times_L$ Hopf algebra makes a bridge with some representation theoretical characterizations of Hopf algebras, like the following result.

**Theorem 3.44.** *[13] A left bialgebroid $H$ over a base algebra $L$ is a$\times_L$ Hopf algebra if, and only if the strict monoidal forgetful functor ${}_H\mathcal{M} \to {}_L\mathcal{M}_L$ preserves internal Homs.*

# Appendix A

# Some basics of category theory

In this appendix we review the basic facts about category theory. In order to fix the ideas, given a category $\mathcal{C}$, we are going to consider it, unless otherwise stated, as a locally small category, that is, the class of morphisms between any two objects in this category will be a set. Given two objects $A$ e $B$ in a category $\mathcal{C}$, we denote the set of morphisms between $A$ and $B$ by $\mathcal{C}(A, B)$ the identity morphism of an object $A$ in $\mathcal{C}$ also will be denoted simply by $A$, instead of $I_A$ or $\mathrm{Id}_A$. In the proofs throughout this text, we always consider the functors $F : \mathcal{C} \to \mathcal{D}$ to be covariant. Obviously, the proofs can be made for contravariant functors as well, in this case we can consider the covariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$ associated to $F$. The identity functor of the category $\mathcal{C}$ will be denoted by $\mathrm{Id}_{\mathcal{C}}$.

**Definition A.1.** A natural transformation between two functors $F, G : \mathcal{C} \to \mathcal{D}$ is a family of morphisms

$$\alpha = (\alpha_A \in \mathcal{D}(F(A), G(A)))_{A \in \mathcal{C}}$$

such that, for each morphism $f \in \mathcal{C}(A, B)$ the following diagram commutes.

$$\begin{tikzcd}
F(A) \arrow[r, "\alpha_A"] \arrow[d, "F(f)"'] & G(A) \arrow[d, "G(f)"'] \\
F(B) \arrow[r, "\alpha_B"'] & G(B)
\end{tikzcd}$$

We denote this natural transformation by $\alpha : F \Rightarrow G$. If, in addition, each morphism $\alpha_A$ is an isomorphism, we say that $\alpha$ is a natural isomorphism between the functors $F$ e $G$.

In category theory, the problem of classification of categories is more delicate than the classification problem of other algebraic structures, like modules, rings, algebras, etc. A first attempt is to say when two categories are isomorphic. We say that the categories $\mathcal{C}$ and $\mathcal{D}$ are isomorphic when there exist two functors

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"]
\end{tikzcd} \tag{A.1}$$

such that $\mathrm{Id}_{\mathcal{C}} = GF$ e $FG = \mathrm{Id}_{\mathcal{D}}$. This concept is in general useless for many porpuses in category theory, relevant and nontrivial examples of isomorphic categories are rare. One weaker version of isomorphism is given by the concept of equivalence between categories. We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if there exist two functors, as denoted in (A.1), and two natural isomorphisms $\mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ e $FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$. There is an alternative characterization of categorical equivalence given by the following theorem.

**Theorem A.2.** *Two categories $\mathcal{C}$ and $\mathcal{D}$ are equivalent if, and only if, there is a functor $F : \mathcal{C} \to \mathcal{D}$ satisfying the following conditions:*

*(i) $F$ is essentially surjective, that is, for any object $B \in \mathcal{D}$ there exists an object $A \in \mathcal{C}$ such that $Y \cong F(X)$.*

*(ii) $F$ is faithful, that is, for each pair of objects $A, B \in \mathcal{C}$ the map $F : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ is injective.*

*(iii) $F$ is full, that is, for each pair of objects $A, B \in \mathcal{C}$ the map $F : \mathcal{C}(A, B) \to \mathcal{D}(F(A), F(B))$ is surjective.*

**Example A.3.** Given two rings $R$ and $S$, when their categories of modules ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ are equivalent we say that the rings are Morita equivalent. A necessary and sufficient condition to $R$ and $S$ to be Morita equivalent is the existence of an exact Morita context, this is the content of the Morita Theorem. A Morita context between $R$ and $S$ is a quadruple $(M, N, \sigma, \tau)$, in which $M$ is an $R - S$ bimodule, $N$ is an $S - R$ bimodule, $\sigma : M \otimes_S N \to R$ is a morphism of $R$ bimodules and $\tau : N \otimes_R M \to S$ is a morphism of $S$ bimodules satisfying the following compatibility conditions:

1. For any $m_1 \otimes n \otimes m_2 \in M \otimes_S N \otimes_R M$ we have
$$\sigma(m_1 \otimes n) \cdot m_2 = m_1 \cdot \tau(n \otimes m_2).$$

2. For any $n_1 \otimes m \otimes n_2 \in N \otimes_R M \otimes_S N$ we have
$$\tau(n_1 \otimes m) \cdot n_2 = n_1 \cdot \sigma(m \otimes n_2).$$

A Morita context is said to be exact if the morphisms $\sigma$ and $\tau$ are epimorphisms. The equivalence between ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ is performed by the functors

$$\begin{tikzcd} {}_R\mathcal{M} \arrow[r, shift left, "N\otimes_R \_"] & {}_S\mathcal{M} \arrow[l, shift left, "M\otimes_S \_"] \end{tikzcd}$$

It is not difficult to prove that any ring $R$, is Morita equivalent to the matrix ring $M_n(R)$ for any integer $n \geq 1$.

**Example A.4.** Another famous equivalence between categories is the Serre-Swan Theorem, which proves that the category of (complex) vector bundles over a compact topological Hausdorff space $X$ is equivalent to the category of finitely generated projective modules over the algebra $C(X)$ of the $\mathbb{C}$ valued continuous functions defined on $X$. Given a vector bundle $E \xrightarrow{\pi} X$, one can define the set $\Gamma(E)$ of continuous sections on the vector bundle $E$. The set $\Gamma(E)$ is in fact a module over $C(X)$ with the sum and multiplication by functions given pointwise. In fact, because of the compactness of $X$ one can prove that $\Gamma(E)$ is a finitely generated projective $C(X)$ module. This construction defines a functor $\Gamma : \underline{\text{VectBun}}_X \to {}_{C(X)}\underline{\text{FGProj}}$ and this functor is fully faithful and essentially surjective, and this means that the categories are equivalent.

**Example A.5.** We say that two categories $\mathcal{C}$ and $\mathcal{D}$ are dual if $\mathcal{D}$ is equivalent to $\mathcal{C}^{op}$, that is the equivalence is performed by contravariant functors. The most trivial example of duality is given in the category of finite dimensional vector spaces and the functor is the dualization, which associates to any finite dimensional vector space its dual space and to each linear transformation its dual transformation, namely its transpose, this functor is contravariant. If one applies the dualization functor twice then a classical result gives us the natural isomorphism between the identity functor $Id$ and the bidual $(\ )^{**}$.

**Example A.6.** Finally, another less trivial example of duality of categories is given by the Theorem of Gelfand and Naimark, which states that the category of commutative unital $C^*$ algebras is dual to the category of compact Hausdorff topological spaces. Given a commutative unital $C^*$ algebra $A$ we associate the space $Spec(A)$ which is the set of all non vanishing characters of of $A$, a character is a multiplicative linear functional defined on $A$. The set $Spec(A)$, endowed with the weak$^*$ topology is a compact Hausdorff topological space. On the other hand, given a compact Hausdorff topological space $X$ , one can construct the unital commutative $C^*$ algebra $C(X)$. Then one can prove that $C(Spec(A)) \cong A$ as $C^*$ algebras and the space $X$ is homeomorphic to $Spec(C(X))$.

Even when two categories are not equivalent, the very existence of a pair of functors, as in (A.1), satisfying weaker properties than those required for categorical equivalence, allows one to prove that some results or some constructions in one category are also valid in the other.

**Definition A.7.** We say that two functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ form an adjunction, or that they are adjoint, if there exist two natural transformations $\eta : \mathrm{Id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{Id}_{\mathcal{D}}$ such that for each object $A \in \mathcal{C}$ and $B \in \mathcal{D}$ the following diagrams commute

$$\begin{tikzcd} F(A) \arrow[r, "F(\eta_A)"] \arrow[dr, "F(A)"'] & FGF(A) \arrow[d, "\epsilon_{F(A)}"] \\ & F(A) \end{tikzcd} \qquad \begin{tikzcd} G(B) \arrow[r, "\eta_{G(B)}"] \arrow[dr, "G(B)"'] & GFG(B) \arrow[d, "G(\epsilon_B)"] \\ & G(B) \end{tikzcd}$$

These two natural transformations $\eta$ e $\epsilon$ are called, respectively, the unit and the counit of the adjunction. The functor $F$ is said to be the left adjoint of $G$, respectively, the functor $G$ is the right adjoint of $F$.

There is another equivalent characterization of an adjunction.

**Theorem A.8.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ a pair of functors. Then the following are equivalent:*

*(i) The functors $F$ and $G$ form an adjunction, with $F$ being the left adjoint of $G$.*

*(ii) There exist a natural isomorphism $\phi : \mathcal{D}(F(\_), \_) \Rightarrow \mathcal{C}(\_, G(\_))$ (these two functors are defined from $\mathcal{C} \times \mathcal{D}$ to the category of sets).*

**Example A.9.** The forgetful functor $U : \underline{\mathrm{Grp}} \to \underline{\mathrm{Set}}$ is the functor which associates to each group $G$ its underlying set $G$, and sees each group homomorphism $f : G \to H$ only as a function. This functor has a left adjoint, the free group, which associates to each set the unique group $\mathcal{F}(X)$, with a function $i : X \to \mathcal{F}(X)$ with the following universal property: For each group $G$ and each function $f : X \to X$ there is a unique group homomorphism $\widehat{f} : \mathcal{F}(X) \to G$ such that $\widehat{f} \circ i = f$. The universal property garantees the uniquenes of the free group, the functorial nature of $\mathcal{F} : \underline{\mathrm{Set}} \to \underline{\mathrm{Grp}}$ and the fact that $\mathcal{F}$ is the left adjoint of the forgetful functor $U$.

**Example A.10.** A famous example of adjunction is the Hom tensor relation: Consider a ring $R$ and a left $R$ module $M$, in particular $M$ is an abelian group. For any abelian group $X$, the set of additive maps between $M$ and $P$, $\mathrm{Hom}_{\mathbb{Z}}(M, P)$, has the structure of a right $R$ module, the addition is given pointwise and the right $R$ action is given by $(f \cdot r)(m) = f(r \cdot m)$. This, in fact defines a covariant functor

$$\mathrm{Hom}_{\mathbb{Z}}(M, \_) : \underline{\mathrm{Ab}} \to \mathcal{M}_R.$$

This functor has a left adjoint, given by the tensor product functor

$$\_ \otimes_R M : \mathcal{M}_R \to \underline{\mathrm{Ab}}.$$

This is because for any right $R$ module $N$ and any abelian group $P$ there is an isomorphism

$$\mathrm{Hom}_{\mathbb{Z}}(N \otimes_R M, P) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_{\mathbb{Z}}(M, P)).$$

and this isomorphism is natural in both $N$ and $P$.

**Example A.11.** Given a group $G$ and a subgroup $H \leq G$, any $kG$ module $W$ can be viewed naturally as a $kH$ module, let us denote it by $(W)_H$. It is easy to see that this defines a functor $(\_)_H : {}_{kG}\mathcal{M} \to {}_{kH}\mathcal{M}$. On the other hand, given a $kH$ module $V$ one can construct a $kG$ module out of it by tensoring with $kG$, then we have $(V)^G = kG \otimes_{kH} V$, again it is easy to see that $(\_)^G : {}_{kH}\mathcal{M} \to {}_{kG}\mathcal{M}$ is a functor. Now we have the isomorphism of $k$ vector spaces

$$ {}_{kG}\mathrm{Hom}((V)^G, W) \cong {}_{kH}\mathrm{Hom}(V, (W)_H) $$

given in the following way: For each $kG$ morphism $F : (V)^G \to W$ we associate the map $F^\flat : V \to (W)_H$ defined as $F^\flat(v) = F(e \otimes v)$ we leave to the reader the verification that $F^\flat$ is a $kH$ module morphism. On the other hand for each $kH$ morphism $f : V \to (W)_H$, we associate the map $f^\sharp : (V)^G \to W$ given by $f^\sharp(\sum_{g\in G} a_g g \otimes v) = \sum_{g\in G} a_g g \cdot f(v)$. Again it is easy to see that this is a morphism of $kG$ modules. Let us verify that they are mutually inverse. Take $F \in {}_{kG}\mathrm{Hom}((V)^G, W)$ and and element $\sum_{g\in G} a_g g \otimes v \in (V)^G$, then

$$ (F^\flat)^\sharp(\sum_{g\in G} a_g g \otimes v) = \sum_{g\in G} a_g g \cdot F^\flat(v) = \sum_{g\in G} a_g g \cdot F(e \otimes v) = F(\sum_{g\in G} a_g g \otimes v). $$

Now, take $f \in {}_{kH}\mathrm{Hom}(V, (W)_H)$ and $v \in V$, then

$$ (f^\sharp)^\flat(v) = f^\sharp(e \otimes v) = e \cdot f(v) = f(v). $$

It is easy to see that this isomrphism is natural in $V$ and $W$. Therefore the functor $(\_)^G$ is the left adjoint of $(\_)_H$.

Obviously, every equivalence of categories is an adjunction, but in general we have adjunctions that are not equivalences. Nevertheless, there are some useful results on adjunctions which gives conditions to the functors of an adjunction to be fully faithful or even to be an equivalence.

**Proposition A.12.** *Given an adjunction*

$$ \begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"] \end{tikzcd} $$

*The functor $F$ is fully faithful if, and only if, the unit of the adjunction is bijective. The functor $G$ is fully faithful if, and only if, the counit of the adjunction is bijective. There is an equivalence of categories between $\mathcal{C}$ and $\mathcal{D}$ if and only if, the unit and the counit of the adjunction are bijective.*

# Bibliography


[1] E. Batista and J. Vercruysse, *Dual constructions for partial actions of Hopf algebras*, arxiv:1403.1399 (2014).

[2] G. Böhm, *Hopf algebroids*, Handbook of Algebra, vol. **6** (2009), 173–235.

[3] T. Brzezinski and G. Militaru, *Bialgebroids, $\times_A$-Bialgebras and Duality*, J. Algebra **251** (2002) 279–294.

[4] T. Brzezinski and R. Wisbauer, *Corings and comodules*, London Math. Soc. Lecture Note Series **309**, Cambridge Univ. Press. (2003).

[5] S. Caenepeel and J. Vercruysse, *Hopf Algebras*, Lecture Notes, Vrije Universiteit Brussels (2012).

[6] P. Etingof, O. Schiffmann, *Lectures on Quantum Groups*, International Press, 2nd edition (2002).

[7] N. Jacobson, *Basic Algebra*, vol. II, 2nd Edition, Dover (2009).

[8] A. Joyal, R. Street, *Braided Tensor Categories*, Advances in Mathematics **102** Issue 1 (1993), 20–78.

[9] C. Kassel,*Quantum Groups*, Graduate Texts in Mathematics, Springer-Verlag (1995).

[10] J. Kock, *Elementary remarks on units in monoidal categories*, Math. Proc. Cambridge Phil. Soc. **144** (2008), 53–76.

[11] S. Mac Lane,*Categories for the Working Mathematician*, Graduate Texts in Mathematics, Springer-Verlag (2010).

[12] J. J. Rotman, *An introduction to Homological Algebra*, Universitext, 2nd edition, Springer-Verlag (2008).

[13] P. Schauenburg, *Bialgebras over noncommutative rings and structure theorems for Hopf bimodules*, Appl. Categorical Structures **6** (1998) 192–222.

[14] P. Schauenburg, *Weak Hopf algebras and quantum groupoids*, Banach Center Publications **61**, Polish Acad. Sci. (2003) 171–188.

[15] M. Takeuchi,*Groups of algebras over $A \otimes \overline{A}$*, J. Math. Soc. Japan **29** (1977) 459–492.

[16] P. Xu, *Quantum groupoids*, Commun. Math. Phys. **216** (2001) 539–581.