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A NOVEL SPIN-STATISTICS THEOREM IN $(2+1)$D CHERN–SIMONS GRAVITY

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It has been known for some time that topological geons in quantum gravity may lead to a complete violation of the canonical spin-statistics relation: There may be no connection between spin and statistics for a pair of geons. We present an algebraic description of quantum gravity in a $(2+1)$D manifold of the form $\Sigma \times \mathbb{R}$, based on the first-order canonical formalism of general relativity. We identify a certain algebra describing the system, and obtain its irreducible representations. We then show that although the usual spin-statistics theorem is not valid, statistics is completely determined by spin for each of these irreducible representations, provided one of the labels of these representations, which we call *flux*, is superselected. We argue that this is indeed the case. Hence, a new spin-statistics theorem can be formulated.

1. Introduction

In general relativity, although the metric of space–time is a dynamical entity determined by Einstein's field equations, the underlying topology is not *a priori* determined. On a closer inspection, however, one actually finds that once one imposes that space–time should possess some physically reasonable geometrical conditions, the presence of nontrivial topology is constrained. Simple examples are the well-known constraints on the space–time topology in Robertson–Walker models. Also, in classical general relativity, when some standard types of energy conditions are valid, nontrivial spatial topology may lead to singularities in space–time: Gannon's theorem[1] (see also Ref. 2) implies that, in a space–time satisfying the weak energy condition, if one attempts to develop Cauchy initial data on a spatial three-manifold[a] with a non-simply connected topology, the corresponding Cauchy development will be geodesically incomplete to the past or to the future. The so-called active topological censorship theorem[3] formulated more recently states that in a globally hyperbolic, asymptotically flat space–time obeying an averaged null energy condition (ANEC), every causal curve beginning and ending at the boundary at infinity can be homotopically deformed to that boundary. Therefore, an external observer near that boundary would not be able to probe the non-simply connectedness of space–time. This result has been extended to more general contexts than the asymptotically flat case, such as asymptotically anti-de Sitter space–times (see Ref. 4 and references therein).

In spite of such results, there is still much room left for investigation of the physical consequences of having a nontrivial spatial topology, especially in quantum theory. On the one hand, even in the classical case one can have nontrivial *compact* spatial topologies, which evade the conditions of the above cited theorems and also have physical interest, and on the other hand, in quantum theory, the energy conditions to prove these theorems are often violated: for example Wald and Yurtsever[5] show that ANEC is violated by the renormalized stress–tensor of free fields in generic curved space–times. Indeed, it is the existence of this so-called quantum "exotic" matter that permits the violation of the classical area theorem by evaporating black holes,[6] and the existence of "traversable" wormholes, despite the above-mentioned theorems (see, e.g., Ref. 7 for an extensive account). Moreover, it is widely believed that quantum gravity effects will alter the topology of space–time at Planck scales ("space–time foam"). Indeed, some semiclassical calculations indicate that a configuration with the presence of wormholes is energetically favored over the Euclidean one.[8]

Topological geons, which are the subject of this letter, are topological structures with some remarkable properties. They were first studied by Friedmann and Sorkin,[9] as "localized excitations of spatial topology", or "lumps" of nontrivial topology in an otherwise Euclidean spatial background. The idea was to view such entities as particles much in the same way as solitons in a field theory. The presence of geons can give rise to half-integer spin states and fermionic or even fractional statistics, in pure (i.e. without matter) quantum gravity.[9,10] It is common in the literature to refer to such solitonic states as geon states. We follow this usage here.

Geons being soliton-like objects, we can talk about their spin and statistics. In Refs. 11 and 12, it was shown that such states could violate the usual spin-statistics theorem, in $(3+1)$D and $(2+1)$D, if the spatial topology is assumed not to change in time, or more precisely if the topology of the space–time M is of the form $M = \Sigma \times \mathbb{R}$. On a space–time of the form $M = \Sigma \times \mathbb{R}$, the topology of a spatial slice is well-captured by the geons on Σ. For example, in the $(2+1)$D context that

[a]More precisely, a partial Cauchy surface regular near infinity — see Ref. 1 for the appropriate definitions.

we are interested in this letter, the topology of an orientable, connected surface Σ representing space, with at most one asymptotic region, is completely specified by the number of handles. Each handle corresponds to a geon in this simple context. Accordingly, topology changes are always associated with creation and annihilation of geons. It has been suggested[13] that the standard spin-statistics relation can be recovered if geons can be created and annihilated, in other words, topology change may be required in order to establish the full spin-statistics theorem for geons. In this letter we seek instead a relation between spin and statistics assuming a *fixed* spatial topology.

To appreciate the importance of having or not having a spin-statistics connection for geons, one must recall that in ordinary quantum field theories in Minkowski space–time, the particles which arise when we second quantize, for example, have this connection naturally. Now, in a hypothetical quantum theory of gravity, one could think of geons as a "particle", representing the excitations of the topology itself. It seems therefore natural to ask whether they share this connection with normal particles. We find that in the formalism we develop here a different, weaker version of the spin-statistics connection arises, instead of the normal one.

Before we describe our approach to this situation, we examine more carefully what is meant by spin and statistics. Let us assume that we have a configuration space Q describing a pair of identical geons. One such configuration can be visualized as two handles on the plane. Now, the quantization of two geons on the plane is not unique. One has to choose some Hermitian vector bundle B_k over Q whose square-integrable sections (with a suitable measure) serve to define the domains of appropriate observables,[9,11,12] and are the "wave functions" in the quantum theory. The index k labels inequivalent quantizations. The space of these sections is the quantum Hilbert space $\mathcal{H}_k$ of the two-geon system. Physical operations can be implemented as operators on $\mathcal{H}_k$. If we perform a 2π-rotation of one of the geons, described by an operator $C_{2\pi}$, then its eigenstate will change by a phase $e^{i2\pi S}$, where S is the spin. Just like particles in $(2+1)$D, geons can carry fractional spin, i.e. S can be any real number.[10,12] Similarly, if we exchange the position of the two geons, the wave function will change by the action of an operator $\mathcal{R}$ that we call the statistics operator. The standard spin-statistics relation would tell us that the action of $\mathcal{R}$ on a two-geon system should be equivalent to acting with the operator $C_{2\pi}$ on one of the geons. Note that there is no *a priori* reason for this relation to hold since $C_{2\pi}$ and $\mathcal{R}$ correspond to two independent diffeomorphisms of Σ. Now one can ask if such a relation is true for each quantization procedure parametrized by k. The results of Refs. 10–13 shed some light on the problem. The authors show that some quantizations violate the spin-statistics theorem, but leave open the question of which are the ones that do not. Furthermore, as emphasized in Ref. 10, the list of quantum theories derived in Ref. 12 is completely based on kinematic considerations. In other words, only the diffeomorphism constraint is imposed, whereas the Hamiltonian constraint, which gives the dynamical features of gravity, is not considered at the quantum level. Imposing the latter would further restrict the states, and in this sense some of the values of k may not be dynamically allowed.

In this letter we show that, at least for $(2+1)$D gravity in the first-order formalism, there is a generalization of the standard spin-statistics connection relating $\mathcal{R}$ and $C_{2\pi}$, even for a fixed spatial topology, i.e. for space–time manifolds of the form $\Sigma \times \mathbb{R}$. We shall consider Σ to be a one-point compactified two-manifold, i.e. we compactify the spatial manifold with one asymptotic region by adding a "point at infinity". In the quantization scheme given in Ref. 12, one considers the mapping class group M_Σ (the group of "large" *spatial* diffeomorphisms, not connected to the identity of $\mathrm{Diff}(\Sigma)$) and finds a vector bundle B_k for each unitary irreducible representation of M_Σ. Then, one sees no relation between $\mathcal{R}$ and $C_{2\pi}$ for a generic k. The physical significance of this procedure is as follows. Physical states in quantum gravity obey the diffeomorphism constraint, meaning that they are invariant under "small" diffeomorphisms, i.e. the diffeomorphisms connected to the identity of $\mathrm{Diff}(\Sigma)$, which are the ones generated by this constraint. The diffeomorphism constraint means that "small" diffeos should be regarded as gauge, but leaves one free to consider the states either as invariant under the "large" diffeos (those not connected to the identity of $\mathrm{Diff}(\Sigma)$), in which case the "large" diffeomorphisms are also viewed as gauge, or just "covariant", i.e. transforming by a unitary representation of the mapping class group. In this approach, "large" diffeos are regarded as a symmetry of the theory. We adopt the latter view in this work, the former being a special case of this view.

We will look at M_Σ as part of a larger algebra $\mathcal{A}$ of operators describing the quantum theory of geons. It contains the group algebra of M_Σ. Let us give an intuitive account of $\mathcal{A}$. We start by considering the classical (reduced) configuration space $\tilde{Q}$ of $(2+1)$D gravity in the first-order formalism which is based on the $\mathrm{SO}(2,1)$ gauge group. It is well known that this is the space of flat $\mathrm{SO}(2,1)$ bundles over the space manifold Σ. As we will discuss in more detail in the body of the letter, this space admits a natural measure. The wave functions are then taken to be square-integrable functions with respect to this measure. We now describe the algebra $\mathcal{A}$ used for quantization. In building this algebra, we consider only the minimum needed to investigate the spin-statistics connection. First, we comment on its general structure. Its first component consists of the operators of "position" type on the space $\tilde{Q}$ and corresponds to the commutative algebra $\mathcal{F}(\tilde{Q})$ of continuous functions of compact support $f : \tilde{Q} \to \mathbb{C}$. Next we consider the operators corresponding to the symmetries of the theory. The gauge group $\mathrm{SO}(2,1)$ acting on $\tilde{Q}$ induces an action on the functions. Again, instead of $\mathrm{SO}(2,1)$, we take its group algebra $\mathcal{G}$. Finally, we also include the algebra $\mathcal{U}$ of (suitable) remaining operators acting on $\mathcal{F}(\tilde{Q})$. In other words, $\mathcal{A}$ has the structure

$$\mathcal{A} = (\mathcal{U} \otimes \mathcal{G}) \ltimes \mathcal{F}(\tilde{Q})\,. \tag{1.1}$$

We then choose the algebra $\mathcal{U}$ to be the group algebra of M_Σ. It contains all the operations necessary to investigate the spin-statistics connection.

Another important feature is that the first-order formalism naturally takes into account the dynamical constraints. The possible quantizations are given by irreducible $*$-representations Π_r of $\mathcal{A}$, where the index r parametrizes inequivalent quantizations. We show that there is a large class of quantizations Π_r such that statistics is totally determined by spin according to the formula

$$\Pi_r(\mathcal{R}) = e^{i(2\pi S - \theta[r])}\mathbb{I}\,, \tag{1.2}$$

on state vectors of spin S. Here the extra phase $\theta[r]$ is completely fixed by the choice of the representation Π_r.

The rest of the letter is organized as follows. In Sec. 2 we briefly review the first-order formalism of general relativity and deduce the classical configuration space and the group actions thereon. We then proceed to the construction of the algebra. The geon algebra can be viewed as an example of a *transformation group algebra*, first studied by Glimm,[14] and the representation theory of this algebra is known. In Sec. 3 we analyze more closely the structure of the algebra and classify the irreducible $*$-representations. We then show how a class of states in these representations possesses a spin-statistics connection, namely those states which are eigenstates of a certain charge operator. These states are then argued to be the true physical states, due to a superselection rule. We end the letter with some final remarks.

2. The Connection Formalism

In the first-order formalism, one takes as fundamental variables a triad $e^{(3)a} = e^{(3)a}_\mu\, dx^\mu$, possibly degenerate and an $\mathrm{SO}(2,1)$ connection one-form $A^{(3)a} = \frac{1}{2}\varepsilon^{abc}\omega^{(3)}_{\mu bc}\, dx^\mu$, where $\omega^{(3)a}_{bc}$ is the spin connection.[b] The Einstein–Hilbert action takes the form

$$S = \int_M e^{(3)a} \wedge F^{(3)}_a + \text{boundary terms}\,, \tag{2.1}$$

where $F^{(3)}_a = d_M A^{(3)}_a + \frac{1}{2}\varepsilon_{abc}A^{(3)b} \wedge A^{(3)c}$ is the usual curvature for the connection $A^{(3)}$. In our convention, Lorentz space–time indices are represented by Greek letters, and spatial indices by Latin letters $i, j = 1, 2$. Internal $\mathrm{SO}(2,1)$ indices are represented by Latin letters $a, b = 0, 1, 2$. Boundary terms arise[15,16] in the cases in which the spatial manifold Σ is non-compact, or compact with boundary, and are of course zero for closed Σ.

Upon variation of the action (2.1) with respect to $A^{(3)}$ and $e^{(3)}$, we find the equations of motion

$$\begin{aligned} F^{(3)a} &= 0\,, \\ D_M e^{(3)a} &= 0\,, \end{aligned} \tag{2.2}$$

[b]In our notation, the superscript (3) on the upper right denote fields on the three-dimensional space–time M, of the form $\Sigma \times \mathbb{R}$ and fields without superscript correspond to their pullbacks to Σ.

where D_M denotes covariant differentiation with respect to the connection $A^{(3)}$. Let us consider the equations of motion (2.2) in coordinates. Since M is taken to be of the form $\Sigma \times \mathbb{R}$, we can use a "space + time" splitting. We then obtain the following set of equations for the spatial components:

$$\begin{aligned} F^a_{ij} &= 0\,, \\ D_{[i} e^a_{j]} &= 0\,, \end{aligned} \tag{2.3}$$

which are nothing but the pullback of Eq. (2.2) to Σ by the natural inclusion $\Sigma \hookrightarrow \Sigma \times \mathbb{R}$: $x \mapsto (x, 0)$. The covariant differentiation is now with respect to the pullback A of the connection $A^{(3)}$. Note that Eqs. (2.3) do not involve time derivatives of the basic fields: they are just constraints on the fields e^a and A_a on Σ at any given time, and initial data are a set of basic fields on Σ satisfying these constraints. The remaining equations are the time evolution equations for e^a and A_a. Since we shall not make explicit use of the latter, we omit them here.

A_{aj} and $\varepsilon^{ij} e^a_i$, $i = 1, 2$ are canonically conjugate variables defined on Σ. The pairs (e^a, A^a) obeying the constraints span the (reduced) phase space $\mathcal{P}$ of the theory, which is just the cotangent bundle of the space of $\mathrm{SO}(2,1)$ connections on Σ. The canonical symplectic structure is given by the Poisson brackets coming from (2.1). The only nonvanishing ones are:

$$\{A^a_i(x), e^b_j(y)\} = \frac{1}{2}\delta_{ab}\varepsilon_{ij}\delta^{(2)}(x-y)\,, \tag{2.4}$$

where $x, y \in \Sigma$.

The quantum theory in the "position representation" would be described by wave functionals $\psi[A]$. The constraints can be easily imposed before quantization, and one then quantizes only the physical degrees of freedom. When Σ is a closed (i.e. compact and boundaryless) two-surface, the constraints imply[10,17] that the physical configuration space Q is given by the moduli space of flat connections, i.e. the set of equivalence classes of flat connections on Σ under gauge transformations. When Σ is non-compact, however, one has to specify how fields behave asymptotically. This choice gives rise to boundary terms in (2.1),[15,16] and the physical configuration space is the space of those flat connections which have the appropriate asymptotic behavior, modulo those gauge transformations which preserve this behavior.

The full analysis becomes considerably more complicated in the non-compact case because of the asymptotic considerations involved. To simplify matters we just perform a one-point-compactification of Σ, by adding a point p_∞, the "point at infinity", since the boundary terms in (2.1) will play no role here. "Rotations" of geons will be considered to be about this point, and we also fix a frame there. Thus, Σ is topologically taken to be a closed surface with a marked point and a frame attached there.

Again, just like in the usual closed case, the configuration space is the space of all flat connections. However, gauge transformations which are not trivial at infinity are *not* a symmetry of the theory. Therefore, in our case, configurations which differ by a gauge transformation which is not trivial at p_∞ should not be viewed as equivalent. The reduced configuration space in this case is therefore the moduli space space of flat connections modulo gauge transformations which are trivial at p_∞.

Note also that we only need regular flat initial data on Σ to define the configuration space Q and to quantize. We make no assumption as to geodesic completeness, and in particular, the formalism can accommodate geodesically incomplete classical solutions. This is important, because in classical general relativity, Gannon's theorems[1] imply that singularities must arise due to the multiple connectivity of Σ, at least when Σ is non-compact, under certain mild physical assumptions. Even if the formation of singularities occurs in our case, this seems not to interfere with the quantization procedure, at least formally. On the other hand, precisely because of this independence, it is not clear at this point what are the implications, if any, of such singularities in the quantum theory.

A connection A on Σ is determined by its holonomies. For each closed curve γ based at p_∞ compute the holonomy $W([\gamma]) = Pe^{\int_\gamma A}$. This quantity is invariant under gauge transformations that are identity at p_∞. Since A is flat, $W([\gamma])$ is invariant under small deformations of γ preserving p_0. In other words, it depends only on the homotopy class $[\gamma]$ of loop γ. In fact, W gives a homomorphism $\pi_1(\Sigma) \to \mathrm{SO}(2,1)$.

Let $\tilde{Q}$ be the set of all such maps. We recall that $W([\gamma])$ changes to $gW([\gamma])g^{-1}$, $g \in \mathrm{SO}(2,1)$, under gauge transformations that are *not* identity (and equal g) at p_∞. For closed surfaces with no marked point, one must make an identification $W \sim gWg^{-1}$ to get the moduli space of flat connections. In other words, $Q = \tilde{Q}/\mathrm{SO}(2,1)$.

In our case, Σ is a two-dimensional surface with a marked point p_∞, which is chosen to be our base point. Gauge transformations which are *not* trivial at p_∞, taking a value g (say) at p_∞, change W to gWg^{-1} as before, but, as explained, *these are no longer equivalent.* We call this action of $\mathrm{SO}(2,1)$ by conjugation the *gauge action.* It corresponds to a Lorentz transformation of our chosen, fixed frame at p_∞. The group $\mathrm{Diff}^\infty(\Sigma)$ of orientation-preserving *spatial* diffeomorphisms (diffeos) which are trivial at p_∞ (and leave a frame there fixed) acts on the holonomies W by changing the curve γ. Its subgroup $\mathrm{Diff}_0^\infty(\Sigma) \subset \mathrm{Diff}^\infty(\Sigma)$, connected to the identity (the group of small diffeos) cannot change the homotopy class of γ. Therefore the formulation is already invariant by small diffeos, and the physical configuration space is $\tilde{Q}$. Large diffeos, on the other hand, act nontrivially on the holonomies. So, we can work with the quotient group $M_\Sigma = \mathrm{Diff}^\infty(\Sigma)/\mathrm{Diff}_0^\infty(\Sigma)$, known as the mapping class group. In particular, the elements $C_{2\pi}$ and $\mathcal{R}$ are large diffeos.[9,11,12] For the sake of simplicity, we will denote the elements of $\mathrm{Diff}^\infty(\Sigma)$ and its classes in M_Σ by the same letters. An important fact is that elements of M_Σ commute with the gauge action.

3. The Geon Algebra

The algebra $\mathcal{A}$ used for quantization has the structure

$$\mathcal{A} = (\mathcal{U} \otimes \mathcal{G}) \ltimes \mathcal{F}(\tilde{Q}) , \tag{3.1}$$

where $\mathcal{G}$ is the group algebra of SO(2, 1) and $\mathcal{F}(\tilde{Q})$ is the space of complex-valued, continuous functions with compact support on $\tilde{Q}$. We choose the algebra $\mathcal{U}$ to be the group algebra of M_Σ. $\mathcal{A}$ contains all the operations necessary to investigate the spin-statistics connection.

Let us give an explicit presentation of $\mathcal{A}^{(1)}$, the algebra $\mathcal{A}$ for a single geon. We choose the generators of $\pi_1(\Sigma)$ to be the homotopy classes of the loops γ_1 and γ_2 of Fig. 1. Each flat connection provides us with a pair of holonomies $(a, b) = (W(\gamma_1), W(\gamma_2))$. Since there are no relations among the generators of $\pi_1(\Sigma)$, any pair of values (a, b) can occur. Therefore $\tilde{Q}$ is SO(2, 1) × SO(2, 1).

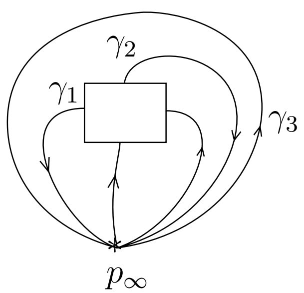

Fig. 1. The figure shows Σ for a single geon (opposite sides of the rectangle are to be identified) and loops γ_i $(1 \le i \le 3)$. The homotopy classes $[\gamma_1]$ and $[\gamma_2]$ generate the fundamental group, while $[\gamma_3]$ is not independent of $[\gamma_1]$ and $[\gamma_2]$.

Instead of working with $\mathcal{F}(\tilde{Q})$ directly, we work with one of its representations. Note that the Haar measure on SO(2, 1) induces a measure on $\tilde{Q}$. Using this measure we may define an inner product on $\mathcal{F}(\tilde{Q})$ in the obvious way. The completion of $\mathcal{F}(\tilde{Q})$ in this norm is a Hilbert space $\mathcal{H}_0$, which is the space of square-integrable functions (with this measure) on $\tilde{Q}$, carrying what we call the defining representation of $\mathcal{F}(\tilde{Q})$. A function $f \in \mathcal{F}(\tilde{Q})$ acts on $\varphi \in \mathcal{H}_0$ as a multiplication operator:

$$(f\varphi)(a, b) = f(a, b)\varphi(a, b) . \tag{3.2}$$

With $g \in$ SO(2, 1), let $\hat{\delta}_g$ denote the generators of the group algebra $\mathcal{G}$. These $\hat{\delta}_g$'s are gauge transformations, and act by conjugating holonomies:

$$(\hat{\delta}_g\varphi)(a, b) = \varphi(g^{-1}ag, g^{-1}bg) . \tag{3.3}$$

The mapping class group of Σ has two generators A and B, which correspond to Dehn twists along the loops. Their effect on loops γ_1 and γ_2 is given by

$$\begin{aligned} (A\varphi)(a, b) &= \varphi(a, ba^{-1}) , \\ (B\varphi)(a, b) &= \varphi(ab^{-1}, b) . \end{aligned} \tag{3.4}$$

The generators of $\mathcal{A}^{(1)}$ are functions $f \in \mathcal{F}(\tilde{Q})$, diffeos A, B of the mapping class group and gauge transformations δ_g.

The mapping class group includes $C_{2\pi}$.[9,11,12,18] Its action on the defining representation is

$$(C_{2\pi}\varphi)(a,b) = \varphi(cac^{-1}, cbc^{-1}), \tag{3.5}$$

where $c := aba^{-1}b^{-1}$. One can verify that $C_{2\pi} = (AB^{-1}A)^4$.

These operators can be encoded in what is called a transformation group algebra.[14] Let G be a group with a left-invariant measure acting on a space X. The transformation group algebra is just the set of continuous functions $\mathcal{F}(G \times X)$, with compact support and with the product

$$(F_1F_2)(g,x) = \int_G F_1(z,x)F_2(z^{-1}g, z^{-1}x)dz\,. \tag{3.6}$$

Here $x \to z^{-1}x$ is the group action on X, $z^{-1}g$ is the group product of z^{-1} and g, and dz is the left-invariant measure on G. The irreducible representations of a transformation group algebra have been worked out in Ref. 14. In our case, $X = \tilde{Q}$ and $G = \mathrm{SO}(2,1) \times M_\Sigma$, where G can be made into a topological group by giving M_Σ the discrete topology. The measure on $\mathrm{SO}(2,1)$ is the Haar measure and the measure on M_Σ is given by

$$\sum_{m \in M_\Sigma} f(m)$$

for any function f on M_Σ with appropriate convergence properties. The measure on G is then the product measure. Finally, $\mathcal{A}^{(1)} = \mathcal{F}(\mathrm{SO}(2,1) \times M_\Sigma \times \tilde{Q})$, where we use the bijection

$$\mathbb{C}(G) \otimes \mathcal{F}(X) \Leftrightarrow \mathcal{F}(G \times X) \tag{3.7}$$

by interpreting $\delta_g \otimes f$ as the distribution

$$\begin{aligned} \delta_g \otimes f:\ (h,x) &\mapsto \delta_g(h)f(x) \\ &\equiv \delta(g,h)f(x) \end{aligned} \tag{3.8}$$

on $G \times X$, δ_g being the δ-function supported at g.

Let $Y = \tilde{Q}/G$ be the set of orbits of G in $\tilde{Q}$, one such orbit being $\mathcal{O}_\omega$. Let us choose one representative $(a_\omega, b_\omega) \in \tilde{Q}$ for each orbit $\mathcal{O}_\omega$, and write $\mathcal{O}_\omega = [(a_\omega, b_\omega)]$. We define the stabilizer group $N_\omega \subset G$ as the set of elements (g,λ) of G such that $(g,\lambda)\cdot(a_\omega,b_\omega) = (a_\omega,b_\omega)$, where the G action has been denoted by a dot. Let α be a unitary irreducible representation of N_ω on some Hilbert space V_α. Now consider the space of square-integrable functions ϕ: $G \to V_\alpha$ such that $\phi(hg, \xi\lambda) = \alpha(g^{-1},\lambda^{-1})\phi(h,\xi)$ for all $(g,\lambda) \in N_\omega$ and $(h,\xi) \in G$. They are called equivariant functions. The set of these functions can be completed into a Hilbert space $L^2(G, V_\alpha)$.[14] The irreducible unitary $*$-representations $\Pi_{(\omega,\alpha)}$ of $\mathcal{F}(G\times\tilde{Q})$ can be realized on the Hilbert spaces $\mathcal{H}_{(\omega,\alpha)} = L^2(G, V_\alpha)$ and, up to unitary equivalence, labeled by $r=(\omega,\alpha)$. This label is a quantum number characterizing a single geon. The action of the operators $\hat{F}=\Pi_r(F)$, $F\in\mathcal{A}^{(1)}$ on a vector $\phi^r\in\mathcal{H}_r$ is given by

$$(\hat{F}\phi^r)(h,\xi)=\int_{\mathrm{SO}(2,1)\times M_\Sigma} F((h,\xi)\cdot(a_\omega,b_\omega),(g,\lambda))\times\phi^r(g^{-1}h,\lambda^{-1}\xi)dz\,, \tag{3.9}$$

for any $h\in\mathrm{SO}(2,1)$ and $\xi\in M_\Sigma$. We find, in particular, that

$$\begin{aligned}(\widehat{\delta_{h'}}\phi^r)(h,\xi)&=\phi^r(h'^{-1}h,\xi)\,,\\ (\hat{A}\phi^r)(h,\xi)&=\phi^r(h,A^{-1}\xi)\,,\\ (\hat{B}\phi^r)(h,\xi)&=\phi^r(h,B^{-1}\xi)\,,\\ (\hat{f}\phi^r)(h,\xi)&=f(h\xi\tilde{q}_y)\phi^r(h,\xi)\,.\end{aligned} \tag{3.10}$$

Now, let Σ be an orientable surface of genus two with a marked point p_∞. It supports a system of two geons. Their algebra $\mathcal{A}^{(2)}$ can be presented in the defining representation space $\mathcal{H}_0\otimes\mathcal{H}_0$ of $\mathcal{A}^{(1)}\otimes\mathcal{A}^{(1)}$. It is generated by elements of $\mathcal{A}^{(1)}\otimes\mathcal{A}^{(1)}$ plus the elements of the mapping class group that mix up the geons, with the proviso that we retain only "diagonal" elements of the form $\delta_g\otimes\delta_g$ from the gauge transformations. There are only two independent generators of M_Σ involving both geons. One of them, the diffeo $\mathcal{R}$ that exchanges the position of the geons, has already been discussed in connection with the spin-statistics relation. The other one is the so-called handle slide H. Unlike the exchange $\mathcal{R}$, the handle slide H has no analogue for particles. Its existence comes from the fact that a geon is an extended object. As the name indicates, it corresponds to the operation of sliding an end of one of the handles through the other handle.

Our description of a pair of geons should be given by an algebra $\mathcal{A}^{(2)}$ which also includes H. *But since H does not enter directly in the spin-statistics relation, we will not include it in $\mathcal{A}^{(2)}$.*

Although $\mathcal{A}^{(1)}$ is not a Hopf algebra, there is an element $R\in\mathcal{A}^{(1)}\otimes\mathcal{A}^{(1)}$ that plays the role of an R-matrix. In other words, we can write $\mathcal{R}=\sigma R$ where $\sigma:\mathcal{H}_0\otimes\mathcal{H}_0\to\mathcal{H}_0\otimes\mathcal{H}_0$ is the flip automorphism $\sigma(f_1\otimes f_2)=f_2\otimes f_1$. The R-matrix turns out to be

$$R=\iint da\,db\,P_{(a,b)}\otimes\delta^{-1}_{aba^{-1}b^{-1}}\,, \tag{3.11}$$

where $P_{(a,b)}(\tilde{q},h,\xi)=\delta(\tilde{q},(a,b))\delta(h,e)\delta(\xi,e)$, the δ's being δ-functions. The existence of the R-matrix is essential to establish the connection between spin and statistics. It relates a diffeo performed on a pair of objects with operators acting on each object individually.

Each geon carries a representation $\mathcal{H}_r$ labeled by quantum numbers $r=(\omega,\alpha)$. However, we only need to consider eigenstates of $\hat{C}_{2\pi}:=\Pi^r(C_{2\pi})$ with spin S. Let $\{\phi_i^{r,S}\}$ be a basis for the eigenspace of spin S in $\mathcal{H}_r$ for some fixed r. Two geons are said to be identical if they carry the same quantum numbers r and S. We consider identical geons, fix an element (a_ω,b_ω) in the corresponding class ω and denote the net flux $a_\omega b_\omega a_\omega^{-1}b_\omega^{-1}$ by c_ω. Consider the characteristic function P_c which at (a,b)

is 1 if $aba^{-1}b^{-1} = c$ and zero otherwise. It is clear that a generic vector $\phi_i^{r,S}$ is not an eigenstate of $\hat{P}_c$. A simple computation shows that $\phi_i^{r,S}$ is an eigenstate of $\hat{P}_c$ if and only if it has support only on points (h,ξ) such that $hc_\omega h^{-1} = c_\omega$.

The quantum state for two identical geons is a linear combination of vectors of the form $\phi_i^{r,S} \otimes \phi_j^{r,S}$. It is enough to show the spin-statistics connection (1.2) for such decomposable vectors. We must act with the operator $\hat{\mathcal{R}} = (\Pi_r \otimes \Pi_r)(\mathcal{R})$ on these vectors. By using Eq. (3.9), we easily see that

$$\hat{P}_{(a,b)}\phi_i^{r,S}(h,\xi) = \delta((a,b),(h,\xi)\cdot(a_\omega,b_\omega))\phi_i^{r,S}(h,\xi) \tag{3.12}$$

for every $(h,\xi) \in \mathrm{SO}(2,1) \times M_\Sigma$. Also,

$$\hat{\delta}_{c^{-1}}\phi_j^{r,S}(h,\xi) = \phi_j^{r,S}(ch,\xi)\,, \tag{3.13}$$

where we have put $c = aba^{-1}b^{-1}$. Using (3.11) and the flip automorphism we conclude that

$$\hat{\mathcal{R}}\phi_i^{r,S}(h_1,\xi_1) \otimes \phi_j^{r,S}(h_2,\xi_2) = \hat{\delta}_{h_2c_\omega^{-1}h_2^{-1}}\phi_j^{r,S}(h_1,\xi_1) \otimes \phi_i^{r,S}(h_2,\xi_2)\,. \tag{3.14}$$

At this point we make the assumption that $\phi_{i,j}^{r,S}$ *are eigenstates of the net flux* $\hat{P}_c$, explaining its physical meaning later. So we can set $h_2c_\omega h_2^{-1} = c_\omega$. But we have

$$\hat{\delta}_{c_\omega^{-1}}\phi_j^{r,S}(h_1,\xi_1) = e^{i2\pi S}\hat{\delta}_{c_\omega^{-1}}\hat{C}_{2\pi}^{-1}\phi_j^{r,S}(h_1,\xi_1) = e^{i2\pi S}\phi_j^{r,S}(c_\omega h_1, C_{2\pi}\xi)\,. \tag{3.15}$$

Note that $\phi_j^{r,S}(c_\omega h_1, C_{2\pi}\xi) = \phi_j^{r,S}(h_1c_\omega, \xi C_{2\pi})$ because of the above assumption, and because c_ω commutes with h_1 and $C_{2\pi}$ commutes with every element of M_Σ. On the other hand, $(c_\omega, C_{2\pi}) \in N_\omega$ and hence we can use the equivariance property of $\phi_j^{r,S}$ to rewrite the R.H.S. of the last equality in (3.15) as

$$\phi_j^{r,S}(c_\omega h_1, C_{2\pi}\xi) = \alpha(c_\omega^{-1}, C_{2\pi}^{-1})\phi_j^{r,S}(h_1,\xi)\,.$$

Now, every δ_g commuting with a_ω and b_ω also commutes with c_ω, while $C_{2\pi}$ is in the center of M_Σ. Therefore, $(c_\omega, C_{2\pi})$ is in the center of N_ω, and by Schur's lemma we conclude that $\widehat{\delta_{c_\omega^{-1}}C_{2\pi}^{-1}}$ is equal to a phase, say $e^{-i\theta(r)}$. Equation (1.2) then follows:

$$\hat{\mathcal{R}}\phi_i^{r,S} \otimes \phi_j^{r,S} = e^{i[2\pi S - \theta(r)]}\phi_j^{r,S} \otimes \phi_i^{r,S}\,. \tag{3.16}$$

We were able to establish a connection between spin and statistics for all eigenstates of the net flux $\hat{P}_c$. In other words, *a spin-statistics exists for states with a definite net flux.* Now why are these states special? The answer is that other vectors in the representation space of r are not physically allowed as a consequence of a superselection rule, which we will discuss below. As a consequence, only vectors which are in the eigenspace, say $\mathcal{H}_c$, of $\hat{P}_c$ are to be viewed as pure quantum states. Linear combinations of vectors in different $\mathcal{H}_c$'s are not pure, much in the same way as one cannot have pure states of different charges in QED, for example.

This superselection is actually very natural. First, note that the net flux of a geon commutes with all elements of the algebra except the gauge transformations at p_∞. Now, the gauge action cannot be viewed as having a local effect from the standpoint of the geons, their effect being limited to performing a transformation on the frame at infinity. The other operators, like those corresponding to the mapping class group operators are "local", in the sense that they correspond to operations on the geons themselves, i.e. operations which can be taken to leave the region outside some ball surrounding the geons invariant (no other, stronger notion of locality is possible here, since we have no fixed background metric). This is mathematically reflected in the fact that all elements of the geon algebra other than the gauge transformations (which are "local" in the above sense) themselves commute with the gauge action.

Therefore, given some eigenspace $\mathcal{H}_c$ of a net flux operator $\hat{P}_c$, all operators other than gauge transformations preserve $\mathcal{H}_c$. Only the gauge transformation, say corresponding to an element $g \in \mathrm{SO}(2,1)$, takes vectors in $\mathcal{H}_c$ into vectors in $\mathcal{H}_{gcg^{-1}}$. That is, gauge transformations do change the net flux, but this change does not correspond to a physical, local operation in the theory; rather it is merely a relabeling of the fluxes. Once one fixes the frame, and considers only local operations, one concludes that the net flux can be regarded as a charge which commutes with all the local operators, and hence is superselected.

4. Final Remarks

In this letter, we have shown a relation between the actions of the diffeomorphisms $\hat{C}_{2\pi}$ and $\mathcal{R}$ on a class of geon states in $(2+1)$D quantum gravity. An algebra describing the system was identified and its representations were explained in detail.

Our discussion can be viewed as a generalization of previous work,[18,19] where a spin-statistics relation was derived for geonic states arising in a Yang–Mills theory coupled to a Higgs field in the Higgs phase, where the symmetry is spontaneously broken down to a finite gauge group H. In Ref. 19 we showed the existence of a class of "localized" states in quantum gravity arising indirectly from the Yang–Mills theory which did obey the spin-statistics relation derived here. However, those states form a very restricted class. The present letter greatly expands the scope of the original version to a much larger class of geonic states in quantum gravity.

In our version of the spin-statistics relation, there appears an extra phase θ_r for each representation, and a natural question is what is its meaning. It turns out to be a somewhat involved problem, which we are presently tackling.[20]

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