Amenability for non-locally compact topological groups

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The focus of the talk

open question (A. Carey, H. Grundling, since at least 2000):

Let X be a smooth closed manifold, $n \geq 2$.
Is the current group $C^\infty(X, SU(n))$ amenable ?

More generally: same can be asked for groups of gauge transformations ($=$ automorphisms of principal G-bundles with compact simple Lie groups as structure groups).

"Yes" $\Rightarrow$ exists a "gauge-invariant vacuum state".

Conjecture (Ping Wong Ng, motivated by the above): Let A be a unital C^*-algebra. Then $U(A)_{norm}$ is amenable $\iff$ A is nuclear and has a tracial state.

$* * *$

thm. $C(\mathbb{I}, SU(n))$ is amenable.

(Here $\mathbb{I} = [0, 1]$.)

Why one needs amenability

thm. (André Weil) If a Polish (= separable completely metrizable) topological group admits an invariant sigma-additive measure, then it is locally compact.

Amenability = poor man's version of invariant integration with finite total volume.

Definitions of amenability of a top. group G

– if G acts continuously on a compact space X, there is an invariant probability measure μ on X:
$\mu(A) = \mu(gA)$ for all Borel $A \subseteq X$, all $g \in G$;

– if G acts continuously by affine transformations on a convex compact subspace C of a locally convex space, it has a fixed point $x \in C$;

– $\exists$ left-invariant mean m on the space $\mathrm{RUCB}\,(G)$ of all right uniformly continuous bounded functions on G:

$f : G \to \mathbb{C}$ is RUC if
$\forall \varepsilon > 0,\ \exists V \ni e,\ xy^{-1} \in V \Rightarrow |f(x) - f(y)| < \varepsilon$.

An invariant mean:
$\phi\colon \mathrm{RUCB}\,(G) \to \mathbb{C}$, linear, positive, $\phi(1) = 1$,
$\phi(f) = \phi(\text{any left translate of } f)$.

Why this is the correct definition: examples

— $U(\ell^2)_{sot}$;

— $\mathrm{Aut}\,(X, \mu)_{coarse}$;

— S_∞, topology of simple convergence on $\mathbb{N}$ discrete;

— $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ with the simple convergence topology on $\mathbb{Q}$ discrete;

— $\mathrm{Homeo}\,\mathbb{R}$, compact-open topology ...

Non-definitions of amenability

– $\exists$ left-invariant mean on $L^\infty(G)$. $\Leftarrow$ makes no sense.

– $\exists$ left-invariant mean on $\mathrm{CB}(G)$ $\Leftarrow$ fails for $U(\ell^2)_{sot}$, $\mathrm{Aut}\,(X,\mu)_{coarse}, \ldots$ (though holds for $\mathbb{I}+$ compact operators);

– $\exists$ bi-invariant mean on $\mathrm{RUCB}(G)$ (same observation)...

– every unitary representation of G is amenable in the sense of Bekka. $\Leftarrow$ there are amenable groups which fail this ($U(\ell^2)_{sot}$) and non-amenable groups which have the property ($\mathrm{Homeo}\,(\mathbb{S}^1)$).

Some observations

If G is locally compact, the list is much longer.

Abelian top. groups, compact groups are amenable ...

If H is dense in G, then G is amenable $\iff$ H is amenable.

example (J. Baez): $C(X, SU(n))$ with the simple convergence topology on X is amenable [dense in the compact group $SU(n)^X$]

Amenability is closed under extensions: $H, G/H$ are amenable $\Rightarrow$ G is amenable.

Some infinite dimensional tools

Unions of chains of amenable groups are amenable.
If $\mathrm{cl}\, H = G$, then G is amenable $\iff$ H is amenable.

ex. $U(\ell^2)_{sot} = \mathrm{cl}\, \cup_{n=1}^{\infty} U(n)$, or of $SU(n)$, $\Rightarrow$ amenable.

ex. $S_\infty = \mathrm{cl}\, \cup_{n=1}^{\infty} S_n$, $\Rightarrow$ amenable.

ex. $\mathrm{Aut}\,(X\,,\mu) = \cup_{n=1}^{\infty} S_n$ ("interval exchange transformations"), by Rokhlin Lemma, $\Rightarrow$ amenable.

ex. $L^0(X\,,\mu; SU(n))$, with the topology of convergence in measure, is the union of groups of simple functions of the form $\mathbb{T}^k$, $\Rightarrow$ amenable.

ex. $C(\text{Cantor set}, SU(n))$ is amenable.

Further applications of the approximation technique

— for a vN algebra M, $U(M)_{ultraweak}$ is amenable $\iff$ M is AFD (Pierre de la Harpe);

— for a C^*-algebra A, $U(A)_{weak}$ is amenable $\iff$ A is nuclear (Petersen).

example: $C(X, SU(n))$ with relative weak topology (induced from $C(X) \otimes M_n(\mathbb{C})$) is amenable

[in fact, it is strongly amenable in the sense of Glasner (Giordano–VP): $\forall$ proximal action on a compact space has f.p.]

Proving non-amenability

A closed subgroup of an amenable group need not be amenable:
$F_2 < U(\ell^2)_{sot}$ as a closed discrete subgroup.

Example (Greg Hjorth): $C(\mathbb{I}, SU(2))$ contains a closed discrete copy of F_2... tells nothing about (non)amenability of the group.

Example: $U(\ell^2)_{uniform}$ is non-amenable (de la Harpe).

Example: $\mathrm{Aut}\,(X, \mu)$ with *uniform* topology

$$d_{unif}(\sigma, \tau) = \mu\{x \in X : \sigma(x) \neq \tau(x)\}$$

is non-amenable (Giordano-VP). $\Leftarrow$ will outline a proof.

SIN groups and amenable representations

A topological group G is SIN if left and right uniform structures coincide; $\cong$ conjugation-invariant V form a basis at identity.

SIN groups: compact, abelian, discrete, $U(\ell^2)_{unif}$, $\mathrm{Aut}\,(X,\mu)_{unif}$.

Non-SIN: $SL_2(\mathbb{R})$, $\mathbb{U}(\ell^2)_{sot}$, $\mathrm{Aut}\,(X,\mu)_{coarse}$, S_∞.

SIN $\Rightarrow \mathrm{RUCB}\,(G) = \mathrm{LUCB}\,(G)$. (The converse is open.)

Lemma: If G is amenable SIN group, then every strongly continuous unitary rep. π of G is Bekka-amenable: there is an invariant mean on $\mathrm{UCB}\,(\mathbb{S}_\pi)$.

$\triangleleft$ Suppose $\mathrm{Aut}\,(X,\mu)_{unif}$ is amenable. $\Rightarrow$ the standard rep. in $L^2_0(X,\mu)$ is amenable. Identify $X \cong SL_3(\mathbb{R})/SL_3(\mathbb{Z})$. Then $SL_3(\mathbb{R}) < \mathrm{Aut}\,(X,\mu)$. $\Rightarrow$ the rep. of $SL_3(\mathbb{R})$ in $L^2_0(X,\mu)$ is amenable $\Rightarrow \exists$ f.-d. subrep $\Rightarrow \exists$ fixed vector. $\lightning$

Extreme amenability and Ramsey theory

G is *extremely amenable* if every continuous action of G on a compact space X has a global fixed point.

Example. $U(\ell^2)_{sot}$ is e.a. (Gromov–Milman, 1983), concentration of measure arguments.

No LC group is such (Granirer–Lau), in fact every LG G acts freely on a compact space (Veech).

Extreme amenability of a group of automorphisms of a structure $S \sim$ a Ramsey-type property of S.

Ex.: $\mathrm{Aut}\,(\mathbb{Q}, \leq)$ is extremely amenable, and this is equivalent to the classical finite Ramsey theorem (VP, 1998).

Amenability of current group

thm. $W^{1,2}(i, SU(n))$ is amenable.

[Differentiable a.e., equal a.e. to the integral of the derivative]

Corollary. $C(\mathbb{I}, SU(n))$ is amenable.

Enough to prove thm. for $G = W_e^{1,2}(\mathbb{I}, SU(n))$ of paths $p(0) = e$.

G is a Banach-Lie group, contractible.

Left logarithmic derivative and product integral

The *left logarithmic derivative:*

$u(t) \mapsto u'(t)u(t)^{-1}$, curve in $SU(n) \mapsto$ curve in $su(n)$.

It has an inverse, the *product integral:*

$u(t) \mapsto \prod_0^t \exp u(\tau)\, d\tau$, curve in $su(n) \mapsto$ curve in $SU(n)$.

For constant functions c it is simply

$$\prod_0^t \exp c\, d\tau = \exp(tc),$$

then extends to piecewise constant functions, etc.

• G is isomorphic to $L^2(\mathbb{I}, su(n))$ with the group operation
$u * v(t) = u(t) + Ad\left(\prod_0^t \exp u(\tau)\, d\tau\right) v(t)$.

Translations $+$ rotations.

Approach to prove amenability

A sequence (or a net) of probability measures μ_α on G which converge to invariance *weakly on the universal compact G-space*: for each $f \in \mathrm{RUCB}(G)$, $\int |f - ^g f| \, d\mu_\alpha \xrightarrow{\alpha} 0$.

Note: the product integral is a uniform isomorphism.
Given translations $t_1, \ldots, t_n$, rotations by $r_1, \ldots, r_k$, an $\varepsilon > 0$ and RUCB-functions $f_1, \ldots, f_n$, find μ so that

$$\int \left| f_i(x) - f_i(r_j x + t_k) \right| d\mu < \varepsilon, \quad \text{for all } i, j, k.$$

W.l.o.g. $\|t_i\| \leq 1$,

f_i 1-Lipschitz w.r.t. the L^2-distance,

t_j are locally constant on small intervals,

r_k are Lipschitz.

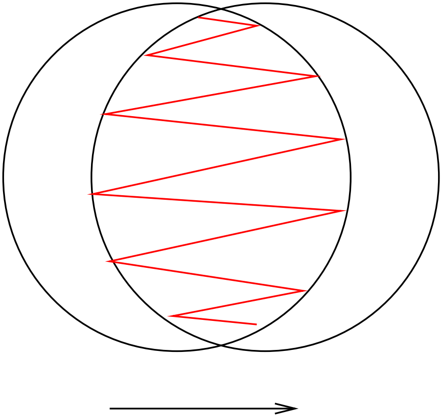
The measure

Divide $\mathbb{I} = [0, 1]$ into N intervals, so that r_k are constant on each to within $\Theta(1/N)$.

Identify $su(n)^N$ with functions constant on intervals, let μ be the uniform measure on the euclidean ball in $su(n)^N$ of radius $R = R(N)$.

Only need to chose $R(N)$.

Choosing the radius $R(N)$: translations



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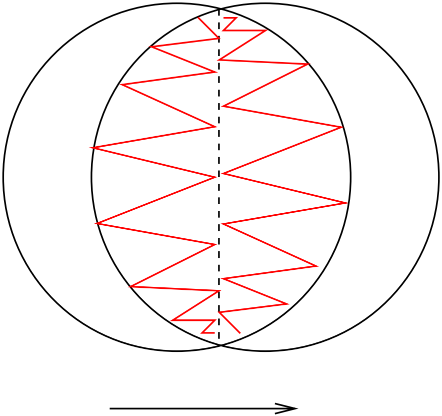

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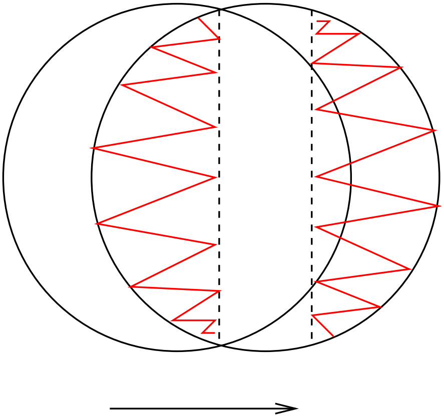
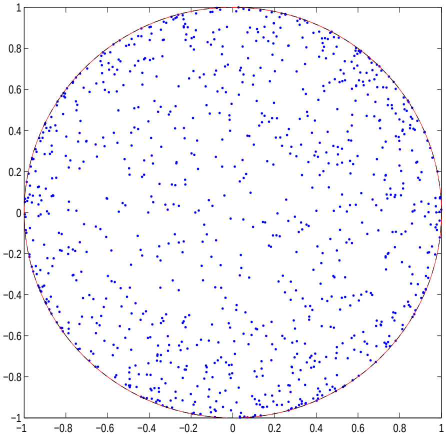


Concentration of measure on the sphere

1000 **random points are sampled from** $\mathbb{S}^d$, **projected to** $\mathbb{R}^2$**.**

$d = 2$

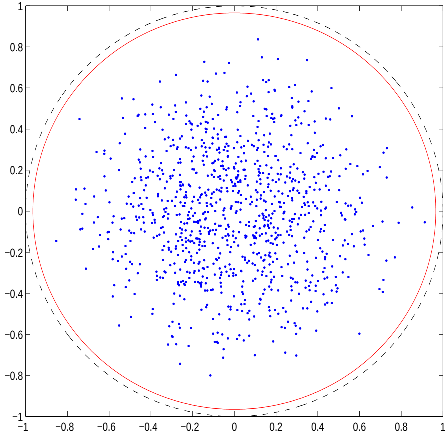


$\text{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}$.

Concentration of measure on the sphere

1000 **random points are sampled from** $\mathbb{S}^d$**, projected to** $\mathbb{R}^2$**.**

$d = 10$

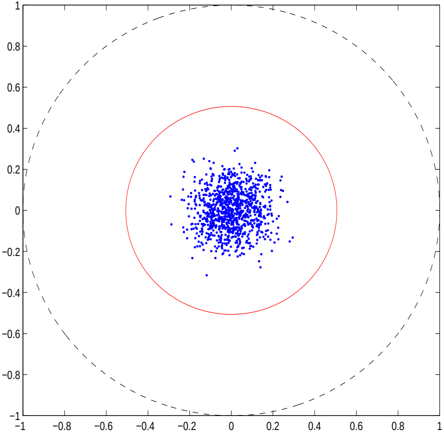


$\text{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}$.

Concentration of measure on the sphere

1000 **random points are sampled from $\mathbb{S}^d$, projected to $\mathbb{R}^2$.**

$d = 100$

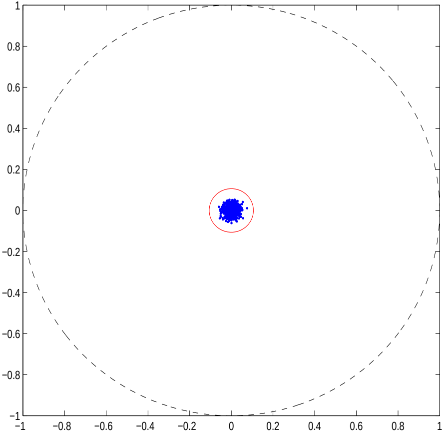


$\mathrm{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}.$

Concentration of measure on the sphere

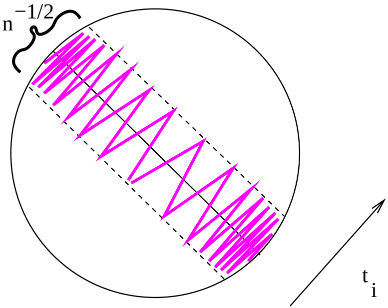
1000 random points are sampled from $\mathbb{S}^d$, projected to $\mathbb{R}^2$.

$d = 2,500$



$\mathrm{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}$.

Choosing the radius $R(N)$: translations



$1/2$ of the mass of the euclidean n-ball is concentrated in an equatorial strip of width $\Theta(1/\sqrt{N})$.

If translation is by unit length, ball of $R = \omega(\sqrt{N})$ is a "Følner set":

$$\forall f \in L^\infty, \quad \int |f - {}^g f| \, d\mu = o(1).$$

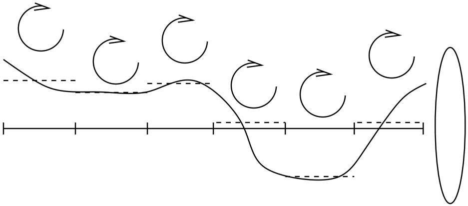
Choosing the radius $R(N)$: rotations

If r_j were locally constant, they would leave μ invariant.

They are only locally constant up to $1/N$.

Need to bound the L^2-mass transportation $d_{MK}(\mu, r_j \cdot \mu)$.

For most points of the ball, all N coordinates are $\Theta(R/\sqrt{N})$.



$d_{MK}(\mu, r_i \cdot \mu) \leq \sqrt{N \cdot (R/N\sqrt{N})^2} = \frac{R}{N} = o(1)$ if $R(N) = o(N)$.

Conclusion

Setting $R(N) = \omega(\sqrt{N}) \cap o(N)$ does it. $\square$

- C^∞ case
- loop groups
- Wiener measure
- $\dim X > 0$
- $C(X, \mathrm{SU}(n))$, X a compact space

O nosso MUITO OBRIGADO aos

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pela conferência memorável!

MAIS UMA, POR FAVOR !