Amenability for non-locally compact topological groups

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The focus of the talk

**open question** (A. Carey, H. Grundling, since at least 2000):
Let $X$ be a smooth closed manifold, $n \geq 2$. Is the current group $C^\infty(X, SU(n))$ amenable?

More generally: same can be asked for groups of gauge transformations (= automorphisms of principal $G$-bundles with compact simple Lie groups as structure groups).

“Yes” $\Rightarrow$ exists a “gauge-invariant vacuum state”.

**Conjecture** (Ping Wong Ng, motivated by the above): Let $A$ be a unital $C^*$-algebra. Then $U(A)_{\text{norm}}$ is amenable $\iff$ $A$ is nuclear and has a tracial state.

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**thm.** $C(\mathbb{I}, SU(n))$ is amenable.

(Here $\mathbb{I} = [0, 1]$.)
Why one needs amenability

\textbf{thm.} (André Weil) If a Polish (= separable completely metrizable) topological group admits an invariant sigma-additive measure, then it is locally compact.

Amenability = poor man’s version of invariant integration with finite total volume.
Definitions of amenability of a top. group $G$

– if $G$ acts continuously on a compact space $X$, there is an invariant probability measure $\mu$ on $X$:

$\mu(A) = \mu(gA)$ for all Borel $A \subseteq X$, all $g \in G$;

– if $G$ acts continuously by affine transformations on a convex compact subspace $C$ of a locally convex space, it has a fixed point $x \in C$;

– $\exists$ left-invariant mean $m$ on the space $\text{RUCB}(G)$ of all right uniformly continuous bounded functions on $G$:

$f : G \to \mathbb{C}$ is RUC if

$\forall \varepsilon > 0, \exists V \ni e, \ xy^{-1} \in V \Rightarrow |f(x) - f(y)| < \varepsilon$.

An invariant mean:

$\phi : \text{RUCB}(G) \to \mathbb{C}$, linear, positive, $\phi(1) = 1$,

$\phi(f) = \phi(\text{any left translate of } f)$. 
Why this is the correct definition: examples

- $U(\ell^2)_{sot}$;
- $\text{Aut} (X, \mu)_{coarse}$;
- $S_\infty$, topology of simple convergence on $\mathbb{N}$ discrete;
- $\text{Aut} (\mathbb{Q}, \leq)$ with the simple convergence topology on $\mathbb{Q}$ discrete;
- $\text{Homeo} \, \mathbb{R}$, compact-open topology ...
Non-definitions of amenability

– ∃ left-invariant mean on $L^\infty(G)$. $\iff$ makes no sense.

– ∃ left-invariant mean on $\text{CB}(G) \iff$ fails for $U(\ell^2)_{sot}$, $\text{Aut}(X, \mu)_{coarse}$, ... (though holds for $\mathbb{I}+$ compact operators);

– ∃ bi-invariant mean on $\text{RUCB}(G)$ (same observation)...

– every unitary representation of $G$ is amenable in the sense of Bekka. $\iff$ there are amenable groups which fail this ($U(\ell^2)_{sot}$) and non-amenable groups which have the property ($\text{Homeo}(\mathbb{S}^1)$).
If $G$ is locally compact, the list is much longer.

Abelian top. groups, compact groups are amenable ...

If $H$ is dense in $G$, then $G$ is amenable $\iff H$ is amenable.

**example** (J. Baez): $C(X, SU(n))$ with the simple convergence topology on $X$ is amenable [ dense in the compact group $SU(n)^X$ ]

Amenability is closed under extensions: $H, G/H$ are amenable $\Rightarrow G$ is amenable.
Some infinite dimensional tools

Unions of chains of amenable groups are amenable. If $\text{cl } H = G$, then $G$ is amenable $\iff H$ is amenable.

**ex.** $U(\ell^2)_{sot} = \text{cl } \bigcup_{n=1}^{\infty} U(n)$, or of $SU(n)$, $\Rightarrow$ amenable.

**ex.** $S_{\infty} = \text{cl } \bigcup_{n=1}^{\infty} S_n$, $\Rightarrow$ amenable.

**ex.** $\text{Aut } (X, \mu) = \bigcup_{n=1}^{\infty} S_n$ (“interval exchange transformations”), by Rokhlin Lemma, $\Rightarrow$ amenable.

**ex.** $L^0(X, \mu; SU(n))$, with the topology of convergence in measure, is the union of groups of simple functions of the form $\mathbb{T}^k$, $\Rightarrow$ amenable.

**ex.** $C(\text{Cantor set, } SU(n))$ is amenable.
Further applications of the approximation technique

— for a vN algebra $M$, $U(M)_{ultraweak}$ is amenable $\iff M$ is AFD (Pierre de la Harpe);
— for a $C^*$-algebra $A$, $U(A)_{weak}$ is amenable $\iff A$ is nuclear (Petersen).

**example:** $C(X, SU(n))$ with relative weak topology (induced from $C(X) \otimes M_n(\mathbb{C})$) is amenable

[ in fact, it is strongly amenable in the sense of Glasner (Giordano–VP): $\forall$ proximal action on a compact space has f.p. ]
A closed subgroup of an amenable group need not be amenable: $F_2 < U(\ell^2)_{sot}$ as a closed discrete subgroup.

**Example** (Greg Hjorth): $C(\mathbb{I}, SU(2))$ contains a closed discrete copy of $F_2$... tells nothing about (non)amenability of the group.

**Example:** $U(\ell^2)_{uniform}$ is non-amenable (de la Harpe).

**Example:** $\text{Aut}(X, \mu)$ with *uniform* topology

$$d_{unif}(\sigma, \tau) = \mu\{x \in X: \sigma(x) \neq \tau(x)\}$$

is non-amenable (Giordano-VP). $\iff$ will outline a proof.
SIN groups and amenable representations

A topological group $G$ is SIN if left and right uniform structures coincide; $\cong$ conjugation-invariant $V$ form a basis at identity.

SIN groups: compact, abelian, discrete, $U(\ell^2)_{\text{unif}}$, $\text{Aut}(X, \mu)_{\text{unif}}$.

Non-SIN: $\text{SL}_2(\mathbb{R})$, $U(\ell^2)_{\text{sot}}$, $\text{Aut}(X, \mu)_{\text{coarse}}$, $S_\infty$.

SIN $\Rightarrow$ RUCB $(G) = \text{LUCB}(G)$. (The converse is open.)

Lemma: If $G$ is amenable SIN group, then every strongly continuous unitary rep. $\pi$ of $G$ is Bekka-amenable: there is an invariant mean on $\text{UCB}(S_\pi)$.

\[ \Diamond \text{ Suppose } \text{Aut}(X, \mu)_{\text{unif}} \text{ is amenable. } \Rightarrow \text{ the standard rep. in } L_0^2(X, \mu) \text{ is amenable. Identify } X \cong \text{SL}_3(\mathbb{R})/\text{SL}_3(\mathbb{Z}). \text{ Then } \text{SL}_3(\mathbb{R}) < \text{Aut}(X, \mu). \Rightarrow \text{ the rep. of } \text{SL}_3(\mathbb{R}) \text{ in } L_0^2(X, \mu) \text{ is amenable } \Rightarrow \exists \text{ f.-d. subrep } \Rightarrow \exists \text{ fixed vector. } \]
Extreme amenability and Ramsey theory

*G* is *extremely amenable* if every continuous action of *G* on a compact space *X* has a global fixed point.

**Example.** *U*(\(\ell^2\))_{sot} is e.a. (Gromov–Milman, 1983), concentration of measure arguments.

No LC group is such (Granirer–Lau), in fact every LG *G* acts freely on a compact space (Veech).

Extreme amenability of a group of automorphisms of a structure *S* \(\sim\) a Ramsey-type property of *S*.

**Ex.:** Aut \((\mathbb{Q}, \leq)\) is extremely amenable, and this is equivalent to the classical finite Ramsey theorem (VP, 1998).
Amenability of current group

thm. $W^{1,2}(i, SU(n))$ is amenable.

[ Differentiable a.e., equal a.e. to the integral of the derivative ]

Corollary. $C(\mathbb{I}, SU(n))$ is amenable.

Enough to prove thm. for $G = W^{1,2}_e(\mathbb{I}, SU(n))$ of paths $p(0) = e$.

$G$ is a Banach-Lie group, contractible.
Left logarithmic derivative and product integral

The *left logarithmic derivative*:
\[ u(t) \mapsto u'(t)u(t)^{-1}, \text{ curve in } SU(n) \mapsto \text{ curve in } su(n). \]
It has an inverse, the *product integral*:
\[ u(t) \mapsto \prod_{0}^{t} \exp u(\tau) d\tau, \text{ curve in } su(n) \mapsto \text{ curve in } SU(n). \]
For constant functions \( c \) it is simply
\[ \prod_{0}^{t} \exp c d\tau = \exp(tc), \]
then extends to piecewise constant functions, etc.

- \( G \) is isomorphic to \( L^2(\mathbb{I}, su(n)) \) with the group operation
  \[ u \ast v(t) = u(t) + Ad \left( \prod_{0}^{t} \exp u(\tau) d\tau \right) v(t). \]
Translations + rotations.
Approach to prove amenability

A sequence (or a net) of probability measures $\mu_\alpha$ on $G$ which converge to invariance \textit{weakly on the universal compact G-space}: for each $f \in \text{RUCB}(G)$, $\int |f - g f| \, d\mu_\alpha \xrightarrow{\alpha} 0$.

\textbf{Note:} the product integral is a uniform isomorphism. Given translations $t_1, \ldots, t_n$, rotations by $r_1, \ldots, r_k$, an $\varepsilon > 0$ and RUCB -functions $f_1, \ldots, f_n$, find $\mu$ so that

$$\int |f_i(x) - f_i(r_j x + t_k)| \, d\mu < \varepsilon, \quad \text{for all } i, j, k.$$  

W.l.o.g. $\|t_i\| \leq 1$,

$f_i$ 1-Lipschitz w.r.t. the $L^2$-distance,

$t_j$ are locally constant on small intervals,

$r_k$ are Lipschitz.
The measure

Divide \([0, 1]\) into \(N\) intervals, so that \(r_k\) are constant on each to within \(\Theta(1/N)\).

Identify \(su(n)^N\) with functions constant on intervals, let \(\mu\) be the uniform measure on the euclidean ball in \(su(n)^N\) of radius \(R = R(N)\).

Only need to chose \(R(N)\).
Choosing the radius $R(N)$: translations
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Concentration of measure on the sphere

1000 random points are sampled from $\mathbb{S}^d$, projected to $\mathbb{R}^2$.

$$d = 2$$

\[
\text{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}.
\]
Concentration of measure on the sphere

1000 random points are sampled from $S^d$, projected to $\mathbb{R}^2$.

\[ d = 10 \]

\[
\text{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}.
\]
Concentration of measure on the sphere

1000 random points are sampled from $\mathbb{S}^d$, projected to $\mathbb{R}^2$.

\[ d = 100 \]

\[
\text{Prob}\{x \text{ is outside the red circle}\} = 10^{-10}.
\]
1000 random points are sampled from $S^d$, projected to $\mathbb{R}^2$.

$d = 2,500$

Prob\{x is outside the red circle\} = $10^{-10}$.
Choosing the radius $R(N)$: translations

$\frac{1}{2}$ of the mass of the euclidean $n$-ball is concentrated in an equatorial strip of width $\Theta(1/\sqrt{N})$.

If translation is by unit length, ball of $R = \omega(\sqrt{N})$ is a “Følner set”:

$$\forall f \in L^\infty, \int |f - g f| \, d\mu = o(1).$$
Choosing the radius $R(N)$: rotations

If $r_j$ were locally constant, they would leave $\mu$ invariant.
They are only locally constant up to $1/N$.
Need to bound the $L^2$-mass transportation $d_{MK}(\mu, r_j \cdot \mu)$.
For most points of the ball, all $N$ coordinates are $\Theta(R/\sqrt{N})$.

\[ d_{MK}(\mu, r_i \cdot \mu) \leq \sqrt{N \cdot (R/N\sqrt{N})^2} = \frac{R}{N} = o(1) \text{ if } R(N) = o(N). \]
Conclusion

Setting $R(N) = \omega(\sqrt{N}) \cap o(N)$ does it.

- $C^\infty$ case
- loop groups
- Wiener measure
- $\dim X > 0$
- $C(X, SU(n))$, $X$ a compact space
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MAIS UMA, POR FAVOR!