PARTIAL ACTIONS OF GROUPS ON ALGEBRAS

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Let $G$ be a group and $R$ a unital $k$-algebra, $k$ a ring. A partial action of $G$ on $R$ is a collection of ideals $S_g$, $g \in G$ of $R$ and isomorphisms $\alpha_g : S_{g^{-1}} \to S_g$ such that

(i) $S_1 = R$ and $\alpha_1$ is the identity mapping of $R$;

(ii) $S_{(gh)^{-1}} \supseteq \alpha_{h^{-1}}^{-1}(S_h \cap S_{g^{-1}})$,

(iii) $\alpha_g \circ \alpha_h(x) = \alpha_{gh}(x)$, for any $x \in \alpha_{h^{-1}}^{-1}(S_h \cap S_{g^{-1}})$.

The property (ii) easily implies that $\alpha_g(S_{g^{-1}} \cap S_h) = S_g \cap S_{gh}$, for all $g, h \in G$. Also $\alpha_{g^{-1}} = \alpha_g^{-1}$, for every $g \in G$. 


Let $\alpha$ be a partial action of $G$ on $R$. The partial skew group ring

$$R *_{\alpha} G$$

is defined as the set of all finite formal sums

$$\sum_{g \in G} a_g u_g, a_g \in S_g$$

for every $g \in G$, where the addition is defined in the usual way and the multiplication is determined by

$$(a_g u_g)(b_h u_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)u_{gh}.$$ 

This algebra may be non-associative. It is an interesting question on whether it is associative.
Given a global action of a group $G$ on an algebra $T$ by automorphisms $\sigma_g$, $g \in G$, and an ideal $R$ of $T$, the restriction of the action on $R$ is given by the following:

Take $S_g = R \cap \sigma_g(R)$, for every $g \in G$, and define

$\alpha_g : S_{g^{-1}} \to S_g$ by $\alpha_g(x) = \sigma_g(x)$, for all $g \in G$.

It is easy to see that this gives a partial action on $R$. 
Given a partial action $\alpha$ of a group $G$ on $R$, an enveloping action is an algebra $T$ together with a global action $\beta = \{ \beta_g \mid g \in G \}$ of $G$ on $T$, where $\beta_g$ is an automorphism of $T$, such that the partial action is given by restriction of the global action. This means that we may consider $R$ as an ideal of $T$ and the following holds:

(i) the subalgebra of $T$ generated by $\bigcup_{g \in G} \beta_g(R)$ coincides with $T$ and we have $T = \sum_{g \in G} \beta_g(R)$;
(ii) $S_g = R \cap \beta_g(R)$, for every $g \in G$;
(iii) $\alpha_g(x) = \beta_g(x)$, for all $g \in G$ and $x \in S_g^{-1}$.
We will always assume that $R$ is a unital algebra.

An important result by M. Dokuchaev and R. Exel shows:

**Theorem.** The partial action $\alpha$ has an enveloping action if and only if all the ideals $S_\beta$ are unital algebras (i.e., they are generated by central idempotents of $R$).

As a consequence, when $\alpha$ has an enveloping action, then the partial skew group ring $R *_\alpha G$ is associative.
Another type of enveloping action can be defined. We say that \((T, \beta)\) is a weak enveloping action of \(\alpha\) if \(T\) is a ring which contains \(R\), \(\beta\) is a global action of \(G\) on \(T\) by automorphisms, and for any \(g \in G\) the map \(\alpha_g\) is the restriction of \(\beta_g\) to the ideal \(S_{g^{-1}}\) of \(R\).

The following results was already proved:

**Theorem.** [F] If \(R\) is a semiprime ring, then any partial action \(\alpha\) on \(R\) possesses a weak enveloping action.

As a consequence, for any semiprime ring the partial skew group ring \(R *_{\alpha} G\) is associative.
The construction of the weak enveloping action in the above result is done in the following way: first we consider the Martindale ring of quotients $Q$ of $R$ and we extend the partial action to a partial action $\alpha^*$ of $Q$.

The ideals $S^*_g$ corresponding to $\alpha^*$ are the closure of the ideals $S_g$. Then they are closed ideals and so generated by central idempotent elements of $Q$. Then we consider the enveloping action $(T, \beta)$ of $\alpha^*$. This is the weak enveloping action of $\alpha$. For the corresponding partial skew group rings we have:

$$R \star_{\alpha} G \subseteq Q \star_{\alpha^*} G \subseteq T \star G.$$
The enveloping action is defined by an universal property. So it is unique, unless equivalence.

**Question** It is an open problem to find a general definition of weak enveloping action in order to have uniqueness. Until now I did not solve this problem.
PARTIAL SKEW POLYNOMIAL RINGS

Let $R$ be an associative ring with an identity element $1_R$, $G$ an infinite cyclic group generated by $\sigma$ and 
$\alpha = \{\alpha_{\sigma^i} : S_{\sigma^{-i}} \to S_{\sigma^i}\}$ a partial action of $G$ on $R$.

The partial skew group ring $R \ast_\alpha G$ can be identified with the set of all the finite sums $\sum_{i=-n}^{m} a_i x^i$, where $a_i \in S_{\sigma^i}$, for any integer number $i$, where the addition and the multiplication are defined as above.

We denote $R \ast_\alpha G$ by $R < x; \alpha >$. The ideal $S_{\sigma^i}$ will be denoted simply by $S_i$, $i \in \mathbb{Z}$.
Assume that for all $i$, the ideal $S_i$ is generated by a central idempotent $1_i$. In this case $\alpha$ has an enveloping action which will be denoted by $(T, \sigma)$, where $\sigma$ is an automorphism of $T$.

The skew group ring $T \ast G$ is the skew Laurent polynomial ring $T < x; \sigma >$ and $R < x; \alpha >$ is a subring of $T < x; \sigma >$.

We define the partial skew polynomial ring $R[x; \alpha]$ as the subring of $R < x; \alpha >$ whose elements are the polynomials $\sum_{i=0}^{n} a_i x^i$, $a_i \in S_i$, for every $i \geq 0$. Thus the partial skew polynomial ring is an associative ring, contained in the skew polynomial ring $T[x; \sigma]$. 
PRIME IDEALS OF $R < x; \alpha >$ AND $R[x; \alpha]$

**Proposition.** [C, F] (i) There is a one-to-one correspondence, via contraction, between the set of all prime ideals of $R[x; \alpha]$ and the set of all prime ideals of $T[x; \sigma]$ which do not contain $R$.

(ii) There is a one-to-one correspondence, via contraction, between the set of all prime ideal of $R < x; \alpha >$ and the set of all prime ideals of $T < x; \sigma >$.

Using this and the known results about prime ideals in $T[x; \sigma]$ and $T < x; \sigma >$ we can obtain a complete description of prime ideals of $R[x; \alpha]$ and $R < x; \alpha >$. 
Proposition. [C, F] Let $P$ be a prime ideal of $R[x; \alpha]$ (resp. $R < x; \alpha >$). Then we have one of the following possibilities:

(i) $P = Q \oplus \sum_{i \geq 1} S_i x^i$, where $Q$ is a prime ideal of $R$
(resp. $P = Q \oplus \sum_{i \neq 0} S_i x^i$, where $Q$ is a prime ideal of $R$ with $S_j \subseteq Q$, for any $j \neq 0$).

(ii) $1_i x^i \notin P$, for some $i \geq 1$.

The description of the prime ideals of the case (ii) is quite technical and we will omit here.
An ideal $I$ of $R$ is said to be an $\alpha$-ideal if $\alpha_i(I \cap S_{-i}) \subseteq I \cap S_i$, for all $i \geq 0$, and is said to be an $\alpha$-invariant ideal if $\alpha_i(I \cap S_{-i}) = I \cap S_i$, for all $i \in \mathbb{Z}$.

If $I$ is an $\alpha$-ideal of $R$, then the set of all the polynomials $\sum_{i \geq 0} a_i x^i$, where $a_i \in I \cap S_i$, is an ideal of $R[x; \alpha]$. Similar for an $\alpha$-invariant ideal of $R < x, \alpha >$.

Let $Q$ be an $\alpha$-invariant ideal of $R$.

(i) $Q$ is said to be $\alpha$-prime if $IJ \subseteq Q$, for $\alpha$-invariant ideals $I$ and $J$ of $R$, implies that either $I \subseteq Q$ or $J \subseteq Q$.

(ii) $Q$ is said to be strongly $\alpha$-prime if for any $m \geq 1$ there exists $j \geq m$ such that $1_j \notin Q$ and for any ideal $I$ and $\alpha$-ideal $J$ of $R$, $IJ \subseteq Q$ implies either that $I \subseteq Q$ or $J \subseteq Q$. 13
Corollary. Let $P$ be an ideal of $R < x; \alpha >$. Then $P$ is prime if and only if $P \cap R$ is $\alpha$-prime and either $P = (P \cap R) < x; \alpha >$ or $P$ is maximal amongst the ideals $N$ of $R < x; \alpha >$ such that $N \cap R = P \cap R$.

Corollary. Let $P$ be an ideal of $R[x; \sigma]$ such that $1, x^i \notin P$, for some $i \geq 1$. Then $P$ is prime if and only if $P \cap R$ is strongly $\alpha$-prime and either $P = (P \cap R)[x; \alpha]$ or $P$ is maximal amongst the ideals $N$ of $R[x; \alpha]$ with $N \cap R = P \cap R$. 
The maximal ideals can be classified into two types:

(i) If $M$ is a maximal ideal which contains $\sum_{i \geq 1} S_i x^i$, then we have $M = (M \cap R) \oplus \sum_{i \geq 1} S_i x^i$, where $M \cap R$ is a maximal ideal of $R$ and conversely.

(ii) If $M$ is a maximal with $1, x^i \not\in M$, for some $i \geq 1$:

**Theorem.** Assume that $M$ is a prime ideal of $R[x; \alpha]$ such that $1, x^i \not\in M$, for some $i \geq 0$. Then $M$ is maximal if and only if the corresponding prime ideal $M'$ of $T[x; \sigma]$ such that $M' \cap R[x; \alpha] = M$ is a maximal ideal. If this is the case $T[x; \sigma]/M'$ has an identity element.
The $\alpha$-pseudo radical $ps_{\alpha}(R)$ of $R$ is defined as the intersection of all non-zero $\alpha$-prime ideals of $R$.

**Definition** An element $a \in R$ is said to be $\alpha$-invariant if $\alpha_{\sigma_j}(a_{1-j}) = a_1^j$, for all $j \in \mathbb{Z}$.

**Definition** An element $a \in R$ is said to be $\alpha_{\sigma_m}$-normalizing if for all $r \in R$ we have $ra = a\alpha_{\sigma_m}(r1_{-m})$. 

Theorem The following are equivalent:

(i) There exists an $R$-disjoint maximal ideal ideal $M$ of $R[x;\alpha]$ such that $1_ix_i \not\in M$, for some $i \geq 0$.

(ii) There exists a $T$-disjoint ideal $M'$ of $T[x;\sigma]$ such that $T[x;\sigma]/M'$ is simple with identity and $T[x;\sigma]x \not\in M'$.

(iii) $T$ is $\sigma$-prime and $ps_{\sigma}(T)$ contains a non-zero element which is $\sigma$-invariant and $\sigma^m$-normalizing, for some $m \geq 1$.

(iv) $R$ is $\alpha$-prime and $ps_{\alpha}(R)$ contains a non-zero $\alpha$-invariant element which is $\alpha^m$-normalizing, for some $m \geq 1$. 

**Corollary** Assume that $\alpha$ is a partial action on $R$ and $Q$ is an $\alpha$-invariant ideal. Then the following conditions are equivalent:

(i) There exists a maximal ideal $M$ of $R[x; \alpha]$ such that $M \cap R = Q$ and $1, x^i \notin M$, for some $i \geq 0$.

(ii) $R/Q$ is $\alpha$-prime and $\psi_\alpha(R/Q)$ contains a non-zero $\alpha$-invariant element which is $\alpha^m$-normalizing, for some $m \geq 1$. 
Theorem. Under the same assumptions as above, the Brown-McCoy radical of a partial skew polynomial ring can be obtained:

\[ U(R[x; \alpha]) = U(T[x; \sigma]) \cap R[x; \alpha] = \]

\[ U_\alpha(R) \cap U(R) \bigoplus \sum_{i \geq 1} (U_\alpha(R) \cap S_i)x^i. \]
PARTIAL SKEW POLYNOMIAL RINGS OF SEMIPRIME RINGS

Assume that $R$ is semiprime and $\alpha$ is arbitrary, i.e., the ideals $S_i$ do not necessarily have identity element. In this case we define the partial skew Laurent polynomial ring as:

$$R < x; \alpha > = \{ \sum_{-n \leq i \leq m} a_i x^i | a_i \in S_i \},$$

where the addition and the multiplication are defined as above. So we have

$$R < x; \alpha > \subseteq Q < x; \alpha^* > \subseteq T < x; \sigma >,$$

where $(T, \sigma)$ denotes the enveloping action of $(Q, \alpha^*)$. 

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Then $R \prec x; \alpha \succ$ is an associative ring and $R[x; \alpha]$ can be naturally defined as above:

$$R[x; \alpha] = \{ \sum_{0 \leq i \leq n} a_i x^i \} \subseteq R \prec x; \alpha \succ .$$

The essential difference in studying the first case and the second one is that here we have to go from $R[x; \alpha]$ to $Q[x; \alpha^*]$ and then from this to $T[x; \sigma]$, and come back.
PARTIAL SKEW POLYNOMIAL RINGS AND GOLDIE RINGS

Recall that a ring $R$ is said to be a right Goldie ring if $R$ has finite right uniform dimension and satisfies ACC on right annihilators.

The right singular ideal $Z(R)$ of $R$ is the set of all the elements $a \in R$ such that the right annihilator $r\text{Ann}_R(a) = \{x \in R | ax = 0\}$ is an essential right ideal.

A semiprime ring $R$ is right Goldie if and only if $R$ has finite right uniform dimension and the right singular ideal of $R$ is zero ($R$ is right non-singular).
It is well-known that if $T$ is a semiprime ring and $\sigma$ is an automorphism of $T$, then $T$ is right Goldie if and only if $T < x; \sigma >$ is right Goldie if and only if $T[x; \sigma]$ is right Goldie. In addition, $T < x; \sigma >$ and $T[x; \sigma]$ are also semiprime.

For partial skew Laurent polynomial rings and partial skew polynomial rings over semiprime rings we proved the following results [C, F, M]

**Proposition 1.** If $R$ is a semiprime right non-singular ring, then $R < x; \alpha >$ and $R[x; \alpha]$ are also right non-singular.
Proposition 2. If $R$ is a semiprime ring, then the right uniform dimension of $R$, $R < x; \alpha >$ and $R[x; \alpha]$ are equal.

So we have the following:

Theorem. If $R$ is a semiprime ring, then the following conditions are equivalent:

(i) $R$ is right Goldie;
(ii) $R < x; \alpha >$ is right Goldie;
(iii) $R[x; \alpha]$ is right Goldie.

We also proved that in this case $R < x; \alpha >$ is semiprime, but $R[x; \alpha]$ is not necessarily semiprime.
There is natural question related with these results: Is the weak enveloping action $T$ of $\alpha$ also right Goldie when $R$ is right Goldie? The answer is NO in general. To have an affirmative answer we must have a rather special partial action. Assume that $R$ is semiprime.

**Definition** We say that the partial action $\alpha$ on $R$ is of finite type if for any uniform (two-sided) ideal $I$ of $R$ there exist an ideal $H$ of $R$ and a positive integer $i$ such that $r\text{Ann}_R(I) = r\text{Ann}_R(H)$ and $H \subseteq S_{-i}$.

The above definition works well for skew polynomial rings, but now we have a more general definition and some equivalences.
Theorem. Assume that \( R \) is semiprime right Goldie ring and \( \alpha, (Q, \alpha^*) \) and \( (T, \sigma) \) are as above. Then the following conditions are equivalent:

(i) \( \alpha \) is of finite type;
(ii) There exists \( N > 0 \) such that \( \sigma^N(Q) = Q \);
(iii) \( T \) has an identity element;
(iv) \( T \) has a finite number of central idempotent elements;
(v) The right uniform dimension of \( T \) is finite.

Under the above situation \( T, T < x; \sigma > \) and \( T[x; \sigma] \) are also right Goldie and \( R[x; \alpha] \) is semiprime.
PARTIAL ACTIONS OF FINITE TYPE

Recall that an ideal $H$ of $R$ is said to be essential if for any non-zero ideal $I$ we have $I \cap H \neq 0$

**Definition** A partial action $\alpha = \{\alpha_g : S_{g^{-1}} \to S_g\}$ on $R$ is said to be of finite type if there exist $g_1, \ldots, g_n \in G$ such that for any $g \in G$ we have that $\sum_{1 \leq i \leq n} S_{gg_i}$ is an essential ideal of $R$.

This condition is a natural one to have good finiteness conditions. When $\alpha$ has an enveloping action $(T, \beta)$ we have the following form of the condition.
**Theorem** The following conditions are equivalent:

(i) \( \alpha \) is of finite type;

(ii) There exist \( g_1, \ldots, g_n \in G \) such that
\[
\sum_{1 \leq i \leq n} S_{gg_i} = R, \text{ for any } g \in G;
\]

(iii) There exist \( g_1, \ldots, g_n \in G \) with
\[
T = \sum_{1 \leq i \leq n} \beta_{g_i}(R);
\]

(iv) \( T \) has an identity element.

Using a result by M. Dokuchaev and R. Exel we have

**Corollary** Assume that \( \alpha \) is a partial action of finite type on \( R \) and there exists an enveloping action \((T, \beta)\). Then \( R \star_{\alpha} G \) and \( T \star G \) are Morita equivalent.
In the case $R$ is semiprime the above definition have the following equivalences.

**Theorem** Assume that $R$ is semiprime, $Q$ is the Martindale ring of right quotients of $R$ and $(T, \beta)$ is the enveloping action of $\alpha^*$, the extension of $\alpha$ to $Q$. The following conditions are equivalent:

(i) $\alpha$ is of finite type.
(ii) $\alpha^*$ is of finite type;
(iii) $T$ has an identity element.
We give an example to see that the definition is natural.

**Theorem** Assume that $R$ is a right noetherian ring, $G$ is a polycyclic by finite group, $\alpha$ is a partial action of $G$ on $R$ and $(T, \beta)$ is an enveloping action of $\alpha$. Then the following conditions are equivalent:

(i) $\alpha$ is of finite type;
(ii) $T$ is right noetherian;
(iii) $T$ has a finite number of central idempotents;
(iv) $\{\beta_g(R) | g \in G\}$ is a finite set.
The converse of the above is also true.

**Theorem** Assume that $T$ is a right noetherian ring, $G$ is a polycyclic by finite group and $(\beta_g)_{g \in G}$ is an action of $G$ on $T$. If $e$ is a central idempotent of $T$ and $R = Te$, then the induced partial action of $\beta$ on $R$ is of finite type.
FIXED RINGS AND THE TRACE MAP

For a finite group $G$ and action $\beta$ of $G$ on $T$ the fixed subring of $T$ under the action is defined by

$$T^G = \{ x \in R | \beta_g(x) = x, \forall g \in G \}$$

Similarly, for any partial action $\alpha$ of $G$ on $R$ we define the invariant subring of $\alpha$ by

$$R^\alpha = \{ x \in R | \alpha_g(xa) = xa_\alpha(a), \forall g \in G \forall a \in S_g^{-1} \}$$

Assume that every ideal $S_g$ has an identity element $1_g$. Then we easily have

$$R^\alpha = \{ x \in R | \alpha_g(x1_g^{-1}) = x1_g, \forall g \in G \}$$
For a global actions of a finite group \( G \) on \( T \) the trace map is defined by \( \text{tr}(x) = \sum_{g \in G} \beta_g(x) \).

It is easy to see that \( \text{tr}(T) \subseteq T^G \).

When the action \( \alpha \) on \( R \) is partial and it has enveloping action, denoting as above the identities of the ideals by \( 1_g, \ g \in G \), we have also a trace map:

\[
\text{tr}_\alpha(x) \sum_{g \in G} \alpha_g(x1_{g^{-1}})
\]

and as in the global case we have \( \text{tr}_\alpha(R) \subseteq R^\alpha \)
PARTIAL SKEW GROUP RINGS, by [F, L]

In this section we assume that $G$ is a finite group and also that every ideal $S_g$ has an identity element $1_g$.

**Proposition** Assume that $R$ is right noetherian (artinian). Then $R \star_\alpha G$ is right noetherian (artinian).

**Theorem** (Maschke Theorem 1) If $R$ is a semisimple ring and $|G|^{-1} \in R$, then $R \star_\alpha G$ is semisimple.

We can prove another version of Maschke’s Theorem:
**Theorem** (Maschke Theorem 2) Assume that $tr_\alpha(1_R)$ is invertible in $R$. If $M$ is any left module over $R \star_\alpha G$ and $N$ is a $R \star_\alpha G$-submodule of $M$ which is a direct summand as an $R$-submodule, then $N$ is a direct summand as $R \star_\alpha G$-submodule.

So, as in the classical case we have

**Corollary** If $R$ is semisimple and $tr_\alpha(1_R)$ is invertible in $R$, then $R \star_\alpha G$ is semisimple.

The assumptions in the above two Maschke’s theorem are independent.
Denote by $J(R)$ the Jacobson radical of $R$. We have

**Proposition** For any group $G$ (not necessarily finite) and partial action $\alpha$ of $G$ on $R$ we have \[ J(R \ast_\alpha G) \cap R \subseteq J(R). \]

An ideal $I$ of $R$ is said to be $\alpha$-invariant if \[ \alpha_g(I \cap S_g) = I \cap S_g \]
for any $g \in G$. If $I$ is an $\alpha$-invariant ideal of $R$, then \[ I \ast_\alpha G = \sum_{g \in G} (I \cap S_g)u_g \]
is an ideal of $R \ast_\alpha G$.

It is easy to see that $J(R)$ is $\alpha$-invariant. So $J(R) \ast_\alpha G$ is well defined and an ideal of $R \ast_\alpha G$. 
Now assume that $G$ is finite of cardinality $n$. We have the following which generalizes a well-known result.

**Theorem** $J(R \star \alpha G)^n \subseteq J(R) \star \alpha G \subseteq J(R \star \alpha G)$

**Corollary** Under the same assumption as above we have:

(i) $J(R \star \alpha G) \cap R = J(R)$.

(ii) $J(R)$ is nilpotente if and only if $J(R \star \alpha G)$ is nilpotente.
Another result we can prove is the following (again $|G| = n$).

**Proposition** If $R$ is semiprime and has no (additive) $n$-torsion, then $R \star_\alpha G$ is semiprime.

Recall that a ring $R$ is said to be von Neumman regular if for any $a \in R$ there exists $b \in R$ such that $bab = b$.

**Theorem** Assume that $R$ is von Neumman regular and either $n$ or $tr_\alpha(1_R)$ is invertible in $R$. Then $R \star_\alpha G$ is von Neumman regular.
There are many results given relations between a ring $T$ and the invariant subring $T^G$. We generalize several of this results.

In the following $\alpha$ is a partial action of $G$ on $R$ which has an enveloping action $(T, G)$. One main result is the following

**Theorem** If $G$ is a finite group, then $R^\alpha = T^G 1_R$. In particular, $R^\alpha$ and $T^G$ are isomorphic.

For infinite groups we have $R^\alpha \subset T^G 1_R$. We can have an strict inclusion.
**Corollary** In $G$ is a finite group and $\gamma$ is any radical of rings, then $\gamma(R^\alpha) = \gamma(T^G)1_R$.

**Corollary** If $R$ is semiprime and do not have (additive) $n$-torsion, then $R^\alpha$ is semiprime.

**Corollary** If $R$ is semisimple and $n$ is invertible in $R$, then $R^\alpha$ is semisimple.

**Theorem** If $R$ is semiprime and $tr_\alpha(1_R)$ is not a zero divisor in $R$, then $R^\alpha$ is semiprime and for any non-zero $\alpha$-invariant left (right) ideal $I$ of $R$, $tr_\alpha(I) \neq 0$. 
Assume that $I$ is a left ideal of $R$ which is $\alpha$-invariant and put $K = R^\alpha \cap I$.

**Theorem** If $tr_\alpha(1_R)$ is invertible in $R$ we have:

(i) $R/I$ is artinian (noetherian) as a left $R$-module, then $R^\alpha/K$ is artinian (noetherian) as a left $R^\alpha$-module.

(ii) If the left $R$-module $R/I$ has a composition series of length $l$, then the left $R^\alpha$-module $R^\alpha/K$ has a composition series of length $\leq l$.

**Corollary** If $R$ is left artinian (noetherian) and $tr_\alpha(1_R)$ is invertible in $R$, then $R^\alpha$ is left artinian (noetherian).
We proved also results on radicals.

**Theorem** If \( n = |G| \) is invertible in \( R \), then \( J(R^\alpha) = J(R) \cap R^\alpha \). In particular, if \( R \) is \( J \)-semisimple, then \( R^\alpha \) is \( J \)-semisimple.

**Theorem** For \( n = |G| \) we have \( nJ(R^\alpha) \subseteq J(R) \). In particular, if \( R \) is \( J \)-semisimple and do not have (additive) \( n \)-torsion, then \( R^\alpha \) is \( J \)-semisimple.

**Corollary** If \( R \) is semisimple, do not have (additive) \( n \)-torsion and \( tr_\alpha(1_R) \) is invertible in \( R \), then \( R^\alpha \) is semisimple.
We denote the prime radical of $R$ by $\mathcal{P}(R)$. We have

**Theorem** For any group $G$, $\mathcal{P}(R) \cap R^\alpha \subseteq \mathcal{P}(R^\alpha)$.

**Theorem** If $G$ is finite and either $tr_\alpha(1_R)$ is not a zero divisor in $R$ or $R$ do not have (additive) $n$-torsion, then $\mathcal{P}(R^\alpha) = \mathcal{P}(R) \cap R^\alpha$. 
We say that $\alpha$ have non-degenerated partial trace if $R^\alpha$ is semiprime and $tr_\alpha(I) \neq 0$ for any non-zero $\alpha$-invariant left ideal $I$ of $R$.

**Theorem** If $\alpha$ have non-degenerated partial trace and $R$ is a left Goldie ring, then $R^\alpha$ is a left Goldie ring.

If in the above result we assume, in addition, that $R$ do not have (additive) $n$-torsion, then we have:

(i) $R^\alpha$ is a left Goldie ring if and only if $R$ is a left Goldie ring.

(ii) $R^\alpha$ is semisimple if and only if $R$ is semisimple.
A semigroup $H$ (always with identity) is said to be an inverse semigroup if for any $x \in H$ there exists $x^* \in H$ such that $xx^*x = x$ and $x^*xx^* = x^*$.

The set of all partially defined bijective maps on a set $X$ is an inverse semigroup, which we will denote by $I(X)$. The composition of maps is given by composition of partial maps in the largest domain where it makes sense.

**Definition** An action of an inverse semigroup $H$ on $X$ is a homomorphism of inverse semigroups $\phi : H \to I(X)$. 
For a group $G$, R. Exel defined a universal inverse semigroup $S(G)$, associated to $G$. Then he proved that there is a one-to-one correspondence between partial action of a group $G$ on $X$ and actions of the inverse semigroup $S(G)$ on $X$.

We consider an algebraic version of this result.

Let $R \subseteq S$ be a ring extension and $I_R(S)$ the inverse semigroup of all $R$-isomorphisms between ideals of $S$ which are generated by central idempotents.

It is easy to see that $I_R(S)$ is an inverse semigroup and $\text{Aut}_R(S) \subseteq I_R(S)$. 

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In a recent paper we study a ring extension under the action of an inverse semigroup $H$.

Assume that $H$ is an inverse subsemigroup of $I_R(S)$. We denote by $e_h$ the central idempotent of $S$ such that $Se_h$ is the domain of $h^{-1}$.

The fixed subring is defined as above:

$$S^H = \{ s \in S \mid h(se_{h^{-1}}) = se_h, \forall h \in H \}$$
It seems to be clear that when considering partial actions of groups on a ring extension $R \subseteq S$ (with identity elements) the inverse semigroup $I_R(S)$ is as relevant as the group $Aut_R(S)$ is when leading with global actions. So the computation of $I_R(S)$ is of interest.

In our paper we compute this semigroup as function of the group of automorphisms.

For any central idempotent $e \in S$ and automorphism $\sigma \in Aut_R(S)$ we define a partial isomorphism by

$$h_{e,\sigma} = \sigma|s\sigma^{-1}(e) : S\sigma^{-1}(e) \rightarrow Se$$
In the set of all the pairs \((e, \sigma)\), as above, we can define an structure of inverse semigroups by

\[(e, \sigma)(f, \tau) = (e\sigma(f), \sigma\tau)\]

**Question** Determine conditions under which any action of an inverse semigroup on \(S\) can be obtained as restriction of an inverse semigroup of pairs \((e, \sigma)\), as above.

We proved the following:
Theorem [B, C, F, P] Assume that $S$ is a finitely generated and projective algebra over $R$ and $H$ is a finite inverse subsemigroup of $I_R(S)$ with $SH = R$. Then

(i) For any $h \in I_R(S)$ there exists $\sigma \in Aut_R(S)$ and a central idempotent $e$ of $S$ such that $h = \sigma|S\sigma^{-1}(e)$.

(ii) There exists a finite subgroup $G$ of $Aut_R(S)$ such that $S^G = R$. 

MAIN REFERENCES


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