A NEW LOOK AT THE CROSSED PRODUCT OF A C*-ALGEBRA BY A SEMIGROUP OF ENDMORPHISMS

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Abstract. Let $G$ be a group and let $P \subseteq G$ be a subsemigroup. In order to describe the crossed product of a C*-algebra $A$ by an action of $P$ by unital endomorphisms we find that we must extend the action to the whole group $G$. This extension fits into a broader notion of interaction groups which consists of an assignment of a positive operator $V_g$ on $A$ for each $g$ in $G$, obeying a partial group law, and such that $(V_g, V_{g^{-1}})$ is an interaction for every $g$, as defined in a previous paper by the author. We then develop a theory of crossed products by interaction groups and compare it to other endomorphism crossed product constructions.

1. Introduction.

Roughly five years ago I wrote a similarly titled paper [7] in which a new notion of crossed product by an endomorphism was introduced. One of the reasons for doing so was a certain dissatisfaction with the existing crossed product theories because none of these could give the “correct” answer in the example of Markov subshifts.

I explain. Let $A$ be an $n \times n$ matrix with 0–1 entries and $K \subseteq \{1, \ldots, n\}^\mathbb{N}$ be the corresponding Markov space. Denoting by $T$ the shift on $K$ given by $T(x)_n = x_{n+1}$, one may define an endomorphism $\alpha$ on the C*-algebra $C(K)$ by setting $\alpha(f) = f \circ T$, for all $f$ in $C(K)$.

It has always been my impression that, whatever crossed product construction one could possibly conceive of, it should give the Cuntz–Krieger algebra $O_A$ if applied to the endomorphism described above. This is of course a matter of personal opinion, but one must concede that the close relationship between Markov subshifts and Cuntz-Krieger algebras definitely suggests this. The construction described in [7] is therefore the result of my search for a theory of crossed products which fulfills these expectations.

The basic starting point in the construction of any C*-algebra associated to a given endomorphism

$$\alpha : A \to A$$

is to decide what is one’s idea of covariant representation. In other words one imagines that $A$ is a subalgebra of some bigger C*-algebra where the given endomorphism can be implemented by means of some algebraic operation.

A very popular proposal is to require the covariance condition

$$\alpha(a) = SaS^*, \quad (1.1)$$

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where $S$ is an isometry in this bigger universe, that is, an element satisfying $S^*S = 1$. This expression is very appealing because, after all, the correspondence $a \mapsto SaS^*$ is a *-homomorphism for any isometry $S$!

However, even though the algebra of continuous functions on Markov’s space is a subalgebra of $O_A$ in a standard way, i.e. as the diagonal of the standard AF-subalgebra, there is no isometry $S$ in $O_A$ which implements the Markov subshift according to the above covariance condition. Please see the Appendix for a detailed proof of this statement.

When one studies nonunital endomorphisms, especially those with hereditary range, condition (1.1) becomes a powerful tool leading to very interesting applications, but unfortunately it must be abandoned if one deals with unital endomorphisms.

Carefully studying the case of the Cuntz-Krieger algebra I eventually settled for the following substitute of (1.1):

$$Sa = \alpha(a)S,$$
$$S^*aS = L(a).$$

While the first condition has already been discussed in the literature, the appearance of the transfer operator $L$ was a new feature. It should be stressed that $L$ needs to be given in advance; it is an important ingredient of the construction as much as the endomorphism $\alpha$ itself. See [7] for more details.

The resulting theory of crossed products based on transfer operators seems to be getting more and more acceptance and hence it is natural to try to apply the same ideas for semigroups of endomorphisms. Suppose, therefore, that $A$ is a unital C*-algebra, $P$ is a semigroup and

$$\alpha : P \to \text{End}(A),$$

is a semigroup homomorphism, a.k.a. an action by endomorphisms.

In case the range of every $\alpha_g$ is a hereditary subalgebra of $A$ the natural covariance condition is definitely to ask for a semigroup of isometries $\{S_g\}_{g \in P}$ such that

$$\alpha_g(a) = S_g a S_g^*,$$

as done e.g. in [15: 2.2]. But, as seen above, this is not appropriate for unital endomorphisms.

Given that we are now dealing with many endomorphisms at the same time, we should also need many transfer operators. However one must decide how to put these together in a harmonious way.

If one resorts to the single endomorphism case for guidance, given $\alpha$ and $L$, it is easy to see that $L^n$ is a transfer operator for $\alpha^n$. It is also interesting to notice that one then gets a collection $\{V_n\}_{n \in \mathbb{Z}}$ of bounded linear maps on $A$ given by

$$V_n = \begin{cases} 
\alpha^n, & \text{if } n \geq 0, \\
L^{-n}, & \text{if } n < 0.
\end{cases}$$

(1.2)
Thus, not only the positive integers are assigned to an operator on \( A \) (namely \( \alpha^n \)), but the negative integers are similarly given a role. This is related to the discussion in the introduction of [8] regarding past time evolution for irreversible dynamical systems.

Assuming that \( P \) is a subsemigroup of a group \( G \) it is then tempting to generalize by requiring that one is given an operator \( \mathcal{L}_h \) for each \( h \in P^{-1} \), such that \( \mathcal{L}_{g^{-1}} \) is a transfer operator for \( \alpha_g \), for every \( g \) in \( P \). This is essentially what is done in [16].

However it seems a bit awkward to me that elements of \( P \cup P^{-1} \) are allowed to act on \( A \), i.e., are assigned an operator on \( A \) (either an endomorphism or a transfer operator) but the other elements in the group are not given any role. On totally ordered groups, that is, when \( G = P \cup P^{-1} \), there are really no other such elements but one may be interested in more general group-subsemigroup pairs such as \((\mathbb{Z}^2, \mathbb{N}^2)\).

We therefore face a crucial question. Given \( g \) not in \( P \cup P^{-1} \), should we assign to it an endomorphism or a transfer operator? It was precisely my interest in solving this puzzle that led me to introduce the notion of interactions [8]. These are operators on \( \text{C}^* \) algebras which simultaneously generalize endomorphisms and transfer operators.

To cut a long story short I was led to reformulating the very notion of semigroups of endomorphisms: it should be replaced by an assignment of an operator \( V_g \) for every \( g \) in \( G \), such that the pair \((V_g, V_{g^{-1}})\) is an interaction.

It could perhaps be the case that \( G \) contains a subsemigroup \( P \) such that \( V_g \) is an endomorphism for every \( g \) in \( P \), in which case \( V_{g^{-1}} \) will necessarily be a transfer operator for \( V_g \). But then again this may not be so. From this point on I decided not to rely on the existence of subsemigroups any more and fortunately the price to be payed was not so high.

Of course we have so far neglected to consider which kind of dependence the maps \( V_g \) should have on the variable \( g \). Obviously it would be too much to require that \( V_g V_h \) coincide with \( V_{gh} \), as this is not even true for (1.2). On the other hand it would be crazy not to require any compatibility between the group law and the composition of the \( V_g \).

Resorting again to the single endomorphism case I was able to prove that the \( V \) of (1.2) is a partial representation of \( \mathbb{Z} \) on \( A \), meaning that

\[
V_n V_m V_{-m} = V_{n+m} V_{-m} \\
V_{-n} V_n V_m = V_{-n} V_{n+m}.
\]

This says that the group law \( "V_n V_m = V_{n+m}" \) does hold, as long as it is right-multiplied by \( V_{-m} \), or left-multiplied by \( V_{-n} \).

Following this I was led to the definition of what we shall call an interaction group: it is a triple \((A, G, V)\), where \( A \) is a \( \text{C}^* \) algebra, \( G \) is a group, and \( V \) is a partial representation of \( G \) on \( A \) such that \((V_g, V_{g^{-1}})\) is an interaction for every \( g \) in \( G \).

I am well aware that the search for a new definition is always a risky business unless one has valuable examples under one’s belt. In fact it was precisely this fear that led me to keep the findings of this paper to myself for quite some time now. However the problem of finding an appropriate definition for the notion of crossed product by semigroups of unital endomorphisms has been around for almost thirty years now [3; §2]. In addition, recent
discussions with Jean Renault suggested the existence of some quite interesting examples which we hope to describe in a forthcoming paper.

A few words must now be said about the crossed product. As already mentioned the starting point should be the covariance condition. Using the ideas of [8] it is not so hard to arrive at the following: a covariant representation of an interaction group \((A,G,V)\) in a C*-algebra \(B\) consists of a \(*\)-homomorphism

\[ \pi : A \to B \]

and a \(*\)-partial representation

\[ v : G \to B, \]

such that

\[ v_g \pi(a) v_{g^{-1}} = \pi (V_g(a)) v_g v_{g^{-1}}, \]

for every \(g\) in \(G\), and every \(a\) in \(A\). Once this is accepted the experienced reader will immediately think of the universal C*-algebra for covariant representations, which we shall denote \(T(A,G,V)\).

However this is definitely too big an object since among its representations one finds bizarre things such as \(v_g \equiv 0\). This algebra must therefore be recognized as just a first step in the construction of the crossed product. A fine tuning is necessary to restrict its wild collection of representations.

This fine tuning consists in eliminating the “redundancies”, as done in [7] and cleverly recognized by Brownlowe and Raeburn [2] as the passage from the Toeplitz-Cuntz-Pimsner algebra to the Cuntz-Pimsner algebra in the case of a correspondence [17: 3.4 and 3.12].

There are in fact several other instances in the literature where ideas with a similar flavor appeared. Among these we cite Larsen’s redundancies [16: 1.2] and Fowler’s Cuntz-Pimsner covariance in the context of discrete product systems [12: 2.5].

Having barely no examples in hand to help decide what the correct notion of redundancies should be, I had to search for a clue somewhere. Recall that, in the context of a single endomorphism/transfer operator pair \((\alpha, L)\), one has a very concrete model of the crossed product, provided one is given a faithful invariant state. In this case \(A \rtimes_{\alpha, L} \mathbb{N}\) essentially\(^1\) coincides with the algebra of operators on the GNS space of the given state, generated by \(A\) and the isometry naturally associated to \(\alpha\) [10: 6.1].

One nice feature of interaction groups is that, given an invariant state, one can also easily construct a covariant representation \((\pi, v)\) on the GNS space of the given state (11.4). I therefore thought it would be nice if the crossed product ended up isomorphic to the concrete algebra of operators generated by \(\pi(A)\) and \(v(G)\) when the state is faithful, precisely as in the case described above. Of course the latter algebra could not be taken as the definition of the crossed product, unless one postulated the existence of a faithful invariant state.

\(^1\) One in fact needs to tensor this with the regular representation of \(\mathbb{Z}\).
I therefore set out to find the right notion of redundancies guided by the requirement that, once these are moded out, one should arrive at a quotient of $\mathcal{T}(A,G,V)$ which in turn is isomorphic to the above concrete algebra of operators provided a faithful invariant state exists.

This was a rather interesting clue to follow because, on the one hand, it suddenly put our very abstract theory in close contact with a very concrete algebra of operators and, on the other, it led to the highly complex notion of redundancies given in Definition (6.1) which I would not have imagined otherwise. I would even go as far as to suggest that this notion of redundancies could have some impact in a possible strengthening of the notion of Cuntz-Pimsner covariance in the context of discrete product systems.

We feel that the questions we address here are complex enough to warrant avoiding extra difficulties. For that reason we chose to simplify things as much as possible by assuming two extra standing hypotheses, namely that interactions preserve the unit (3.1) and that the conditional expectations $V_gV_g^{-1}$ one gets from an interaction are faithful (3.3). In the same way that the Cuntz–Pimsner construction [17] was later refined by Katsura [13] we believe that our results may be similarly extended to avoid these special assumptions.

The organization of the paper is as follows: in section (2) we give a few basic facts about partial representations and in section (3) we introduce the central notion of interaction groups. In section (4) we study covariant representations and collect some crucial technical tools in preparation for the introduction of the notion of redundancies.

The Toeplitz algebra of an interaction group is defined next and in section (6) the all important notion of redundancies is finally given, together with the definition of the crossed product.

In section (7) we show that the crossed product possesses a natural grading over the group $G$.

Up to this point the development is admittedly abstract but in section (8) we study at length certain Hilbert modules and operators between them to be used in the following section with the purpose of finally giving a concrete representation of the crossed product. This may be thought of as a generalization of the notion of regular representation. As a byproduct we show that the natural map from $A$ to $A \rtimes \nu G$ is injective.

In section (10) we study the general problem of faithfulness of representations of the crossed product culminating with Theorem (10.9) in which we show that the regular representation of the crossed product is faithful provided $G$ is amenable.

In section (11) we assume the existence of a faithful invariant state and show in Theorem (11.7) the result already alluded to above, according to which $A \rtimes \nu G$ is isomorphic to a concrete algebra of operators on a certain amplification of the GNS space, again under the assumption that $G$ is amenable. We hope that this theorem will convince the skeptical reader that our theory is not as outlandish as it might seem.

In section (12) we finally return to considering semigroups of endomorphisms as well as the question of whether such a semigroup can be extended to give an interaction group. Assuming the existence of certain states we show in Theorem (12.3) that, when such an...
extension exists, it is unique and that $A times_{\nu} G$ is isomorphic to a natural concretely defined endomorphism crossed product. This could support the argument that interaction groups underlie certain semigroups of endomorphisms.

In the following section we compare our theory with [16] giving a necessary and sufficient condition for Larsen’s dynamical systems to be extended to an interaction group. In section (14) we show, by means of an example, that Larsen’s systems do not always extend to an interaction group.

Given a substantial difference in the respective notions of redundancies we do not believe that our crossed product coincides with Larsen’s. However it might be that these will coincide after Larsen’s redundancies are replaced by some more general notion involving several semigroup elements at the same time.

I would like to acknowledge some very interesting conversations with Jean Renault, based on which I gathered enough courage to finish a rough draft of this work which was laying dormant for roughly three years now. I would also like to express my thanks to Vaughan Jones for having suggested the very interesting example described in section (14). Thanks go also to Joachim Cuntz, Jean Renault, Mikael Rørdam, and Klaus Thomsen, for helping me to sort out relevant bibliography. Last but not least, I would like to thank the referee for pointing out many mistakes and for numerous suggestions resulting in shorter proofs and increased clarity.

2. Partial representations.

In this section we discuss some elementary facts about partial representations which will be used in the sequell. The reader is referred to [6] for more information.

2.1. Definition. A partial representation of a group $G$ in a unital algebra $A$ is a map

$$v : G \to A$$

such that for all $g$ and $h$ in $G$ one has that

(i) $v_1 = 1$,

(ii) $v_g v_h v_{h^{-1}} = v_{gh} v_{h^{-1}}$, and

(iii) $v_{g^{-1}} v_g v_h = v_{g^{-1}} v_{gh}$.

The following is a useful basic result about partial representations:

2.2. Proposition. Given a partial representation $v$ let

$$e_g = v_g v_{g^{-1}}.$$

Then for all $g$ and $h$ in $G$,

(i) $e_g$ is an idempotent,

(ii) $v_g e_h = e_h v_g$,

(iii) $e_g$ commutes with $e_h$. 
Proof. See [6: 2.4].

Let us now study one-sided invertible elements in partial representations:

2.3. Lemma. Fix a partial representation \( v \).

(i) Given \( g \) in \( G \) suppose that \( u \) is a right-inverse of \( v_g \), in the sense that \( v_g u = 1 \). Then \( v_g^{-1} \) is also a right-inverse of \( v_g \).

(ii) If \( v_g \) is left-invertible then \( v_g^{-1} \) is a left-inverse of \( v_g \).

(iii) If \( v_g \) is right-invertible then \( v_g v_h = v_{gh} \), for all \( h \) in \( G \).

(iv) If \( v_g \) is left-invertible then \( v_h v_g = v_{hg} \), for all \( h \) in \( G \).

Proof. We have

\[ v_g v_g^{-1} = v_g v_g^{-1} v_g u = v_g u = 1. \]

This proves (i) while (ii) may be proved in a similar way. As for (iii) we have by (i) that

\[ v_g v_h = v_g v_g^{-1} v_g v_h = v_g v_g^{-1} v_g h = v_{gh}. \]

The last point follows similarly.

The following is often useful when one deals with ordered groups:

2.4. Lemma. Let \( P \) be a subsemigroup of \( G \) such that \( G = P^{-1}P \), and let \( v \) be a partial representation of \( G \) in \( A \) such that \( v_g \) is left-invertible for every \( g \) in \( P \), then

\[ v_{x^{-1}y} = v_{x^{-1}} v_y, \quad \forall x, y \in P. \]

Moreover, if \( v' \) is another partial representation of \( G \) in \( A \) such that \( v'_g = v_g \) for all \( g \in P \cup P^{-1} \), then \( v' = v \).

Proof. That \( v_{x^{-1}y} = v_{x^{-1}} v_y \) follows from (2.3.iv). That \( v' = v \) is then obvious.

By a word in \( G \) we will mean a finite sequence

\[ \alpha = (g_1, g_2, \ldots, g_n) \]

of elements in \( G \). Given a word \( \alpha \) as above we will let

\[ \alpha^{-1} = (g_n^{-1}, g_{n-1}^{-1}, \ldots, g_1^{-1}). \]

Fixing, for the time being, a partial representation \( v \) of \( G \) on the algebra \( A \), we will let for every word \( \alpha \) as above,

\[ v_\alpha = v_{g_1} v_{g_2} \ldots v_{g_n}, \quad (2.6) \]

and

\[ e_\alpha = v_\alpha v_{\alpha^{-1}}. \]

2.7. Proposition. Let \( \alpha = (g_1, g_2, \ldots, g_n) \) be a word in \( G \). Then

(i) \( e_\alpha v_\alpha = v_\alpha \).

(ii) \( e_\alpha = e_{g_1} e_{g_1 g_2} e_{g_1 g_2 g_3} \ldots e_{g_1 g_2 g_3 \ldots g_n} \), and hence \( e_\alpha \) is idempotent.

(iii) \( e_\alpha = e_{g_1 g_2 \ldots g_n} e_\beta \), where \( \beta = (g_1, g_2, \ldots, g_{n-1}) \).
Proof. In order to prove (iii) observe that

\[ e_\alpha = v_\beta v_{g_n} v_\beta^{-1} v_\beta^{-1} = v_\beta e_{g_n} v_\beta^{-1} = \cdots \]

Applying (2.2.ii) repeatedly we conclude that the above equals

\[ \cdots = e_{g_1 g_2 \cdots g_n} v_\beta v_\beta^{-1} = e_{g_1 g_2 \cdots g_n} e_\beta, \]

thus proving (iii). It is now easy to show that (ii) follows from (iii) and by induction. As for (i) we have by (iii) that

\[ e_\alpha v_\alpha = e_{g_1 g_2 \cdots g_n} e_\beta v_\beta v_{g_n} = \cdots \]

By induction the above equals

\[ \cdots = e_{g_1 g_2 \cdots g_n} v_\beta v_{g_n} = v_\beta v_{g_n} = v_\alpha. \]

If \( \alpha = (g_1, g_2, \ldots, g_n) \) is a word in \( G \) we will denote by \( \hat{\alpha} \) the product of all components of \( G \), namely

\[ \hat{\alpha} = g_1 g_2 g_3 \cdots g_n. \]  

(2.8)

We will also let \( \mu(\alpha) \) be the subset of \( G \) given by

\[ \mu(\alpha) = \{1, g_1, g_1 g_2, g_1 g_2 g_3, \ldots, g_1 g_2 g_3 \cdots g_n\}, \]  

(2.9)

so that \( \mu(\alpha) \) consists of the \( \hat{\beta} \) for all initial segments \( \beta \) of \( \alpha \). Notice that we have included \( \hat{\beta} \) for the empty initial segment \( \beta \), namely 1.

Employing this notation we observe that (2.7.ii) can be stated as \( e_\alpha = \prod_{h \in \mu(\alpha)} e_h \), while (2.7.iii) reads \( e_\alpha = e_\alpha e_\beta \).

The following elementary property of words will be useful later on:

2.10. Proposition. Given words \( \alpha \) and \( \beta \) in \( G \) denote by \( \alpha \beta \) the concatenated word. Then

\[ \mu(\alpha \beta) = \mu(\alpha) \cup \hat{\alpha} \mu(\beta). \]

Another curious fact which will become relevant is:

2.11. Proposition. Let \( \alpha \) be a word in \( G \). Then

(i) \( \mu(\alpha) = \hat{\alpha} \mu(\alpha^{-1}) \),

(ii) If \( \hat{\alpha} = 1 \), then \( \mu(\alpha) = \mu(\alpha^{-1}) \).
Proof. Let $\alpha = (g_1, \ldots, g_n)$. Since
\[
g_1 g_2 \cdots g_{k-1} g_k \cdots g_{n-1} g_n = \hat{\alpha},
\]
we have that
\[
g_1 g_2 \cdots g_{k-1} = \hat{\alpha} g_{n}^{-1} g_{n-1}^{-1} \cdots g_{k-1}^{-1}, \quad \forall \ k = 1, \ldots, n,
\]
from which (i) follows. Obviously (i) implies (ii).

If $\alpha = (g_1, \ldots, g_n)$ is a word in $G$ recall that
\[
v_\alpha = v_{g_1} v_{g_2} \cdots v_{g_n},
\]
while
\[
v_\alpha^* = v_{g_1 g_2 \cdots g_n}.
\]
In our next Proposition we will relate these.

2.12. Proposition. For every word $\alpha$ in $G$ one has $v_\alpha = e_\alpha v_\alpha$

Proof. Let $\alpha = (g_1, \ldots, g_n)$. If $n = 1$ the statement follows immediately from (2.1.ii) with $g = g_1$ and $h = g_1^{-1}$. Proceeding by induction on $n$ let $\beta = (g_2, \ldots, g_n)$ and observe that
\[
v_\alpha = v_{g_1} v_\beta = v_{g_1} e_\beta v_\alpha = v_{g_1} v_{g_1^{-1} v_\beta} \quad \text{(2.2.iii)} = v_{g_1} e_\beta v_{g_1^{-1} v_\beta} \quad \text{(2.1.iii)} = v_{g_1} v_\alpha v_{\beta^{-1} v_\alpha} = v_\alpha v_{\alpha^{-1} v_\alpha} = e_\alpha v_\alpha.
\]

2.13. Proposition. If $\alpha$ and $\beta$ are words in $G$ such that $\mu(\alpha) \subseteq \mu(\beta)$ then $e_\alpha \geq e_\beta$ in the usual sense that $e_\alpha e_\beta = e_\beta$.

Proof. Follows immediately from (2.7.ii).

The following simple result will be of crucial importance later on:

2.14. Corollary. Let $\alpha$ and $\beta$ be words in $G$ such that $\hat{\alpha} = 1$ and $\mu(\alpha) \subseteq \mu(\beta)$. Then $v_\alpha v_\beta = v_\beta$.

Proof. Since $v_\alpha = v_{g_1} = 1$ we have by (2.12) that $v_\alpha = e_\alpha$. Thus
\[
v_\alpha v_\beta = e_\alpha v_\beta = e_\alpha e_\beta v_\beta = e_\beta v_\beta = v_\beta.
\]

2.15. Definition. If $A$ is a $C^*$-algebra and $v : G \to A$ is a partial representation we will say that $v$ is a *-partial representation if
\[
v_g^* = v_g^{-1}, \quad \forall \ g \in G.
\]

2.16. Proposition. If $v$ is a *-partial representation then for every word $\alpha$ in $G$ one has that $v_\alpha$ is a partial isometry and $e_\alpha$ is a projection (self-adjoint idempotent).

Proof. It is clear that $v_\alpha^* = v_{\alpha^{-1}}$. So $e_\alpha = v_\alpha v_\alpha^*$ is clearly self-adjoint. Moreover
\[
v_\alpha v_\alpha^* v_\alpha = e_\alpha v_\alpha \quad \text{(2.7.i)} = v_\alpha,
\]
so $v_\alpha$ is indeed a partial isometry.
3. Interaction groups.

In this section we introduce the main object of this work. Given a Banach space \( X \) we denote by \( \mathcal{B}(X) \) the algebra of all bounded linear operators on \( X \). Below we will refer to \( \mathcal{B}(A) \), where \( A \) is a C*-algebra.

3.1. Definition. An interaction group is a triple \((A, G, V)\) such that \( A \) is a unital C*-algebra, \( G \) is a group, and

\[
V : G \to \mathcal{B}(A)
\]

is a partial representation such that, for every \( g \) in \( G \),

(i) \( V_g \) is a positive map,
(ii) \( V_g(1) = 1 \),
(iii) for every \( a \) and \( b \) in \( A \), such that either \( a \) or \( b \) belongs to the range of \( V_g^{-1} \), one has that \( V_g(ab) = V_g(a)V_g(b) \).

Observe that for each \( g \) in \( G \), the pair \((V_g, V_g^{-1})\) is an interaction according to [8: Definition 3.1].

We will always denote by \( R_g \) the range of \( V_g \), and we will let

\[
E_g = V_g V_g^{-1}.
\]

The following is a direct consequence of [8: 2.6, 2.7 and 3.3]:

3.2. Proposition. Given an interaction group \((A, G, V)\), for every \( g \) in \( G \) one has that

(i) \( V_g \) is completely positive and completely contractive,
(ii) \( R_g \) is a closed *-subalgebra of \( A \),
(iii) \( E_g \) is a conditional expectation onto \( R_g \),
(iv) \( V_g \) restricts to a *-isomorphism from \( R_g^{-1} \) onto \( R_g \), whose inverse is the corresponding restriction of \( V_g^{-1} \).

Recall that a conditional expectation \( E \) is said to be faithful if

\[
E(a^*a) = 0 \Rightarrow a = 0.
\]

3.3. Definition. An interaction group \((A, G, V)\) will be said to be faithful if for every \( g \in G \) one has that \( E_g \) is a faithful conditional expectation.

From now on, and throughout the rest of this paper, we will fix a faithful interaction group \((A, G, V)\).

If \( \alpha = (g_1, \ldots, g_n) \) is a word in \( G \), in accordance with (2.6), we will let

\[
V_\alpha = V_{g_1} \cdots V_{g_n},
\]

and

\[
E_\alpha = V_\alpha V_\alpha^{-1}.
\]

3.4. Proposition. Given a word \( \alpha = (g_1, \ldots, g_n) \) in \( G \) one has that the range of \( V_\alpha \), which we will henceforth denote by \( R_\alpha \), coincides with

\[
R_\alpha = R_{g_1} \cap R_{g_1 g_2} \cap \cdots \cap R_{g_1 g_2 \cdots g_n}.
\]
Proof. By (2.7.ii) we have that
\[ E_\alpha = E_{g_1}E_{g_2} \cdots E_{g_1g_2 \cdots g_n}. \]
Therefore the range of \( E_\alpha \) is precisely the intersection referred to in the statement.

Since \( E_\alpha = V_\alpha V_{\alpha^{-1}} \) we have that the range of \( E_\alpha \) is contained in the range of \( V_\alpha \). By (2.7.i) we have that \( V_\alpha = E_\alpha V_\alpha \), so the reverse inclusion also holds. \( \square \)

3.5. Proposition. For every word \( \alpha \) in \( G \) one has that \( (V_\alpha, V_{\alpha^{-1}}) \) is an interaction.

Proof. Clearly \( V_\alpha \) and \( V_{\alpha^{-1}} \) are positive bounded linear maps. That
\[ V_\alpha V_{\alpha^{-1}} V_\alpha = V_\alpha, \quad \text{and} \quad V_{\alpha^{-1}} V_\alpha V_{\alpha^{-1}} = V_{\alpha^{-1}} \]
follows from (2.7.i). We must now show that \( V_\alpha(ab) = V_\alpha(a)V_\alpha(b) \), if either \( a \) or \( b \) belong to \( R_{\alpha^{-1}} \). Suppose, without loss of generality, that \( a \in R_{\alpha^{-1}} \), and let \( \alpha = (g_1, \ldots, g_n) \). Observing that by (3.4) one has that \( a \in R_{g_n^{-1}} \), and if we let \( \beta = (g_1, \ldots, g_{n-1}) \), one has
\[ V_\alpha(ab) = V_\beta V_{g_n}(ab) = V_\beta \left( V_{g_n}(a)V_{g_n}(b) \right), \]
In order to complete the proof we could use induction if we knew that \( V_{g_n}(a) \in R_{\beta^{-1}} \). In order to prove that this is in fact true observe that by (2.2.ii) and (2.7.ii) we have that \( E_{\beta^{-1}} V_{g_n}(a) = V_{g_n}(a) \), thus proving that \( V_{g_n}(a) \) lies in \( R_{\beta^{-1}} \) and hence completing the proof. \( \square \)


In this section we will consider the natural notion of representations for interaction groups.

4.1. Definition. A covariant representation of \((A,G,V)\) in a unital C*-algebra \( B \) is a pair \((\pi,v)\), where \( \pi : A \to B \) is a unital *-homomorphism and \( v : G \to B \) is a *-partial representation such that
\[ v_g \pi(a) v_{g^{-1}} = \pi \left( V_g(a) \right) v_g v_{g^{-1}}. \]
Throughout this section we fix a covariant representation \((\pi,v)\) of the interaction group \((A,G,V)\) in the C*-algebra \( B \).

The following is a generalization of the covariance condition of (4.1) to words in \( G \).

4.2. Proposition. For every \( a \in A \), and for every word \( \alpha \) in \( G \) one has that
\[ v_\alpha \pi(a) v_{\alpha^{-1}} = \pi \left( V_\alpha(a) \right) v_\alpha v_{\alpha^{-1}}. \]
Proof. Let \( \alpha = (g_1, g_2, \ldots, g_n) \) and put \( \beta = (g_1, g_2, \ldots, g_{n-1}) \). Then \( v_\alpha = v_\beta v_{g_n} \), and hence
\[
v_\alpha \pi(a) v_{\alpha^{-1}} = v_\beta v_{g_n} \pi(a) v_{\beta^{-1}} v_{\beta^{-1}} = v_\beta \pi(V_{g_n}(a)) e_{g_n} v_{\beta^{-1}} = \cdots
\]
Observing that \( e_{g_n} v_{\beta^{-1}} = (2.7.i) \) \( e_{g_n} e_{\beta^{-1}} v_{\beta^{-1}} = e_{\beta^{-1}} e_{g_n} v_{\beta^{-1}} \), we see that the above equals
\[
\cdots = v_\beta \pi(V_{g_n}(a)) v_{\beta^{-1}} = v_{\beta^{-1}} e_{g_n} v_{\beta^{-1}} = \cdots
\]
By induction on the length of \( \alpha \) this equals
\[
\cdots = \pi(V_\beta(V_{g_n}(a))) v_\beta v_{\beta^{-1}} e_{g_n} v_{\beta^{-1}} = \pi(V_\alpha(a)) v_\beta e_{g_n} v_{\beta^{-1}} = \pi(V_\alpha(a)) v_\beta v_{\beta^{-1}} = \pi(V_\alpha(a)) v_\alpha v_{\alpha^{-1}}. \]
\[ \square \]
4.3. Proposition. Let \( \alpha \) be a word in \( G \). Then

(i) For every \( a \in \mathcal{R}_\alpha \) one has that \( \pi(a) \) commutes with \( e_\alpha \).

(ii) The map
\[
a \in \mathcal{R}_\alpha \mapsto \pi(a)e_\alpha \in B
\]
is a (not necessarily unital) *-homomorphism.

Proof. Let \( \alpha = (g_1, \ldots, g_n) \). In order to prove (i) it is enough to show that \( \pi(a) \) commutes with \( e_{g_1 g_2 \cdots g_k} \), for all \( k \leq n \), given that \( e_\alpha = e_{g_1} e_{g_1 g_2} \cdots e_{g_1 g_2 \cdots g_n} \), by (2.7.ii). Noticing that \( a \in \mathcal{R}_{g_1 g_2 \cdots g_k} \), for all \( k \), it suffices to show that for every \( h \) in \( G \), and for every \( a \in \mathcal{R}_h \), one has that \( \pi(a) \) commutes with \( e_h \). To see this notice that
\[
e_h \pi(a) e_h = v_h v_{h^{-1}} \pi(a) v_h v_{h^{-1}} = \pi(V_h(V_{h^{-1}}(a))) v_h v_{h^{-1}} v_h v_{h^{-1}} = \pi(a) e_h.
\]
Applying the same argument to \( a^* \) we conclude that \( e_h \pi(a^*) e_h = \pi(a^*) e_h \). By taking adjoints we then have that \( e_h \pi(a) e_h = \pi(a) \), thus proving (i).

Point (ii) then follows immediately from (i). \[ \square \]

4.4. Proposition. For each word \( \alpha \) in \( G \) let \( \mathcal{M}_\alpha \) and \( \mathcal{K}_\alpha^0 \) be the subsets of \( B \) given by
\[
\mathcal{M}_\alpha = \overline{\pi(A) v_\alpha \pi(A)} \quad \text{and} \quad \mathcal{K}_\alpha^0 = \overline{\pi(A) e_\alpha \pi(A)}.
\]
Then for every \( \alpha \) one has that

(i) \( \mathcal{M}_\alpha^* = \mathcal{M}_\alpha^{-1} \),

(ii) \( \mathcal{M}_\alpha \) is a ternary ring of operators in the sense that \( \mathcal{M}_\alpha \mathcal{M}_\alpha^* \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha \),

(iii) \( \mathcal{K}_\alpha^0 \) is a closed *-subalgebra of \( B \),

(iv) \( \mathcal{K}_\alpha^0 \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha \),

(v) \( \mathcal{M}_\alpha \mathcal{K}_\alpha^0 \mathcal{M}_\alpha^{-1} \subseteq \mathcal{M}_\alpha \),

(vi) \( \overline{\mathcal{M}_\alpha \mathcal{M}_\alpha^*} = \mathcal{K}_\alpha^0 \), and

(vii) \( \overline{\mathcal{M}_\alpha^* \mathcal{M}_\alpha} = \mathcal{K}_\alpha^0 \).
Proof. We leave the easy proof of (i), (vi) and (vii) to the reader. We begin by proving (ii). For this let \( x_1, x_2, x_3 \in M_\alpha \), have the simple form \( x_i = \pi(a_i)v_\alpha \pi(b_i) \), for \( i = 1, 2, 3 \). Then
\[
x_1x_2^*x_3 = \pi(a_1)v_\alpha \pi(b_1)^*v_\alpha^{-1}\pi(a_2)^*v_\alpha \pi(b_3) = \\
= \pi(a_1)\pi(V_\alpha(b_1b_2^*))v_\alpha v_\alpha^{-1}v_\alpha \pi(V_\alpha^{-1}(a_2^*a_3))\pi(b_3) = \\
= \pi(a_1)\pi(V_\alpha(b_1b_2^*))v_\alpha \pi(V_\alpha^{-1}(a_2^*a_3))\pi(b_3) \in M_\alpha.
\]
This proves (ii) which in turn immediately implies (iii) – (v). \( \Box \)

As a consequence one should notice that \( K^0_\alpha \) and \( K^0_{\alpha^{-1}} \) are Morita–Rieffel equivalent \( C^* \)-algebras and that \( M_\alpha \) is an imprimitivity bimodule.

4.5. Definition. Given a word \( \alpha = (g_1, \ldots, g_n) \) in \( G \) we will let \( Z_\alpha \) be the closed linear subspace of \( B \) (recall that \( B \) is the codomain of our covariant representation) spanned by the set
\[
\{ \pi(a_0)v_{g_1}\pi(a_1)v_{g_2}\pi(a_2)\ldots\pi(a_{n-1})v_{g_n}\pi(a_n) : a_0, \ldots, a_n \in A \}.
\]
If \( \alpha \) is the empty word we set \( Z_\alpha = \pi(A) \), by default.

If the reader has indeed understood the subtle difference between the definitions of \( M_\alpha \) and \( Z_\alpha \), he or she will recognize that \( M_\alpha \subseteq Z_\alpha \), but not necessarily vice-versa.

In the next Proposition we will employ the set \( \mu(\alpha) \) introduced in (2.9).

4.6. Proposition. If \( \alpha \) and \( \beta \) are words in \( G \) such that \( \mu(\alpha^{-1}) \subseteq \mu(\beta) \) then
\[
Z_\alpha M_\beta \subseteq M_{\alpha\beta}.
\]

Proof. We need to prove that for every \( z \in Z_\alpha \) and \( y \in M_\beta \), one has that \( zy \in M_{\alpha\beta} \). Let \( \alpha = (g_1, \ldots, g_n) \) and \( \beta = (h_1, \ldots, h_m) \) and put \( \alpha' = (g_1, \ldots, g_{n-1}) \) and \( \beta' = g_n\beta \) (concatenation), so that \( \alpha\beta = \alpha'\beta' \).

By density we may clearly suppose that
\[
z = z'v_{g_n}\pi(a),
\]
and
\[
y = \pi(b)v_\beta \pi(c),
\]
where \( z' \in Z_{\alpha'} \), and \( a, b, c \in A \). Observing that
\[
g_n^{-1} \in \mu(\alpha^{-1}) \subseteq \mu(\beta),
\]
it follows from (2.7.ii) that \( e_{g_n^{-1}}e_\beta = e_\beta \), and by (2.7.i), that \( e_{g_n^{-1}}v_\beta = v_\beta \). So
\[
zy = z'v_{g_n}\pi(a)\pi(b)e_{g_n^{-1}}v_\beta \pi(c) = z'v_{g_n}\pi(ab)v_{g_n^{-1}}v_{g_n}v_\beta \pi(c) = z'\pi(V_{g_n}(ab))v_{g_n}v_\beta \pi(c) = \\
= z'v_{g_n}v_\beta \pi(c) \in M_{\alpha\beta}.
\]
\[ = \pi' \pi(V_n(ab)) v_{\beta'} \pi(e) \in \mathcal{Z}_{\alpha'} \mathcal{M}_{\beta'}. \]  

On the other hand notice that
\[ \mu(\beta) \supseteq \mu(\alpha^{-1}) = \mu(g_n^{-1} \alpha'^{-1}) \overset{(2.10)}{=} \{1, g_n^{-1}\} \cup g_n^{-1} \mu(\alpha'^{-1}), \]
so
\[ \mu(\alpha'^{-1}) \subseteq g_n \mu(\beta') \overset{(2.10)}{\subseteq} \mu(g_n \beta) = \mu(\beta'). \]
By induction on \( n \) one has that
\[ Z_{\alpha'} \mathcal{M}_{\beta'} \subseteq \mathcal{M}_{\alpha' \beta'} = \mathcal{M}_{\alpha}, \]
so \( v_{\beta'} \in \mathcal{M}_{\alpha} \) by \((\dagger)\). \( \square \)

4.7. **Proposition.** Given a finite subset \( X \subseteq G \), denote by
\[ \mathcal{Z}^X = \sum_{\mu(\alpha^{-1}) \subseteq X} \mathcal{Z}_{\alpha} \]
\( \mu(\alpha^{-1}) \subseteq X \)
\[ \hat{\alpha} = 1 \]
Then \( \mathcal{Z}^X \) is a closed \(^*\)-subalgebra of \( B \).

**Proof.** Let \( W \) be the set of all words \( \alpha \) such that \( \mu(\alpha^{-1}) \subseteq X \) and \( \hat{\alpha} = 1 \). We claim that \( W \) is closed both under inversion and concatenation. In fact, given \( \alpha \in W \) we have by \((2.11.ii)\) that \( \mu((\alpha^{-1})^{-1}) = \mu(\alpha^{-1}) \subseteq X \) and \( (\alpha^{-1})^{-1} = 1 \), so \( \alpha^{-1} \in W \). Moreover, given \( \alpha_1, \alpha_2 \in W \), observe that \( (\alpha_1 \alpha_2)^{-1} = \hat{\alpha}_1 \hat{\alpha}_2 = 1 \), and
\[ \mu((\alpha_1 \alpha_2)^{-1}) = \mu(\alpha_2^{-1} \alpha_1^{-1}) \overset{(2.10)}{=} \mu(\alpha_2^{-1}) \cup \hat{\alpha}_2^{-1} \mu(\alpha_1^{-1}) = \mu(\alpha_2^{-1}) \cup \mu(\alpha_1^{-1}) \subseteq X. \]
This proves the claim. Now, in order to prove the statement it is enough to realize that
\[ \mathcal{Z}_{\alpha_1} \mathcal{Z}_{\alpha_2} \subseteq \mathcal{Z}_{\alpha_1 \alpha_2}, \]
and
\[ (\mathcal{Z}_\alpha)^* = \mathcal{Z}_{\alpha^{-1}}. \]

4.8. **Definition.** If \( \alpha \) is a word in \( G \) we will let \( K_\alpha = \mathcal{Z}^{\mu(\alpha)} \).

In other words, \( K_\alpha \) is the closed sum of the \( \mathcal{Z}_\beta \), for all words \( \beta \) such that \( \hat{\beta} = 1 \) and \( \mu(\beta^{-1}) \subseteq \mu(\alpha) \).

4.9. **Proposition.** For every word \( \alpha \) in \( G \) one has that
\[ K_\alpha \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha. \]

**Proof.** It clearly suffices to prove that \( \mathcal{Z}_\beta \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha \) for all words \( \beta \) such that \( \hat{\beta} = 1 \) and \( \mu(\beta^{-1}) \subseteq \mu(\alpha) \). Given such a \( \beta \) observe that
\[ \mu(\beta) \overset{(2.11.ii)}{=} \mu(\beta^{-1}) \subseteq \mu(\alpha). \]
Thus we have that \( v_{\beta'} = v_{\beta} v_{\alpha} = v_{\alpha} \), by \((2.14)\). Therefore
\[ \mathcal{Z}_\beta \mathcal{M}_\alpha \overset{(4.6)}{\subseteq} \mathcal{M}_{\beta \alpha} = \pi(A) v_{\beta \alpha} \pi(A) = \pi(A) v_{\alpha} \pi(A) = \mathcal{M}_\alpha. \]
\( \square \)
5. The Toeplitz algebra of an interaction group.
We shall now define a “Toeplitz algebra” as an auxiliary step before introducing the main construction, i.e. the crossed product algebra.

5.1. Definition. The Toeplitz algebra of an interaction group \((A,G,V)\) is the universal unital C*-algebra \(\mathcal{T}(A,G,V)\) generated by a copy of \(A\) and a set \(\{\hat{s}_g : g \in G\}\) subject to the relations

(i) \(\hat{s}_1 = 1,\)
(ii) \(\hat{s}_g^* = \hat{s}_g^{-1},\)
(iii) \(\hat{s}_g \hat{s}_h \hat{s}_h^{-1} = \hat{s}_g \hat{s}_h \hat{s}_h^{-1},\)
(iv) \(\hat{s}_g a \hat{s}_g^{-1} = V_g(a) \hat{s}_g \hat{s}_g^{-1},\)

for all \(a \in A\) and \(g, h \in G.\)

In order to help the reader to keep track of the objects related to the Toeplitz algebra we have chosen to decorate these with a hat. The rule of thumb is therefore that everything that has a hat on it is related to the Toeplitz algebra.

Observe that by (5.1.ii-iii) one also has that \(\hat{s}_g^{-1} \hat{s}_g \hat{s}_h = \hat{s}_g^{-1} \hat{s}_g \hat{s}_h,\) for all \(g\) and \(h\) in \(G.\)
Therefore, the correspondence

\(g \in G \mapsto \hat{s}_g \in \mathcal{T}(A,G,V)\)

is a *-partial representation of \(G.\) It is therefore clear that:

5.2. Proposition. If

\(\hat{j} : A \to \mathcal{T}(A,G,V)\)

denotes the canonical map then \((\hat{j}, \hat{s})\) is a covariant representation of \((A,G,V),\) henceforth referred to as the Toeplitz covariant representation.

For future reference we state the universal property of \(\mathcal{T}(A,G,V)\) in a form that emphasizes its role relative to covariant representations:

5.3. Proposition. If \((\pi, v)\) is a covariant representation of \((A,G,V)\) in a C*-algebra \(B\) then there exists a unique *-homomorphism \(\hat{\pi} \times \hat{v} : \mathcal{T}(A,G,V) \to B\) such that the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{\hat{j}} & \mathcal{T}(A,G,V) \\
\downarrow & \searrow \pi & \\
\mathcal{T}(A,G,V) & \xrightarrow{\hat{\pi} \times \hat{v}} & B \\
\uparrow \hat{s} & \nearrow v \\
G
\end{array}
\]
6. Redundancies.

As before we fix a faithful interaction group \((A, G, V)\). We will now make extensive use of the spaces \(\mathcal{M}_\alpha\) of (4.4), the \(\mathcal{Z}_\alpha\) of (4.5), and the \(\mathcal{K}_\alpha\) of (4.8), all of these relative to the Toeplitz covariant representation. According to our rule of thumb we will denote these by \(\hat{\mathcal{M}}_\alpha\), \(\hat{\mathcal{Z}}_\alpha\), and \(\hat{\mathcal{K}}_\alpha\), respectively.

6.1. Definition. Let \(\alpha\) be a word in \(G\). By an \(\alpha\)-redundancy we shall mean any element \(k \in \hat{K}_\alpha\) such that \(k \hat{\mathcal{M}}_\alpha = \{0\}\). This is of course equivalent to

\[
k \hat{j}(b) \hat{s}_\alpha = 0, \quad \forall b \in A.
\]

Observe that, unlike [16: 1.2] and [12: 2.5], our notion of redundancies involves many group elements at once in the sense that the definition of \(\hat{K}_\alpha\) involves many \(\hat{s}_g\).

The following is the main concept introduced by this article:

6.2. Definition. Let \((A, G, V)\) be an interaction group. The crossed product of \(A\) by \(G\) under \(V\) is the C*-algebra \(A \rtimes_V G\), obtained by taking the quotient of \(\mathcal{T}(A, G, V)\) by the ideal generated by all redundancies. The quotient map will be denoted by \(q: \mathcal{T}(A, G, V) \rightarrow A \rtimes_V G\), the composition

\[
A \xrightarrow{\hat{j}} \mathcal{T}(A, G, V) \xrightarrow{q} A \rtimes_V G
\]

will be denoted by \(j\), and for every \(g\) in \(G\) we will let

\[
s_g = q(\hat{s}_g).
\]

It is therefore evident that \((j, s)\) is a covariant representation of \((A, G, V)\) in \(A \rtimes_V G\), henceforth referred to as the fundamental covariant representation.

Recall from (5.3) that every covariant representation \((\pi, v)\) induces a *-homomorphism \(\hat{\pi} \times \hat{v}\) defined on \(\mathcal{T}(A, G, V)\). There is no reason, however, for \(\hat{\pi} \times \hat{v}\) to vanish on redundancies.

6.3. Definition. Let \((\pi, v)\) be a covariant representation of \((A, G, V)\) in a C*-algebra \(B\). We will say that \((\pi, v)\) is strongly covariant if \(\hat{\pi} \times \hat{v}\) vanishes on all redundancies. In this case one therefore has a *-homomorphism \(\pi \times v: A \rtimes_V G \rightarrow B\) such that the following diagram commutes

\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & A \rtimes_V G \\
\uparrow j & & \uparrow s \\
A \rtimes_V G & \xrightarrow{\pi \times v} & B \\
\uparrow s & & \uparrow v \\
G
\end{array}
\]
It is evident that the fundamental covariant representation $(\jmath, s)$ is strongly covariant.

7. Grading of the crossed product.

Recall that a C*-algebra $B$ is said to be graded over the group $G$ if one is given an independent family $\{B_g\}_{g \in G}$ of closed linear subspaces of $B$, called the grading subspaces, such that $\bigoplus_{g \in G} B_g$ is dense in $B$ and such that for every $g$ and $h$ in $G$ one has that $B_g B_h \subseteq B_{gh}$ and $B_g^* = B_{g^{-1}}$. According to [5: 3.4] a grading is said to be a topological grading if there exists a conditional expectation from $B$ onto $B_1$ which vanishes on $B_g$, for all $g \neq 1$.

In this section we will prove that $A \rtimes_v G$ admits a canonical topological grading such that for all $g$ in $G$, one has that $s_g$ lies in the grading subspace associated to $g$.

In the following result we will employ the full C*-algebra of $G$, denoted $C^*(G)$, as well as the universal representation of $G$

$$u : G \rightarrow C^*(G).$$

7.1. Proposition. Let $(\pi, v)$ be a covariant representation of $(A, G, V)$ in a C*-algebra $B$. Consider the maps

$$\pi' = \pi \otimes 1 : A \rightarrow B \otimes C^*(G)$$

and

$$v' = v \otimes u : G \rightarrow B \otimes C^*(G).$$

Then $(\pi', v')$ is a covariant representation. Moreover if $(\pi, v)$ is strongly covariant then so is $(\pi', v')$. We will refer to $(\pi', v')$ as the amplification of $(\pi, v)$.

Proof. We leave everything up to the reader, except for the last statement. If $\beta = (h_1, \ldots, h_n)$ is a word in $G$ and $z$ is any element in $\hat{K}_\beta$ notice that

$$\hat{\pi' \times v'}(z) = \hat{\pi \times v}(z) \otimes u_h,$$

where $h = \hat{\beta}$. In case $\hat{\beta} = 1$ we therefore have that

$$\hat{\pi' \times v'} = \hat{\pi \times v} \otimes 1$$

on $\hat{Z}_\beta$. This will clearly also be the case on $\hat{K}_\alpha$ for every word $\alpha$. Given any $\alpha$-redundancy $k \in \hat{K}_\alpha$ we conclude that

$$\hat{\pi' \times v'}(k) = (\hat{\pi \times v})(k) \otimes 1.$$

If $(\pi, v)$ is strongly covariant then $\hat{\pi \times v}(k) = 0$, for all redundancies $k$ and hence the same is true for $\hat{\pi' \times v'}(k)$. \qed

7.2. Proposition. $A \rtimes_v G$ admits a topological $G$-grading $\{C_g\}_{g \in G}$ such that $j(A) \subseteq C_1$ and $s_g \in C_g$, for every $g$ in $G$. 
Proof. Throughout this proof whenever we speak of the $Z_\alpha$ of (4.5) it will be with respect to the fundamental covariant representation (see (6.2)). For every $g$ in $G$ let

$$C_g = \sum_{\alpha = g} Z_\alpha.$$ 

We leave it for the reader to prove that $\{C_g\}_{g \in G}$ satisfies [5: 3.3.i–iii]. Next we will provide a bounded linear map

$$F : A \rtimes_v G \to C_1$$

which is the identity on $C_1$ and which vanishes on all $C_g$ for $g \neq 1$.

For this consider the fundamental covariant representation $(\jmath, s)$ and let $(\jmath', s')$ be the corresponding amplification given by (7.1). Since $(\jmath, s)$ is strongly covariant then so is $(\jmath', s')$, and hence by (6.3) we get

$$\jmath' \times s' : A \rtimes_v G \to (A \rtimes_v G) \otimes C^* (G).$$

It is elementary to verify that

$$\jmath' \times s'(z) = z \otimes u_g, \quad \forall z \in C_g.$$

Let $tr$ be the standard trace on $C^* (G)$ and set

$$F = (id \otimes tr) \circ (\jmath' \times s').$$

For $z \in C_g$ we therefore have

$$F(z) = (id \otimes tr)(z \otimes u_g) = \begin{cases} 
  z, & \text{if } g = 1, \\
  0, & \text{if } g \neq 1.
\end{cases}$$

We therefore see that $F$ satisfies the required properties and hence we may invoke [5: 3.3] to conclude that $\{C_g\}_g$ is a topological grading for $A \rtimes_v G$.

Finally it is easy to see that $\jmath (A) \subseteq C_1$, and $s_g \in Z_g \subseteq C_g$, as desired.  \qed
8. Several Hilbert modules.

So far nothing guarantees that $T(A, G, V)$ or $A \rtimes_v G$ contain a single non-zero element! In order to show that these algebras have any substance at all we need to do some work. This section is dedicated to obtaining several technical results in preparation for the description of certain representations of $T(A, G, V)$ and $A \rtimes_v G$.

8.1. Proposition. For all $a$ in $A$ one has that $V_g(a^*)V_g(a) \leq V_g(a^*a)$.

Proof. Follows easily from (3.2.i) and [19: IV.3.8].

Recall that by (2.2.iii) the $E_g$ commute amongst themselves. It then easily follows that the composition of any number of $E_g$'s is again a conditional expectation onto the intersection of the $R_g$'s involved.

8.2. Definition. For each finite subset $X \subseteq G$ we will let

$$R_X = \bigcap_{g \in X} R_g, \quad \text{and} \quad E_X = \prod_{g \in X} E_g,$$

so that $E_X$ is a conditional expectation onto $R_X$.

Since each $E_g$ is faithful we clearly have that the $E_X$ are faithful as well.

8.3. Proposition. For every $g$ in $G$ and every finite subset $X \subseteq G$ we have that

(i) $V_g(R_X) \subseteq R_g X$,

(ii) if $1 \in X$ then the inclusion in (i) becomes an equality.

Proof. Observing that

$$V_g E_h = E_{gh} V_g,$$

by (2.2.ii) we have

$$V_g(R_X) = V_g \left( \prod_{h \in X} E_h(A) \right) = \prod_{h \in X} E_{gh}(V_g(A)) \subseteq \prod_{h \in X} E_{gh}(A) = R_g X.$$

If $1 \in X$ then

$$R_g X = \prod_{h \in X} E_{gh}(A) = \prod_{h \in X} E_{gh} E_g(A) = \prod_{h \in X} E_{gh} V_g V_{g^{-1}}(A) \subseteq$$

$$\subseteq \prod_{h \in X} E_{gh} V_g(A) = V_g \left( \prod_{h \in X} E_h(A) \right) = V_g(R_X).$$

8.4. Definition. For every finite subset $X \subseteq G$ we will let $H_X$ be the right Hilbert $R_X$–module obtained by completing $A$ under the $R_X$–valued inner-product defined by

$$\langle a, b \rangle_X = E_X(a^*b), \quad \forall a, b \in A.$$
Notice that the standing hypothesis according to which the $E_g$ are faithful implies that the above inner-product is non-degenerated.

**8.5. Proposition.** For every $g$ in $G$, and every finite subset $X \subseteq G$, there exists a bounded linear map

$$
\tilde{s}_g : \mathcal{H}_X \to \mathcal{H}_gX
$$

such that $\tilde{s}_g(a) = V_g(a)$, for all $a$ in $A$ (notice that we are not using any special decoration to denote the image of an element $a \in A$ within $\mathcal{H}_X$). If moreover $1 \in X$ then

$$
\langle \tilde{s}_g(\xi), \eta \rangle_{gX} = V_g(\langle \xi, \tilde{s}_{g^{-1}}(\eta) \rangle_X), \quad \forall \xi \in \mathcal{H}_X, \quad \forall \eta \in \mathcal{H}_gX.
$$

**Proof.** For $a \in A$ we have

$$
\langle V_g(a), V_g(a) \rangle_{gX} = E_{gX} (V_g(a^*) V_g(a)) \overset{(8.1)}{=} E_{gX} (V_g(a^*a)) = V_g(E_{X}(a^*a)) = V_g(\langle a, a \rangle_X),
$$

so that $\|V_g(a)\|_{gX} \leq \|a\|_X$ and hence the correspondence $a \mapsto V_g(a)$ extends to a bounded linear map $\tilde{s}_g : \mathcal{H}_X \to \mathcal{H}_gX$ such that $\tilde{s}_g(a) = V_g(a)$, for all $a$ in $A$.

Assuming that $1 \in X$ we have that $g \in gX$ and hence $E_{gX} = E_{gX} E_g$, so that given $a, b \in A$ we have

$$
\langle V_g(a), b \rangle_{gX} = E_{gX} (V_g(a^*)b) = E_{gX} (V_g(a^*)b) =
$$

$$
= E_{gX} (V_g(a^*)E_g(b)) = E_{gX} (V_g(a^*)V_g(V_{g^{-1}}(b))) =
$$

$$
= E_{gX} (V_g(a^*V_{g^{-1}}(b))) = V_g(E_{X}(a^*V_{g^{-1}}(b))) = V_g(\langle a, V_{g^{-1}}(b) \rangle_X).
$$

The conclusion then follows from the density of $A$ both in $\mathcal{H}_X$ and in $\mathcal{H}_gX$. \qed

**8.6. Definition.** Given $g$ in $G$, let $\mathcal{L}_g(\mathcal{H}_X)$ denote the set of all maps $T : \mathcal{H}_X \to \mathcal{H}_gX$, for which there exists a map $S : \mathcal{H}_gX \to \mathcal{H}_X$ satisfying

$$
\langle T\xi, \eta \rangle_{gX} = V_g(\langle \xi, S\eta \rangle_X), \quad \forall \xi \in \mathcal{H}_X, \quad \forall \eta \in \mathcal{H}_gX.
$$

Observe that because $V_g$ preserves adjoint, the equation displayed above is equivalent to

$$
\langle \eta, T\xi \rangle_{gX} = V_g(\langle S\eta, \xi \rangle_X), \quad \forall \xi \in \mathcal{H}_X, \quad \forall \eta \in \mathcal{H}_gX.
$$

Observe also that when $1 \in X$, the map $\tilde{s}_g$ of (8.5) lies in $\mathcal{L}_g(\mathcal{H}_X)$.

**8.7. Proposition.** When $T \in \mathcal{L}_g(\mathcal{H}_X)$ and $g^{-1} \in X$, one has that the map $S$ mentioned in (8.6) is uniquely determined. We therefore denote it by $T^*$ and call it the adjoint of $T$. 


We claim that if $\zeta \in \mathcal{H}_X$ is such that
\[ V_g (\langle \xi, \zeta \rangle_X) = 0, \quad \forall \xi \in \mathcal{H}_X, \]
then $\zeta = 0$. In order to prove this claim plug $\xi = \zeta$ above so that
\[ 0 = V_{g^{-1}} V_g (\langle \zeta, \zeta \rangle_X) = E_{g^{-1}} (\langle \zeta, \zeta \rangle_X) = \langle \zeta, \zeta \rangle_X, \]
where the last equality follows from the fact that the range of $R$ which in turn is contained in $\mathcal{R}_X$ because $g^{-1} \in X$. Therefore $\zeta = 0$.

If $S_1$ and $S_2$ are maps satisfying the conditions of (8.6) we have for all $\xi \in \mathcal{H}_X$ and $\eta \in \mathcal{H}_{gX}$ that
\[ 0 = V_g (\langle \xi, S_1(\eta) - S_2(\eta) \rangle_X), \]
so that $S_1(\eta) = S_2(\eta)$ by the claim. \qed

We thank the referee for pointing out that the assumption that $g^{-1} \in X$ is unnecessary because we are assuming that $E_{g^{-1}}$ is faithful. However, if a generalization of this work is to be obtained for non faithful interaction groups, it is interesting to see that at least the above result survives.

Combining (8.5) and (8.7) and supposing that $\{1, g^{-1}\} \subseteq X$ we have that $(\bar{s}_g)^* = \bar{s}_{g^{-1}}$.

8.8. Proposition. Any $T \in \mathcal{L}_g(\mathcal{H}_X)$ is bounded and $\mathbb{C}$-linear. If moreover $g^{-1} \in X$ then
\[ T(\xi a) = T(\xi)V_g(a), \quad \forall \xi \in \mathcal{H}_X, \quad \forall a \in \mathcal{R}_X. \]

Proof. Given $\lambda \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathcal{H}_X$, and $\eta \in \mathcal{H}_{gX}$ we have
\[ \langle T(\xi_1 + \lambda \xi_2), \eta \rangle_{gX} = V_g (\langle \xi_1 + \lambda \xi_2, T^* \eta \rangle_X) = \]
\[ = V_g (\langle \xi_1, T^* \eta \rangle_X) + \lambda V_g (\langle \xi_2, T^* \eta \rangle_X) = \langle T(\xi_1), \eta \rangle_{gX} + \lambda \langle T(\xi_2), \eta \rangle_{gX} = \]
\[ = \langle T(\xi_1) + \lambda T(\xi_2), \eta \rangle_{gX}, \]
whence $T(\xi_1 + \lambda \xi_2) = T(\xi_1) + \lambda T(\xi_2)$.

In order to prove that $T$ is bounded suppose that $\lim_n \xi_n = 0$, while $\lim_n T(\xi_n) = \zeta$. Then
\[ \langle \zeta, \zeta \rangle_{gX} = \lim_n \langle T(\xi_n), \zeta \rangle_{gX} = \lim_n V_g (\langle \xi_n, T^* \zeta \rangle_X) = 0, \]
which implies that $\zeta = 0$, and hence the conclusion follows by the Closed Graph Theorem.

Given $\xi \in \mathcal{H}_X$, $\eta \in \mathcal{H}_{gX}$, and $a \in \mathcal{R}_X$ we have that
\[ \langle T(\xi a), \eta \rangle_{gX} = V_g (\langle \xi a, T^* \eta \rangle_X) = V_g (a^* \langle \xi, T^* \eta \rangle_X) = \cdots \]
Assuming that $g^{-1} \in X$ we have that $\langle \xi, T^* \eta \rangle_X \in \mathcal{R}_{g^{-1}}$, and so the above equals
\[ \cdots = V_g (a^* V_g (\langle \xi, T^* \eta \rangle_X) = V_g (a^*) \langle T\xi, \eta \rangle_{gX} = \langle T(\xi)V_g(a), \eta \rangle_{gX}, \]
thus proving that $T(\xi a) = T(\xi)V_g(a)$. \qed
8.9. Proposition. Suppose that \( \{1, g^{-1}\} \subseteq X \). Then for every \( T \in \mathcal{L}_g(\mathcal{H}_X) \) one has that

(i) \( T^* \in \mathcal{L}_{g^{-1}}(\mathcal{H}_{gX}) \),
(ii) \( (T^*)^* = T \),
(iii) \( \|T^*\| = \|T\| \),
(iv) \( \|T^*T\| = \|T\|^2 \).

Proof. Initially observe that the hypothesis that \( 1 \in X \) says that \( g \in gX \), in which case the adjoint of an operator in \( \mathcal{L}_{g^{-1}}(\mathcal{H}_{gX}) \) is well defined by (8.7).

Given that \( g^{-1} \in X \) we have that \( E_X = E_{g^{-1}}E_X \) and hence \( \mathcal{R}_X \subseteq \mathcal{R}_{g^{-1}} \). Therefore for any \( \xi \in \mathcal{H}_X \) and \( \eta \in \mathcal{H}_{gX} \) we have that \( \langle \xi, T^*\eta \rangle_X \in \mathcal{R}_{g^{-1}} \) so

\[
\langle \xi, T^*\eta \rangle_X = V_{g^{-1}}V_g \left( \langle \xi, T^*\eta \rangle_X \right) = V_{g^{-1}} \left( \langle T\xi, \eta \rangle_{gX} \right).
\]

This proves (i) and (ii). Next observe that for all \( \xi \in \mathcal{H}_X \) we have

\[
\|T\|_2 \leq \|T^*T\| \|\xi\|^2,
\]
proving that \( \|T\|^2 \leq \|T^*T\| \). From this (iii) and (iv) follow without difficulty. \( \square \)

8.10. Proposition. Suppose that \( \{g^{-1}, g^{-1}h^{-1}\} \subseteq X \) and let \( T \in \mathcal{L}_g(\mathcal{H}_X) \) and \( S \in \mathcal{L}_h(\mathcal{H}_{gX}) \). Then \( ST \in \mathcal{L}_{gh}(\mathcal{H}_X) \) and \( (ST)^* = T^*S^* \).

Proof. Initially observe that \( h^{-1} \in gX \) whence \( S^* \) is well defined by (8.7). For \( \xi \in \mathcal{H}_X \) and \( \zeta \in \mathcal{H}_{ghX} \) we have

\[
\langle ST\xi, \zeta \rangle_{ghX} = V_h \left( \langle T\xi, S^*\zeta \rangle_{gX} \right) = V_h \left( V_g \left( \langle \xi, S^*S^{-1} \zeta \rangle_{X} \right) \right) = \ldots
\]

Letting \( a = \langle \xi, T^*S^*\zeta \rangle_X \) we have that \( a \in \mathcal{R}_X \subseteq \mathcal{R}_{g^{-1}} \), so the above equals

\[
\ldots = V_hV_g(a) = V_hV_gV_{g^{-1}}V_g(a) = V_{gh}V_{g^{-1}}V_g(a) = V_{gh}(a) = V_{gh}(\langle \xi, T^*S^*\zeta \rangle_X),
\]
proving the statement. \( \square \)

8.11. Proposition. If \( \{1, g^{-1}\} \subseteq X \) then for every \( T \in \mathcal{L}_g(\mathcal{H}_X) \) one has that \( T^*T \) is a positive element in the \( \mathcal{C}^* \)-algebra \( \mathcal{L}(\mathcal{H}_X) = \mathcal{L}_1(\mathcal{H}_X) \).

Proof. First note that since \( g^{-1} \in X \) we have by (8.9.i) that \( T^* \in \mathcal{L}_{g^{-1}}(\mathcal{H}_{gX}) \). Applying (8.10) (with \( h = g^{-1} \)) we then conclude that \( T^*T \in \mathcal{L}_1(\mathcal{H}_X) \). For \( \xi \in \mathcal{H}_X \) we have

\[
\langle T^*T\xi, \xi \rangle_X = V_{g^{-1}} \left( \langle T\xi, T\xi \rangle_{gX} \right) \geq 0,
\]
and hence \( T^*T \) is positive. \( \square \)
Let $X$ and $Y$ be finite subsets of $G$ such that $X \subseteq Y$. Then $E_Y = E_Y E_X$, so for every $a \in A$ we have

$$\| \langle a, a \rangle_Y \| = \| E_Y (a^* a) \| = \| E_Y E_X (a^* a) \| \leq \| E_X (a^* a) \| = \| \langle a, a \rangle_X \|,$$

and hence the correspondence $a \in A \mapsto a \in H_Y$ extends to give a contractive\(^2\) linear map

$$\iota : H_X \to H_Y,$$

whose restriction to $A$ (or rather to the canonical dense copy of $A$ within $H_X$) is the identity.

Because

$$\langle \iota(a), \iota(a) \rangle_Y = E_Y (a^* a) = E_Y E_X (a^* a) = E_Y (\langle a, a \rangle_X), \quad \forall a \in A,$$

we have that

$$\langle \iota(\xi), \iota(\xi) \rangle_Y = E_Y (\langle \xi, \xi \rangle_X), \quad \forall \xi \in H_X.$$  \hspace{1cm} (8.12)

Since $E_Y$ is faithful one sees that $\iota$ is an injective map. It will often be convenient to think of $H_X$ as a dense subspace of $H_Y$. Nevertheless care must be taken to account for the fact that $H_X$ and $H_Y$ are quite different objects, having different norms and being Hilbert modules over different algebras.

8.13. Proposition. Let $X$ and $Y$ be finite subsets of $G$, let $g \in G$ be such that $\{1, g^{-1}\} \subseteq X \subseteq Y$, and let $T \in L_g(H_X)$. Then there exists a unique bounded operator $\tilde{T} : H_Y \to H_{gY}$ such that the diagram

$$\begin{array}{ccc}
H_X & \xrightarrow{T} & H_{gX} \\
\downarrow \iota & & \downarrow \iota \\
H_Y & \xrightarrow{\tilde{T}} & H_{gY}
\end{array}$$

commutes. Moreover $\tilde{T} \in L_g(\mathcal{H}_Y)$, $\| \tilde{T} \| \leq \| T \|$, and $(\tilde{T})^* = \tilde{T}^*$.

Proof. We should first observe that we are denoting by $\iota$ both inclusions $H_X \to H_Y$ and $H_{gX} \to H_{gY}$, leaving for the context to distinguish which is which.

By (8.11) we have that $T^* T \in L_1(\mathcal{H}_X)$ and by (8.9.iv) it follows that $T^* T \leq \| T \|^2$. Therefore for all $\xi \in H_X$ we have

$$\langle T \xi, T \xi \rangle_{gX} = V_g (\langle T^* T \xi, \xi \rangle_X) \leq \| T \|^2 \| V_g (\langle \xi, \xi \rangle_X) \|.$$

So by (8.12) we have that

$$\langle \iota T \xi, \iota T \xi \rangle_{gY} = E_{gY} (\langle T \xi, T \xi \rangle_{gX}) \leq \| T \|^2 \| E_{gY} V_g (\langle \xi, \xi \rangle_X) \|.$$

\(^2\) We use the term “contractive” to mean “non-expansive”, i.e. that $\| \iota(\xi) \| \leq \| \xi \|$, for all $\xi$ in $H_X$.\]
\[ \|T^2\| V_g \mathcal{E}_g (\langle \xi, \xi \rangle_X) = \|T^2\| V_g (\langle \xi, \xi \rangle_Y), \]

from where one deduces that \(\|T\xi\| \leq \|T\| \|\xi\|\). Given that \(\iota(\mathcal{H}_X)\) is dense in \(\mathcal{H}_Y\) we have that the correspondence \(\iota \xi \mapsto \iota T \xi\) extends to a bounded linear map \(\widetilde{T} : \mathcal{H}_Y \to \mathcal{H}_{gY}\) such that the diagram above commutes and such that \(\|\widetilde{T}\| \leq \|T\|.\) Again because \(\iota(\mathcal{H}_X)\) is dense in \(\mathcal{H}_Y\) we have that \(\widetilde{T}\) is uniquely determined.

Since \(\{1, g\} \subseteq gX\), the above reasoning applies to \(T^*\) so we may speak of \(\widetilde{T}^*\) as well. Given \(\xi \in \mathcal{H}_X\) and \(\eta \in \mathcal{H}_{gX}\) we have

\[ \langle \widetilde{T} \iota \xi, \iota \eta \rangle_{gY} = V_g (\langle \iota \xi, \iota T^* \eta \rangle_{gX}) = \mathcal{E}_g (\langle \xi, \iota \iota^* \eta \rangle_{gY}). \]

Since \(\iota(\mathcal{H}_X)\) is dense in \(\mathcal{H}_Y\) and \(\iota(\mathcal{H}_{gX})\) is dense in \(\mathcal{H}_{gY}\) it follows that \(\iota \in \mathcal{H}_{gY}\) whence \(\iota \in \mathcal{H}_{gY}\) and \(\iota \mathcal{H}_{gY}\) it follows that \(\langle \widetilde{T} \iota \xi, \iota \eta \rangle_{gY} = V_g (\langle \xi, \iota \iota^* \eta \rangle_{gY})\), \(\forall \xi \in \mathcal{H}_Y, \forall \eta \in \mathcal{H}_{gY}\),

whence \(\widetilde{T} \in \mathcal{L}_g(\mathcal{H}_Y)\) and \((\widetilde{T})^* = \widetilde{T}^*\).

\section{8.14. Proposition.} Suppose that \(\{1, g^{-1}, g^{-1} h^{-1}\} \subseteq X \subseteq Y\). Then for every \(T \in \mathcal{L}_g(\mathcal{H}_X)\) and \(S \in \mathcal{L}_h(\mathcal{H}_{gX})\) one has that \((ST) = \widetilde{ST}\).

\textbf{Proof.} Initially observe that, by abuse of language, we are denoting by tilde the correspondences given by (8.13) in the all three cases:

\[ \mathcal{L}_g(\mathcal{H}_X) \to \mathcal{L}_g(\mathcal{H}_Y) \text{ (for } T), \]

\[ \mathcal{L}_h(\mathcal{H}_{gX}) \to \mathcal{L}_h(\mathcal{H}_{gY}) \text{ (for } S), \]

\[ \mathcal{L}_{hg}(\mathcal{H}_X) \to \mathcal{L}_{hg}(\mathcal{H}_Y) \text{ (for } ST). \]

Also notice that \(ST \in \mathcal{L}_{hg}(\mathcal{H}_X)\) by (8.10). Considering the diagram

\[
\begin{array}{ccc}
\mathcal{H}_X & \xrightarrow{T} & \mathcal{H}_{gX} & \xrightarrow{S} & \mathcal{H}_{hgX} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{H}_Y & \xrightarrow{\widetilde{T}} & \mathcal{H}_{gY} & \xrightarrow{\widetilde{S}} & \mathcal{H}_{hgY}
\end{array}
\]

we deduce that \((\widetilde{ST}) = \widetilde{ST}\) by the uniqueness part of (8.13).

If we specialize (8.13) and (8.14) to the case in which \(g = h = 1\), and if we suppose that \(1 \in X \subseteq Y\), we conclude that the correspondence

\[ T \in \mathcal{L}_1(\mathcal{H}_X) \mapsto \widetilde{T} \in \mathcal{L}_1(\mathcal{H}_Y) \]

is a \(\mathcal{C}^*\)-algebra homomorphism.

Moreover, since \(\widetilde{T}\) is an extension of \(T\), it is clear that this homomorphism is injective and hence isometric.
8.15. Corollary. If \( \{1, g^{-1}\} \subseteq X \subseteq Y \) and \( T \in \mathcal{L}_g(\mathcal{H}_X) \) then \( \|\tilde{T}\| = \|T\| \).

Proof. By (8.11) it follows that \( T^*T \in \mathcal{L}(\mathcal{H}_X) \). Since \( 1 \in X \) we have by the discussion above that
\[
\|T^*T\| = \|(\tilde{T}^*\tilde{T})\| = \|\tilde{T}^*\tilde{T}\|.
\]
The conclusion then follows from (8.9.iv).

\[\square\]

9. The regular covariant representation.

In this section we will describe representations of \( \mathcal{T}(A, G, V) \) and \( A \rtimes_v G \) which will, among other things, show that the maps \( \hat{\mathfrak{h}} \) and \( \mathfrak{h} \) are injective.

Given \( g \in G \), let \( \mathcal{I}_g \) be the subset of the power set of \( G \) defined by
\[
\mathcal{I}_g = \{ X \subseteq G : X \text{ is finite and } 1, g^{-1} \in X \}.
\]
Ordered by inclusion we have that \( \mathcal{I}_g \) is a directed set. For each \( X, Y \in \mathcal{I}_g \), with \( X \subseteq Y \), let us denote by \( \varepsilon_{YX} \) the correspondence
\[
T \in \mathcal{L}_g(\mathcal{H}_X) \rightarrow \tilde{T} \in \mathcal{L}_g(\mathcal{H}_Y)
\]
of (8.13), which we now know is an isometry.

We want to view the collection of Banach spaces \( \{\mathcal{L}_g(\mathcal{H}_X)\}_{X \in \mathcal{I}_g} \), equipped with the maps \( \{\varepsilon_{YX}\}_{X, Y \in \mathcal{I}_g} \), as a directed system. In order to justify this we must show that, whenever \( X \subseteq Y \subseteq Z \), one has that \( \varepsilon_{ZY} \varepsilon_{YX} = \varepsilon_{ZX} \). But this is easy to see once one realizes that the diagram
\[
\begin{array}{ccc}
\mathcal{H}_X & \xrightarrow{T} & \mathcal{H}_{gX} \\
\downarrow & & \downarrow \\
\mathcal{H}_Y & \xrightarrow{\tilde{T}} & \mathcal{H}_{gY} \\
\downarrow & & \downarrow \\
\mathcal{H}_Z & \xrightarrow{\tilde{\tilde{T}}} & \mathcal{H}_{gZ}
\end{array}
\]
commutes, where \( \tilde{T} = \varepsilon_{YX}(T) \), and \( \tilde{\tilde{T}} = \varepsilon_{ZY}(\tilde{T}) \).

In the next Definition we shall refer to the inductive limit in the category of Banach spaces with isometric linear maps as morphisms. It is easy to prove that this category does indeed admit inductive limits.

9.1. Definition. Given \( g \in G \) we shall denote the inductive limit of the directed family \( \{\mathcal{L}_g(\mathcal{H}_X)\}_{X \in \mathcal{I}_g} \) by \( B_g \). The dense subspace of \( B_g \) formed by the union of the \( \mathcal{L}_g(\mathcal{H}_X) \), as \( X \) ranges in \( \mathcal{I}_g \), will be denoted by \( B_g^0 \).
When $X = \{1\}$ it is clear that $H_X = A$, and $L_1(H_X)$ is $*$-isomorphic to $A$. We therefore have an injective $*$-homomorphism

$$i : A \simeq L_1(H_X) \hookrightarrow B_1,$$

which we will often use to view $A$ as a subalgebra of $B_1$.

Given $T \in B_g^0$ and $S \in B_h^0$ choose $X \in \mathcal{I}_g$ and $Y \in \mathcal{I}_h$ so that $T \in L_g(H_X)$ and $S \in L_h(H_Y)$. We may assume without loss of generality that $Y = gX$ (if not one may replace $X$ by $X \cup g^{-1}Y$ and $Y$ by $Y \cup gX$). By (8.10) we have that $ST \in \mathcal{L}_{hg}(H_X) \subseteq B_{hg}$.

Suppose that instead of choosing the above $X$ and $Y$ we take $X' \supseteq X$ and $Y' \supseteq Y$, still satisfying the relation $Y' = gX'$. In the inductive limit one would then identify $T$ with its extension $\tilde{T} \in \mathcal{L}_g(H_{X'})$, and $S$ with $\tilde{S} \in \mathcal{L}_h(H_{Y'})$. By (8.14) it follows that $\tilde{ST} = \tilde{S}\tilde{T}$ which means that the class of $ST$ in $B_{hg}$ does not depend on the above choices.

We have therefore defined an operation

$$B_h^0 \times B_g^0 \to B_{hg}^0$$

which obviously satisfies $\|ST\| \leq \|S\| \|T\|$ and hence extends to a multiplication operation

$$B_h \times B_g \to B_{hg}.$$

It is also clear that the adjoint operation introduced in (8.7) gives an involution

$$T \in B_g \mapsto T^* \in B_{g^{-1}}.$$

9.3. Proposition. The collection $\mathcal{B} = \{B_g\}_{g \in G}$ is a Fell bundle [11] with the operations defined above.

Proof. Follows from the results of section (8). \qed

Given $g \in G$, observe that by (8.5) we have that $\tilde{s}_g \in \mathcal{L}_g(H_X)$, for any $X \in \mathcal{I}_g$. Henceforth we will identify $\tilde{s}_g$ with its image in $B_g$ and hence we will think of $\tilde{s}_g$ as an element of $C^*(\mathcal{B})$, the cross sectional $C^*$-algebra of $\mathcal{B}$ [11].

9.4. Proposition. The map

$$g \in G \mapsto \tilde{s}_g \in C^*(\mathcal{B})$$

is a partial representation of $G$ in $C^*(\mathcal{B})$ and for every $a \in A \subseteq B_1 \subseteq C^*(\mathcal{B})$ we have that

$$\tilde{s}_g a \tilde{s}_g^{-1} = V_g(a)\tilde{s}_g \tilde{s}_g^{-1}, \forall g \in G.$$
Proof. Let $X = \{1, g\}$ and let us view $\hat{s}_g^{-1} \in \mathcal{L}_{g^{-1}}(\mathcal{H}_X)$, $\hat{s}_g \in \mathcal{L}_g(\mathcal{H}_{g^{-1}X})$, $a \in \mathcal{L}_1(\mathcal{H}_{g^{-1}X})$, and $V_g(a) \in \mathcal{L}_1(\mathcal{H}_X)$. Then, proving the displayed relation above amounts to comparing the following compositions

$$\mathcal{H}_X \xrightarrow{\hat{s}_g^{-1}} \mathcal{H}_{g^{-1}X} \xrightarrow{\hat{s}_g} \mathcal{H}_X \xrightarrow{V_g(a)} \mathcal{H}_X$$

and

$$\mathcal{H}_X \xrightarrow{\hat{s}_g^{-1}} \mathcal{H}_{g^{-1}X} \xrightarrow{a} \mathcal{H}_{g^{-1}X} \xrightarrow{\hat{s}_g} \mathcal{H}_X.$$

Since $A$ is dense in $\mathcal{H}_X$ it is enough to prove that these compositions agree on $A$. Given any $b \in A$ we have

$$\hat{s}_g a \hat{s}_g^{-1}(b) = V_g(a V_g^{-1}(b)) = V_g(a) V_g(V_g^{-1}(b)) = V_g(a) \hat{s}_g \hat{s}_g^{-1}(b).$$

That $\hat{s}$ is a partial representation follows from a similar argument. □

An equivalent way to state the above result is:

9.5. Corollary. Let $i : A \to B_1 \subseteq C^*(\mathcal{B})$ be the canonical inclusion described in (9.2). Then $(i, \hat{s})$ is a covariant representation of $(A, G, V)$ in $C^*(\mathcal{B})$. This will be called the regular covariant representation.

Another important consequence is:

9.6. Corollary. The canonic map $\hat{j} : A \to \mathcal{T}(A, G, V)$ is injective.

Proof. This is an immediate consequence of the fact that the the composition $(\hat{i} \times \hat{s}) \circ \hat{j}$ coincides with $i$ by (5.3), and that $i$ is an injective map. □

In particular, this leads to two examples of covariant representations which are faithful on $A$, namely the Toeplitz covariant representation $(\hat{j}, \hat{s})$ of (5.2) and the regular covariant representation $(i, \hat{s})$ of (9.5).

Our next immediate goal will be to show that $(i, \hat{s})$ is strongly covariant. In preparation for this we need a technical result to be proved below.

9.7. Lemma. If $\beta$ is any word in $G$ and $X \subseteq G$ is any finite set such that $\mu(\beta^{-1}) \subseteq X$, then

$$(\hat{i} \times \hat{s})(\hat{\mathcal{Z}}_{\beta}) \subseteq \mathcal{L}_\beta(\mathcal{H}_X).$$

Proof. Suppose that $\beta = (h_1, \ldots, h_n)$ and let $\gamma = (h_2, \ldots, h_n)$, so that $\beta = h_1\gamma$. Clearly $\hat{\mathcal{Z}}_{\beta} = \hat{\mathcal{Z}}_{h_1} \hat{\mathcal{Z}}_{\gamma}$ (closed linear span of products). Notice that $\mu(\gamma^{-1}) \subseteq \mu(\beta^{-1}) \subseteq X$, so by induction we have that $(\hat{i} \times \hat{s})(\hat{\mathcal{Z}}_{\gamma}) \subseteq \mathcal{L}_\gamma(\mathcal{H}_X)$. This means that the elements of $(\hat{i} \times \hat{s})(\hat{\mathcal{Z}}_{\gamma})$ are operators between the following spaces:

$$\mathcal{H}_X \to \mathcal{H}_{\gamma X}.$$
By hypothesis we have that
\[ \gamma^{-1}h_1^{-1} = h_n^{-1}h_{n-1}^{-1} \cdots h_2^{-1}h_1^{-1} \in \mu(\beta^{-1}) \subseteq X. \]
Therefore \( h_1^{-1} \in \gamma X \). It is also clear that \( \gamma^{-1} \in X \) so that \( 1 \in \gamma X \). In other words \( \{1, h_1^{-1}\} \subseteq \gamma X \), so \( s_{h_1} \in L_{h_1}(H_{\gamma X}) \), which means that \( s_{h_1} \) defines an operator
\[ s_{h_1} : H_{\gamma X} \to H_{h_1\gamma X} = H_{\beta X}. \]
The assertion then follows from (8.10).

9.8. Proposition. **The regular covariant representation \((i, s)\) is strongly covariant.**

Proof. Let \( \alpha = (g_1, \ldots, g_n) \) be a word in \( G \) and let \( k \in \hat{K}_\alpha \) be an \( \alpha \)-redundancy. Recall that \( \hat{K}_\alpha \) is the closure of the sum of the \( \hat{Z}_\beta \), for all words \( \beta \) such that \( \hat{\beta} = 1 \) and \( \mu(\beta^{-1}) \subseteq \mu(\alpha) \).

Since each \( \hat{s}_g \) is mapped under \( \hat{i} \times \hat{s} \) to \( B_g \) it is easy to see that the fact that \( \hat{\beta} = 1 \) implies that \( \hat{i} \times \hat{s} \) maps \( \hat{Z}_\beta \) to \( B_1 \). Recall moreover that \( B_1 \) is defined as the direct limit of the \( L_1(H_X) \), as \( X \) ranges in the collection of all finite subsets of \( G \) containing \( 1 \).

We claim that there exists a single \( X \) such \( \hat{i} \times \hat{s}(\hat{Z}_\beta) \subseteq L_1(H_X) \), for all \( \beta \)'s above. In fact, if \( X \) is any finite subset of \( G \) with \( \mu(\alpha) \subseteq X \) then for every \( \beta \) with \( \hat{\beta} = 1 \) and \( \mu(\beta^{-1}) \subseteq \mu(\alpha) \subseteq X \), we have by (9.7) that
\[ \hat{i} \times \hat{s}(\hat{Z}_\beta) \subseteq L_1(H_X) = L_1(H_X). \]

By definition of \( \hat{K}_\alpha \) we conclude that \( \hat{i} \times \hat{s}(\hat{K}_\alpha) \subseteq L_1(H_X) \). We may therefore regard \( \hat{i} \times \hat{s}(k) \) as an operator on \( H_X \).

On the other hand we would like to show that \( \hat{s}_\alpha \in L_\alpha(H_{\alpha^{-1}X}) \). Since \( \hat{s}_\alpha \in \hat{Z}_\alpha \) and \( \hat{s}_\alpha = \hat{i} \times \hat{s}(\hat{s}_\alpha) \) it is enough to prove that
\[ \hat{i} \times \hat{s}(\hat{Z}_\alpha) \subseteq L_\alpha(H_{\alpha^{-1}X}). \]
But this follows once more from Lemma (9.7) since
\[ \mu(\alpha^{-1} \mathbin{\overset{(2.11.i)}{\Rightarrow}}) \alpha^{-1} \mu(\alpha) \subseteq \alpha^{-1} X. \]
Given that \( k \) is a redundancy, we have that
\[ kb\hat{s}_\alpha = 0, \quad \forall b \in A, \]
which tells us that \( \hat{i} \times \hat{s}(k)b\hat{s}_\alpha = 0 \). By what was said above \( \hat{i} \times \hat{s}(k)b\hat{s}_\alpha \) may be interpreted as the composition of operators
\[ H_{\alpha^{-1}X} \xrightarrow{b\hat{s}_\alpha} H_X \xrightarrow{\hat{i} \times \hat{s}(k)} H_X. \]
Regarding the unit of \( A \) as an element of \( H_{\alpha^{-1}X} \), we therefore have that
\[ 0 = \hat{i} \times \hat{s}(k)b\hat{s}_\alpha(1) = \hat{i} \times \hat{s}(k)(b). \]
Since this holds for every \( b \) in \( A \), and since \( A \) is dense in \( H_X \), we conclude that \( \hat{i} \times \hat{s}(k) = 0 \).
9.9. Corollary. The canonical map
\[
j : A \to A \rtimes_\psi G
\]
(see (6.2)) is an injection.

Proof. By (6.2) we have that \(j = q \circ \hat{j}\). Since \((i, \hat{s})\) is strongly covariant by (9.8) we obtain the \(*\)-homomorphism
\[
i \times \hat{s} : A \rtimes_\psi G \to C^*(B)
\]
which satisfies \((i \times \hat{s}) \circ j = i\), by (6.3). Since \(i\) is injective it follows that \(j\) is injective as well. \(\square\)

We should notice that this does not solve question (1) mentioned after [8: 7.12]. The reason is that a single interaction is not necessarily part of an interaction group.

10. Faithful representations.
As seen in the paragraph following (9.6), strongly covariant representations \((\pi, v)\), where \(\pi\) is faithful, always exist. However there is no reason for \(\pi \times v\) to be faithful. In this section we will look into the question of faithfulness of \(\pi \times v\) carefully.

10.1. Definition. (cf. [8: 3.6]) A covariant representation \((\pi, v)\) of the interaction group \((A, G, V)\) in a C*-algebra \(B\), will be called non-degenerate if for every word \(\alpha\) in \(G\) the map
\[
a \in A \mapsto \pi(a)v_\alpha \in B
\]
is injective.

Given a non-degenerate covariant representation \((\pi, v)\) notice that in particular \(\pi\) must be injective. Therefore we will often view \(A\) as a subalgebra of \(B\) via \(\pi\).

For example we have:

10.2. Proposition. Both the regular covariant representation \((i, \hat{s})\) and the Toeplitz covariant representation \((\hat{j}, \hat{s})\) are non-degenerate.

Proof. We begin by considering \((i, \hat{s})\). Let us therefore be given a word \(\alpha\) in \(G\) and \(a\) in \(A\) such that \(a \hat{s}_\alpha = 0\). Let \(X\) be any finite subset of \(G\) such that \(\mu(\alpha^{-1}) \subseteq X\). By Lemma (9.7) we have that \(\hat{s}_\alpha \in \mathcal{L}_\alpha(\mathcal{H}_X)\).

Thus, if the unit 1 of \(A\) is interpreted as an element of \(\mathcal{H}_X\) we have that
\[
0 = a \hat{s}_\alpha(1) = a.
\]

Given that \(A\) embeds faithfully in each \(\mathcal{H}_X\) we conclude that \(a = 0\).

In order to show that \((\hat{j}, \hat{s})\) is non-degenerate as well it suffices to observe that, if \(a \hat{s}_\alpha = 0\), then
\[
0 = i \times \hat{s}(a \hat{s}_\alpha) = a \hat{s}_\alpha,
\]
which implies that \(a = 0\), by the first part. \(\square\)
Let us write, as usual, \( e_\alpha = v_\alpha v_{\alpha^{-1}} \), where \( \alpha \) is any word in \( G \). Since \( e_\alpha v_\alpha = v_\alpha \) by (2.7.i), we have that for any covariant representation \((\pi, v)\) one has that \( \|\pi(a)v_\alpha\| = \|\pi(a)e_\alpha\| \), so the definition of non-degenerate covariant representations is unchanged if instead we assumed that the map

\[ a \mapsto \pi(a)e_\alpha \]  

(10.3)
is injective.

A very important feature of non-degenerate covariant representations is described next:

**10.4. Proposition.** Let \((\pi, v)\) be a non-degenerate covariant representation of the interaction group \((A, G, V)\) in a C*-algebra \(B\). Then \(\tilde{\pi} \times v\) is isometric on \(\hat{M}_\alpha\) for every word \(\alpha\) in \(G\).

*Proof.* By (10.3) and (4.3.ii) the \(*\)-homomorphism

\[ a \in \mathcal{R}_\alpha \mapsto \pi(a)e_\alpha \in B \]
is injective. Therefore the restriction of \(\tilde{\pi} \times v\) to \(\mathcal{R}_\alpha \hat{e}_\alpha\) is also injective. Since \(\mathcal{R}_\alpha \hat{e}_\alpha = \hat{e}_\alpha \mathcal{A}\hat{e}_\alpha\) is a full hereditary subalgebra of \(\hat{K}_\alpha^0 = \mathcal{A}\hat{e}_\alpha\mathcal{A}\), the restriction of \(\tilde{\pi} \times v\) to \(\hat{K}_\alpha^0\) is injective, and hence isometric. For every \(m \in \hat{M}_\alpha\), notice that \(mm^* \in \hat{K}_\alpha^0\) by (4.4.vi), hence

\[ \|m\|^2 = \|mm^*\| = \|\tilde{\pi} \times v(mm^*)\| = \|\tilde{\pi} \times v(m)\|^2, \]

concluding the proof. \(\Box\)

We thank the referee for suggesting the above proof which is much shorter and elegant than our original proof.

We thus obtain the following crucial faithfulness result:

**10.5. Proposition.** Let \((\pi, v)\) be a non-degenerate strongly covariant representation of \((A, G, V)\) in a C*-algebra \(B\). Then, considering the grading \(\{C_g\}_g \in G\) of \(A \rtimes \nu G\) described in (7.2), one has that \(\tilde{\pi} \times v\) is isometric on \(C_g\), for every \(g \in G\).

*Proof.* We shall begin by proving that \(\tilde{\pi} \times v\) is isometric on \(C_1\). Observe that, by definition,

\[ C_1 = \sum_{\hat{\alpha} = 1} \mathcal{Z}_\alpha. \]

Since \(\mathcal{Z}^X\) is the closed sum of the \(\mathcal{Z}_\alpha\) for all words \(\alpha\) for which \(\hat{\alpha} = 1\) and \(\mu(\alpha^{-1}) \subseteq X\), one has that \(C_1\) is the inductive limit of the \(\mathcal{Z}^X\) for all finite subsets \(X\) of \(G\). Recalling that \(\mathcal{Z}^X\) is a C*-algebra by (4.7), it is therefore enough to prove that \(\tilde{\pi} \times v\) is injective, and hence isometric, on every \(\mathcal{Z}^X\). We therefore suppose that \(k \in \mathcal{Z}^X\) is such that \(\tilde{\pi} \times v(k) = 0\).

Assuming without loss of generality that \(1 \in X\), write \(X = \{x_0 = 1, x_1, \ldots, x_n\}\) and consider the word \(\alpha = (g_1, \ldots, g_n)\), where \(g_i = x_ix_{i-1}^{-1}\), for \(i = 1, \ldots, n\). Clearly \(\mu(\alpha) = X\), so that \(\mathcal{Z}^X = \mathcal{K}_\alpha\) as in (4.8).
Regarding the quotient map \( q : T(A,G,V) \to A \rtimes_v G \), it is elementary to prove that \( q(\hat{\mathcal{K}}_\alpha) = \mathcal{K}_\alpha \). So, there exists \( \hat{k} \in \hat{\mathcal{K}}_\alpha \) such that \( q(\hat{k}) = k \). For every \( \hat{m} \in \hat{\mathcal{M}}_\alpha \), observe that
\[
\pi \times v(\hat{k}\hat{m}) = \pi \times v(\hat{m}q(\hat{k})) = \pi \times v(q(\hat{m}))(\pi \times v(q(\hat{m}))) = 0.
\]

By (4.9) we have that \( \hat{k}\hat{m} \in \hat{\mathcal{M}}_\alpha \) and by (10.4) \( \pi \times v \) is isometric on \( \hat{\mathcal{M}}_\alpha \). It follows that \( \hat{k} = 0 \), and hence that \( \hat{k} \) is an \( \alpha \)-redundancy. Therefore \( k = q(\hat{k}) = 0 \). This proves that \( \pi \times v \) is isometric on \( C_1 \).

Given any \( g \in G \) and \( z \in C_g \) notice that \( z^*z \in C_1 \). So
\[
\|\pi \times v(z)\|^2 = \|\pi \times v(z)^* \pi \times v(z)\| = \|\pi \times v(z^*z)\| = \|z^*z\| = \|z\|^2.
\]

For example we have:

**10.6. Corollary.** Let \( (i, \hat{s}) \) be the regular covariant representation of (9.5). Then \( i \times \hat{s} \) is injective on every \( C_g \).

**Proof.** Follows immediately from (10.5) once we realize that \( (i, \hat{s}) \) is non-degenerate (10.2), and strongly covariant (9.8).

We now want to consider the question of faithfulness of \( \pi \times v \). Observe, however that the hypotheses of (10.5) are not enough to guarantee that \( \pi \times v \) is faithful.

For example, if \( V_g \) is the identity map on \( A \) for every \( g \) in \( G \), then one could conceive of a covariant representation \( (\pi, v) \) in which \( \pi \) is any faithful representation of \( A \) in \( \mathcal{B}(H) \) and \( v_g \equiv 1 \). It is easy to show that this covariant representation satisfies the hypotheses of (10.5) but \( \pi \times v \) is not faithful. To see this notice that, on the one hand, \( \pi \times v(s_g) = v_g = 1 \), but \( s_g \neq 1 \) for \( g \neq 1 \).

To see that in fact \( s_g \neq 1 \), consider the amplification \( (\pi', v') \) of \( (\pi, v) \) described in (7.1). Then \( \pi' \times v'(s_g) = 1 \otimes u_g \), so \( s_g \) could not possibly be equal to 1.

We must therefore give up on the hopes that \( \pi \times v \) be faithful so we will consider its amplification instead. But first let us prove a technical result on graded algebras.

**10.7. Lemma.** Let \( A = \bigoplus_{g \in G} A_g \) and \( B = \bigoplus_{g \in G} B_g \) be topologically graded \( C^* \)-algebras and let \( \varphi : A \to B \) be a graded \( \ast \)-homomorphism (meaning that \( \varphi(A_g) \subseteq B_g \), for all \( g \) in \( G \)) which is injective on \( A_1 \). If \( G \) is amenable then \( \varphi \) is injective.

**Proof.** Let us view \( A = \{ A_g \}_{g \in G} \) and \( B = \{ B_g \}_{g \in G} \) as Fell bundles in the obvious way. Since \( G \) is amenable one has that \( A \) is an amenable Fell bundle by [5: 4.7]. Employing [5 : 4.2] we conclude that \( A \) is isomorphic to the reduced (or full) cross sectional \( C^* \)-algebra \( C^*_r(A) \).

From [5: 2.9 and 2.12] it follows that there exists a faithful conditional expectation
\[
E : A \to A_1
\]
which vanishes on \( A_g \), for all \( g \neq 1 \). Denote by \( F : B \to B_1 \) the corresponding conditional expectation. It is easy to see that for all \( a \) in \( A \) one has that
\[
\varphi(E(a)) = F(\varphi(a)).
\]
If $a \in A$ is such that $\varphi(a) = 0$, then

$$0 = F(\varphi(a^*a)) = \varphi(E(a^*a)).$$

Since $E(a^*a) \in A_1$, and since $\varphi$ is injective on $A_1$ by hypothesis, we conclude that $E(a^*a) = 0$, and hence that $a = 0$ because $E$ is faithful.

10.8. **Theorem.** Assume that $G$ is an amenable group and let $(\pi, v)$ be a non-degenerate strongly covariant representation of $(A, G, V)$ in a $C^*$-algebra $B$. Then $\pi' \times v'$ is injective, where $(\pi', v')$ is the amplification of $(\pi, v)$.

**Proof.** Both $A \rtimes_v G$ and $B \otimes C^*(G)$ are graded algebras and it is easy to see that $\pi' \times v'$ is a graded $*$-homomorphism.

Observe that for every $z \in C_1$ one has that

$$\pi' \times v'(z) = \pi \times v(z) \otimes 1,$$

so $\pi' \times v'$ is injective on $C_1$ by (10.5). The conclusion then follows from (10.7).

Regardless of whether or not $G$ is amenable it is possible to prove that $A \rtimes_v G$ is isomorphic to the full cross sectional $C^*$-algebra of the Fell bundle $C$, while $B \otimes C^*(G)$ is the full cross sectional $C^*$-algebra of the trivial Fell bundle $B \times G$. Moreover it is easy to prove that $\pi' \times v'$ is isometric on each fiber of $C$. This should be enough to prove that $\pi' \times v'$ is injective but, without the hypothesis that $G$ is amenable, I have not been able to find a proof to support this claim! The missing argument is: does a fiberwise isometric Fell bundle homomorphism induces an isomorphism between the full cross sectional $C^*$-algebras? The answer is unfortunately negative (see e.g. [14: 3.2]).

Since $C^*(B)$ is already graded we also have:

10.9. **Theorem.** Assume that $G$ is amenable and let $(i, \hat{s})$ be the regular covariant representation. Then $i \times \hat{s}$ is injective.

**Proof.** Follows from (10.7) and (10.6) as in (10.8).

11. **Invariant states.**
As before we fix a faithful interaction group $(A, G, V)$. Assuming the existence of a faithful invariant state we will give a very concrete model for $A \rtimes_v G$.

11.1. **Definition.** A state $\varphi$ on $A$ is said to be $V$-invariant if for every $g$ in $G$ one has that $\varphi \circ V_g = \varphi$.

From now on we fix a $V$-invariant state $\varphi$ and we will let $\pi$ be the GNS representation of $A$ associated with $\varphi$. The representation space will be denoted by $H$ and the cyclic vector will be denoted by $\xi$. 

11.2. Proposition. For every \( g \in G \) there exists a bounded linear operator \( v_g \) in \( \mathcal{B}(H) \) such that
\[
v_g(\pi(a)\xi) = \pi(V_g(a))\xi, \quad \forall a \in A.
\]
In addition, for every \( g \in G \) one has that \( v_g^* = v_{g^{-1}} \).

Proof. Observe that for every \( a \in A \) we have
\[
\|\pi(V_g(a))\xi\|^2 = \varphi(V_g(a^*V_g(a))) \leq \varphi(V_g(a^*)a) = \varphi(a^*) = \|\pi(a)\xi\|^2.
\]
Therefore the correspondence
\[
\pi(a)\xi \mapsto \pi(V_g(a))\xi
\]
extends to a bounded linear map \( v_g \) on \( H \).

In order to prove the last assertion observe that, for every \( a, b \in A \), one has
\[
\langle v_g(\pi(a)\xi), \pi(b)\xi \rangle = \langle \pi(V_g(a))\xi, \pi(b)\xi \rangle = \varphi(b^*V_g(a)) = \varphi(V_gV_{g^{-1}}(b^*)V_gV_{g^{-1}}V_g(a)) = \varphi(V_gV_{g^{-1}}(b^*)V_{g^{-1}}(b^*)a) = \langle \pi(a)\xi, \pi(V_{g^{-1}}(b))\xi \rangle = \langle \pi(a)\xi, v_{g^{-1}}(\pi(b)\xi) \rangle,
\]
therefore proving that \( v_g^* = v_{g^{-1}} \). \( \square \)

We should remark that \( v_g \) fixes the cyclic vector \( \xi \) for all \( g \) since
\[
v_g(\xi) = v_g(\pi(1)\xi) = \pi(V_g(1))\xi = \pi(1)\xi = \xi. \tag{11.3}
\]

11.4. Proposition. Viewing \( v \) as a map from \( G \) to \( \mathcal{B}(H) \) one has that \( v \) is a \( * \)-partial representation. Moreover the pair \( (\pi, v) \) is a strongly covariant representation of \( (A, G, V) \) in \( H \).

Proof. For every \( g \) and \( h \) in \( G \) and every \( a \in A \) we have
\[
v_{g^{-1}}v_gv_h(\pi(a)\xi) = \pi(V_{g^{-1}}V_gV_h(a))\xi = \pi(V_{g^{-1}}V_g(a))\xi = v_{g^{-1}}v_g(\pi(a)\xi),
\]
so that \( v_{g^{-1}}v_gv_h = v_{g^{-1}}v_{gh} \). This proves that \( v \) is in fact a partial representation. Given \( a, b \in A \) we have
\[
v_g\pi(a)v_{g^{-1}}\pi(b)\xi = \pi(V_g(a)V_{g^{-1}}(b))\xi = \pi(V_g(a)V_{g^{-1}}(b))\xi = \pi(V_g(a))\pi(V_{g^{-1}}(b))\xi = \pi(V_g(a)v_{g^{-1}}\pi(b)\xi).
\]
This shows that \( v_g\pi(a)v_{g^{-1}} = \pi(V_g(a))v_gv_{g^{-1}} \), and hence that \( (\pi, v) \) is a covariant representation.

In order to prove that \( (\pi, v) \) is strongly covariant let \( \alpha \) be a word in \( G \) and let \( k \) be an \( \alpha \)-redundancy. Then, for every \( b \in A \) we have that \( kb\hat{\alpha} = 0 \), whence
\[
0 = \pi\times v(kb\hat{\alpha})\big|_\xi = \pi\times v(k)\pi(b)v_\alpha\big|_\xi = \pi\times v(k)\pi(b)\big|_\xi.
\]
Since \( \xi \) is cyclic we conclude that \( \pi\times v(k) = 0 \). \( \square \)
We now want to consider the question as to whether $\pi \times v$ is an isomorphism onto its range. Clearly there is no hope for this to be true unless $\pi$ is faithful. We therefore assume, from now on, that $\varphi$ is a faithful\footnote{One does not need $\varphi$ to be faithful in order for $\pi$ to be faithful. It would be enough to assume that $\varphi$ satisfies $(\forall x, y \in A : \varphi(xay) = 0) \Rightarrow a = 0$. However we will soon need $\varphi$ to be faithful for other reasons.} state, which is to say that, for every $a$ in $A$

$$\varphi(a^*a) = 0 \Rightarrow a = 0.$$ 

Recall that we are always working under the standing hypothesis according to which $(A, G, V)$ is a faithful interaction group. However, even if this was not assumed in advance, the existence of a faithful invariant state would imply it:

**11.5. Proposition.** Let $(A, G, V)$ be an interaction group which we exceptionally do not suppose to be faithful. If there exists a faithful $V$-invariant state $\varphi$ then $(A, G, V)$ must be faithful.

**Proof.** Let $g \in G$ and $a \in A$ be such that $E_g(a^*a) = 0$. Then

$$0 = V_{g^{-1}}E_g(a^*a) = V_{g^{-1}}V_g V_{g^{-1}}(a^*a) = V_{g^{-1}}(a^*a),$$

so, employing the covariant representation $(\pi, v)$ of (11.4) we would have that

$$0 = \langle \pi(V_{g^{-1}}(a^*a))v_{g^{-1}}v_g \xi, \xi \rangle = \langle v_{g^{-1}}\pi(a^*a)v_g(\xi), \xi \rangle =$$

$$= \langle \pi(a^*a)v_g(\xi), v_g(\xi) \rangle = \langle \pi(a^*a)(\xi), \xi \rangle = \varphi(a^*a),$$

and the faithfulness of $\varphi$ implies that $a = 0$. \hfill \square

Another non-degeneracy property which follows from the existence of a faithful invariant state is as follows:

**11.6. Proposition.** If $\varphi$ is faithful then $(\pi, v)$ is non-degenerate.

**Proof.** Let $\alpha$ be a word in $G$ and let $a \in A$ be such that $\pi(a)v_\alpha = 0$. Then,

$$0 = \|\pi(a)v_\alpha\|^2 = \|\pi(a)\|^2 = \langle \pi(a)\xi, \pi(a)\xi \rangle = \varphi(a^*a),$$

which implies that $a = 0$. \hfill \square

The following important result shows that the abstractly defined crossed product algebra has a very concrete description as an algebra of operators in the presence of a faithful invariant state:

**11.7. Theorem.** Let $(A, G, V)$ be an interaction group and let $\varphi$ be a faithful $V$-invariant state. Let $(\pi, v)$ be the strongly covariant representation obtained from $\varphi$ as in (11.4) and let $(\pi', v')$ be its amplification. If $G$ is amenable then $\pi' \times v'$ is injective. Therefore $A \rtimes_v G$ is isomorphic to the closed $*$-subalgebra of $B(H \otimes \ell_2(G))$ generated by

$$\{\pi(a) \otimes 1 : a \in A\} \cup \{v_g \otimes \lambda_g : g \in G\},$$

where $\lambda$ is the left regular representation of $G$. 
Proof. That $\pi' \times v'$ is injective follows from (10.8) since $(\pi, v)$ is strongly covariant (11.4) and non-degenerate (11.6).

Moreover, since $G$ is amenable, and hence $C^*(G)$ is nuclear, $\mathcal{B}(H) \otimes C^*(G)$ embedds faithfully in $\mathcal{B}(H \otimes \ell_2(G))$ in such a way that $T \otimes u_g$ is mapped to $T \otimes \lambda_g$, for all $T \in \mathcal{B}(H)$ and $g \in G$. \hfill \Box

12. Semigroups of endomorphisms.

In this section we will discuss the relationship between our notion of interaction groups and semigroups of endomorphisms.

We will suppose throughout that $P$ is a subsemigroup of the group $G$. In order to simplify certain technical points we will suppose that $G = P^{-1}P$.

We will also suppose that $A$ is a unital C*-algebra and that $\alpha$ is a semigroup homomorphism from $P$ to the semigroup of all unital *-endomorphisms of $A$. Some will also call $\alpha$ an action by endomorphisms of $P$ on $A$.

Several proposals have appeared in the literature for the notion of crossed product of $A$ by $P$ under $\alpha$.

Under the assumption that there exists a faithful $\alpha$-invariant state $\varphi$ on $A$, an assumption that will be enforced henceforth, the following is perhaps another reasonable proposal: let $(\pi, H, \xi)$ be the GNS representation of $A$ relative to $\varphi$ and, for each $g \in P$, let $v_g$ be the unique isometry on $H$ such that

$$v_g(\pi(a)\xi) = \pi(\alpha_g(a))\xi, \quad \forall a \in A. \tag{12.1}$$

One could then define the crossed product of $A$ by $P$ under $\alpha$ to be the closed *-subalgebra $A \rtimes_\alpha P$ of $\mathcal{B}(H) \otimes C^*(G)$ generated by

$$\{\pi(a) \otimes 1 : a \in A\} \cup \{v_g \otimes u_g : g \in P\},$$

where $u : G \to C^*(G)$ is the universal representation of $G$.

It is not yet clear to what extent does $A \rtimes_\alpha P$ depend on the faithful invariant state $\varphi$ but, based on [10: 6.1] this is perhaps a sensible definition.

Let us now make the main assumption relating our theory with the theory of endomorphism crossed products:

12.2. Hypothesis. We will suppose, from now on, that there exists an interaction group $(A, G, V)$, which also leaves $\varphi$ invariant, and such that

$$V_g = \alpha_g, \quad \forall g \in P.$$

Observe that $(A, G, V)$ must necessarily be faithful by (11.5).

The following result says, among other things that, if $V$ exists, it can somehow be dug out from the triple $(A, \alpha, \varphi)$:
12.3. Theorem. If $V'$ is another interaction group satisfying (12.2), then $V = V'$. In addition, if $G$ is amenable, then $A \rtimes \gamma G$ is isomorphic to $A \rtimes \alpha P$, as defined above.

Proof. Let $(\pi, v)$ and $(\pi, v')$ be the covariant representations on $H$ relative to $V$ and $V'$, respectively, as in (11.4).

Obviously $v_g = v'_g$, for all $g$ in $P$, as these agree with the $v_g$ defined in (12.1). Since $v_{g^{-1}} = v'_g$, we also conclude that $v_g = v'_g$, for all $g$ in $P^{-1}$.

For every $g$ in $P$ we know that $v_g$ is an isometry, and hence left-invertible. So we conclude by (2.4) that $v_g = v'_g$, for all $g$ in $G$. Next notice that, for all $g \in G$, and $a \in A$,

$$v_g \pi(a) v_{g^{-1}} v_g = \pi(V_g(a)) v_{g^{-1}} v_g = \pi(V_g(a)) v_g = \pi(V'_g(a)) v_{g^{-1}} v_g = \pi(V'_g(a)) v_g.$$

It follows that $\pi(V'_g(a) - V_g(a)) v_g = 0$. But since $(\pi, v)$ is non-degenerate by (11.6), we have that $V'_g(a) = V_g(a)$.

Refering to the last part of the statement, and using (11.7), we must compare the subalgebras of $B(H) \otimes C^*(G)$ generated, on the one hand by

$$\{\pi(a) \otimes 1 : a \in A\} \cup \{v_g \otimes \lambda_g : g \in G\},$$

and, on the other by

$$\{\pi(a) \otimes 1 : a \in A\} \cup \{v_g \otimes \lambda_g : g \in P\}.$$

Observe that we are identifying $C^*(G)$ with the reduced $C^*$-algebra of $G$, and the universal representation $\upsilon$ with the regular representation $\lambda$, since $G$ is supposed to be amenable.

Writing $w_g := v_g \otimes \lambda_g$, the problem boils down to whether or not all of the $w_g$ may be generated by those with $g \in P$.

Observing that $w$ is clearly a partial representation, and that $w_g$ is an isometry for all $g$ in $P$, the conclusion follows immediately from the first part of (2.4), since we are assuming that $G = P^{-1} P$.

As already observed, the result above indicates that, in the presence of a faithful invariant state, there is at most one way to extend the semigroup action to an interaction group leaving the state invariant.

The extension question is also relevant in the absence of invariant states:

12.4. Question. Given a semigroup action $\alpha$ of $P$ on a $C^*$-algebra $A$, when is there an interaction group $V$ such that $V_g = \alpha_g$, for all $g$ in $P$?

After the present paper circulated as a preprint we have found [9: Section 14] an example giving a partial (negative) answer to this question, based on the notion of cellular automata.

To put matters in perspective let us think of the simplest possible group-subsemigroup pair, that is, $(\mathbb{Z}, \mathbb{N})$. To give an action of $\mathbb{N}$ on $A$ is clearly equivalent to giving a single endomorphism $\alpha$ of $A$, which we will suppose injective and unit preserving for simplicity. In this case the action may be clearly recovered by iterating $\alpha$. 

12.5. Proposition. Let $\alpha$ be a unital injective endomorphism of $A$.

(i) If there exists an interaction group $V$ such that $V_1 = \alpha$, then $V_1V_{-1}$ is a conditional expectation onto the range of $\alpha$.

(ii) If $E$ is a conditional expectation onto the range of $\alpha$, then the operator $\mathcal{L} := \alpha^{-1} \circ E$ is a transfer operator for $\alpha$ [7: 2.1], and

$$ V_n = \begin{cases} 
\alpha^n, & \text{if } n \geq 0, \\
\mathcal{L}^{-n}, & \text{if } n < 0.
\end{cases} $$

defines an interaction group such that $V_1 = \alpha$.

(iii) Under the hypothesis of (i) one has that $V$ is necessarily given by the expression in (ii) with $E = V_1V_{-1}$.

Proof. (i) follows directly from (3.2). Given $E$ as in (ii) one has by [7: 2.6] that $\mathcal{L} := \alpha^{-1} \circ E$ is transfer operator for $\alpha$. Observe that it makes sense of speaking of $\alpha^{-1}$ here since $\alpha$ is a bijection onto its range, and since the range of $E$ coincides with that of $\alpha$.

Defining $V$ as in the statement one can now prove (a tedious but easy task) that $V$ is an interaction group. Clearly $V_1 = \alpha$.

Referring to (iii) let $V$ be given by (ii) and let $V'$ be another interaction group such that $V'_1 = \alpha$, and

$$ V'_1V'_{-1} = E = V_1V_{-1}. $$

We then have that

$$ \alpha V'_{-1}V'_1 = V'_1V'_{-1}V'_1 = V'_1 = \alpha. $$

Given that $\alpha$ is injective we conclude that $V'_{-1}V'_1$ is the identity. With this observe that

$$ V'_{-1} = V'_{-1}V'_1V'_{-1} = V'_1E = V'_1V_1V_{-1} = V'_1V'_{-1} = V_{-1}. $$

We therefore conclude that $V'_g = V_g$ for all $g \in \{1, -1\}$. Since $V'_1$ is left-invertible and $V'_{-1}$ is right-invertible, we can use (2.3.iv) to prove by induction that

$$ V'_n = (V'_1)^n, \quad \forall n \geq 0, $$

and similarly, using (2.3.iii), deduce that

$$ V'_{-n} = (V'_{-1})^{-n}, \quad \forall n < 0. $$

Therefore $V = V'$. \qed

Thus we see that the extension problem for $(\mathbb{Z}, \mathbb{N})$ has a solution if and only if there exists a conditional expectation onto the range of $\alpha$, and that the collection of all possible solutions is parametrized by these conditional expectations.
13. Larsen’s crossed products.

If we consider more general group-subsemigroup pairs \((G, P)\) the situation may become a lot more complicated. Let us consider, for example, the recent work by Larsen [16] on crossed products by abelian semigroups via transfer operators in which the initial data is a triple \((P, \alpha, \ell)\), where \(P\) is an abelian semigroup, \(\alpha\) is an action of \(P\) on the (non necessarily unital) \(C^*\)-algebra \(A\), and \(\ell\) is an action by transfer operators.

We would actually like to consider a slightly different situation, more general in some aspects, and less general in others: we will suppose that \(A\) is a unital \(C^*\)-algebra (with this we wish to avoid the question of extendibility treated by Larsen), \(P\) is a subsemigroup of the non necessarily abelian group \(G\) such that \(1 \in P\), and \(\alpha\) is an action of \(P\) on \(A\) by means of unital injective endomorphisms. Observe that \(\alpha_1\) is necessarily the identity endomorphism.

We will moreover suppose that we are given a map (action by transfer operators)

\[
\ell : P \to B(A)
\]

such that for every \(g\) in \(P\), \(\ell_g\) is a transfer operator for \(\alpha_g\), and

\[
\ell_g \ell_h = \ell_{hg}, \quad \forall g, h \in P.
\]

Recall that to say that \(\ell_g\) is a transfer operator for \(\alpha_g\) means that \(\ell_g\) is a positive operator on \(A\) such that

\[
\ell_g(a \alpha_g(b)) = \ell_g(a)b, \quad \forall a, b \in G.
\]  \((13.1)\)

In order to avoid trivialities (such as \(\ell_g \equiv 0\)) we will assume that

\[
\ell_g(1) = 1, \quad \forall g \in G.
\]

Plugging \(a = 1\) in \((13.1)\) then implies that

\[
\ell_g \circ \alpha_g = id_A.
\]

In particular \(\ell_1 = id_A\), as well.

Question (12.4) may then be modified to account for the \(\ell_g\) as follows:

13.2. Question. Given \((P, \alpha, \ell)\) as above, is there an interaction group \(V\) such that \(\alpha_g = V_g\), and \(\ell_g = V_{g^{-1}}\), for all \(g\) in \(P\)?

The following is a partial answer:

13.3. Proposition. Suppose that \(G = P^{-1}P\). Then the above question has an affirmative answer if and only if \(\alpha_g \ell_g\) commutes with \(\alpha_h \ell_h\), for every \(g, h \in P\).
Proof. In case $V$ is an interaction group satisfying the requirements of (13.2) we have for all $g$ in $P$ that

$$E_g := V_g V_{g^{-1}} = \alpha_g \ell_g,$$

so the conclusion follows from (2.2.iii). Conversely, given $g$ in $G$ write $g = x^{-1}y$, with $x, y \in P$, and define

$$V_g = \ell_x \alpha_y.$$ 

We claim that $V_g$ does not depend on the particular choice of $x$ and $y$. In order to prove this suppose that $g = z^{-1}w$, with $z, w \in P$, as well. We must then prove that $\ell_x \alpha_y = \ell_z \alpha_w$.

As a first case let us consider the situation in which $z = ux$, for some $u \in P$. Obviously $w = zg = uxx^{-1}y = uy$. So

$$\ell_z \alpha_w = \ell_{ux} \alpha_{uy} = \ell_x \ell_u \alpha_u \alpha_y = \ell_x \alpha_y,$$

because $\ell_u \alpha_u$ is the identity. In the general case pick $u, v \in P$ such that $wy^{-1} = v^{-1}u$, so that $uy = vz$. Observe that $ux = vw^{-1}x = vwg^{-1} = vwv^{-1}z = vz$. So, by the special case already proved, we have

$$\ell_x \alpha_y = \ell_{ux} \alpha_{uy} = \ell_{uz} \alpha_{vw} = \ell_z \alpha_w.$$

Therefore the above $V_g$ is well defined. It is somewhat curious that this is so even without having assumed that $P \cap P^{-1} = \{1\}$!

We next wish to prove that $V$ is a partial representation of $G$ on $A$. Obviously $V_1 = \ell_1 \alpha_1 = \text{id}$.

Given $g, h \in G$ write $g = x^{-1}y$ and $h = z^{-1}w$, with $x, y, z, w \in P$. Pick $u, v \in P$ such that $zy^{-1} = v^{-1}u$ and notice that one has

$$uy = vz.$$

Replacing $(x, y)$ by $(ux, uy)$, and $(z, w)$ by $(vz, vw)$, we may then assume that $y = z$. With an eye on (2.1.ii) we compute

$$V_g V_h V_{h^{-1}} = \ell_x \alpha_y \ell_y \alpha_w \ell_w \alpha_y = \ell_x \alpha_w \ell_w \alpha_y \ell_y \alpha_y = \ell_x \alpha_w \ell_w \alpha_y = V_{gh} V_{h^{-1}}.$$

Speaking of (2.1.iii) we have

$$V_{g^{-1}} V_g V_h = \ell_y \alpha_x \ell_x \alpha_y \ell_y \alpha_w = \ell_y \alpha_y \ell_y \alpha_x \ell_x \alpha_w = \ell_y \alpha_x \ell_x \alpha_w = V_{g^{-1}} V_{gh}.$$ 

Our hypotheses clearly imply (3.1.i-ii). In order to prove (3.1.iii) let $g = x^{-1}y$, with $x, y \in P$. Take $a$ in $A$ and $b$ in the range of $V_{g^{-1}}$, say $b = V_{g^{-1}}(c)$, where $c \in A$. Then

$$V_g(ab) = V_g(a V_{g^{-1}}(c)) = \ell_x \alpha_y (a \ell_y \alpha_x (c)) = \ell_x (\alpha_y(a) \alpha_y \ell_y \alpha_x (c)) =$$

$$= \ell_x (\alpha_y(a) \alpha_y \ell_y \alpha_x \ell_x \alpha_x (c)) = \ell_x (\alpha_y(a) \alpha_y \ell_x \alpha_y \ell_y \alpha_x (c)) =$$

$$= \ell_x (\alpha_y(a) \ell_x \alpha_y \ell_y \alpha_x (c) = V_g(a) V_g(V_{g^{-1}}(c)) = V_g(a)V_g(b).$$

The case in which $a$, instead of $b$, lies in the range of $V_{g^{-1}}$ follows by taking adjoints. □
We refer the reader to the next section for an example which shows that the hypotheses of (13.3) are not always satisfied.

A situation in which the above result applies is for \textit{linearly ordered groups}:

13.4. Proposition. Let \((P, \alpha, \ell)\) be as above and suppose that \(G = P^{-1} \cup P\). Define

\[
V_g = \begin{cases} 
\alpha_g, & \text{if } g \in P, \\
\ell_{g^{-1}}, & \text{if } g^{-1} \in P.
\end{cases}
\]

Then \(V_g\) is well defined and \(V\) is an interaction group.

Proof. We will deduce everything from (13.3). Given \(g, h \in P\) we shall prove that \(\alpha_g \ell_g\) and \(\alpha_h \ell_h\) commute. Suppose, without loss of generality, that \(u := hg^{-1} \in P\), so that \(h = ug\). Then

\[
\alpha_h \ell_h = \alpha_g \ell_{gu} = \alpha_g \alpha_u \ell_u \ell_g.
\]

Therefore

\[
\alpha_h \ell_h \alpha_g \ell_g = \alpha_g \alpha_u \ell_u \ell_g \alpha_g \ell_g = \alpha_g \alpha_u \ell_u \ell_g = \alpha_g \ell_g \alpha_g \alpha_u \ell_u \ell_g = \alpha_g \ell_g \alpha_h \alpha_h.
\]

This verifies the hypotheses of (13.3) and hence the conclusion follows. \(\Box\)

We therefore see that, under the condition that \(G = P^{-1} \cup P\), the dynamical systems of [16] are closely related to interaction groups.

The reader acquainted with Larsen’s paper is undoubtedly curious as to what is the precise relationship between the crossed product defined in [16: 1.3] and our notion of crossed product, whenever (13.2) has an affirmative answer. However, while there may be some close relationship between \(T(A, G, V)\) and the Toeplitz algebra defined in [16: 1.1] it seems that our notion of redundancy is significantly different from that defined in [16: 1.2], so it is unlikely that the crossed products will coincide.

In addition, given the close relationship between the crossed product construction studied in [16] and the Cuntz-Pimsner algebras of product systems (see [16: 3.3]), we also feel that there is a big discrepancy between our notion of strong covariance and the notion of Cuntz–Pimsner covariant representations given in [12: 2.5]. It is therefore unlikely that our crossed product construction could be derived from a product system.
14. Example.

In this section we would like to show an example of a dynamical system \((A, S, \alpha, L)\), as defined by Larsen in \([16: 1]\), for which the hypotheses of (13.3) do not hold. We thank Vaughan Jones for suggesting that such an example could be found by the process of second quantization, as described below.

Having failed to find an example in which \(A\) is a commutative algebra I began wondering if it exists!? See \([9: \text{Section 14}]\) for a related example.

Let \(H\) be a separable infinite dimensional Hilbert space. The CAR algebra of \(H\), denoted \(A(H)\), is the universal C*-algebra generated by the annihilation operators \(\{a(f)\}_{f \in H}\) subject to the canonical anti-commutation relations:

(i) \(a(f)a(g) + a(g)a(f) = 0,\)
(ii) \(a(f)a(g)^* + a(g)^*a(f) = \langle f, g \rangle 1,\)
(iii) \(a(f + \lambda g) = a(f) + \lambda a(g),\)

for all \(f\) and \(g\) in \(H\), and \(\lambda \in \mathbb{C}\). We refer the reader to \([1]\) for a detailed treatment.

In order to fix our notation let us briefly describe the standard representation of \(A(H)\) on the Fermi-Fock space. Denote by \(F(H)\) the full Fock space on \(H\), namely

\[
F(H) = \mathbb{C} \oplus H \oplus H \otimes 2 \oplus \ldots \oplus H \otimes n \oplus \ldots
\]

and let \(P_-\) be the projection defined on each \(H \otimes n\) by

\[
P_-(f_1 \otimes f_2 \otimes \cdots \otimes f_n) = (n!)^{-1} \sum_\pi \varepsilon_\pi f_{\pi_1} \otimes f_{\pi_2} \otimes \cdots \otimes f_{\pi_n},
\]

where the sum ranges over all permutations \(\pi\) of the finite set \(\{1, \ldots, n\}\) and \(\varepsilon_\pi\) is the sign of \(\pi\). The Fermi-Fock space \(F_-(H)\) is defined to be the range of \(P_-\).

The representation of \(A(H)\) on \(F_-(H)\) we referred to is defined as follows: denoting

\[
f_1 \wedge \ldots \wedge f_n = P_-(f_1 \otimes f_2 \otimes \cdots \otimes f_n),
\]

one has that

\[
a^*(f)(f_1 \wedge \ldots \wedge f_n) = (n + 1)^{1/2} f \wedge f_1 \wedge \ldots \wedge f_n.
\]

In order to obtain an expression for inner-products in \(F(H)\) notice that, given

\(f_1, \ldots, f_n, g_1, \ldots, g_n \in H\) we have

\[
\langle f_1 \wedge \ldots \wedge f_n, g_1 \wedge \ldots \wedge g_n \rangle = \langle P_-(f_1 \otimes \cdots \otimes f_n), P_-(g_1 \otimes \cdots \otimes g_n) \rangle =
\]

\[
= \langle P_-(f_1 \otimes \cdots \otimes f_n), g_1 \otimes \cdots \otimes g_n \rangle = (n!)^{-1} \sum_\pi \varepsilon_\pi \langle f_{\pi_1} \otimes \cdots \otimes f_{\pi_n}, g_1 \otimes \cdots \otimes g_n \rangle =
\]

\[
= (n!)^{-1} \sum_\pi \varepsilon_\pi \langle f_{\pi_1}, g_1 \rangle \cdots \langle f_{\pi_n}, g_n \rangle.
\]

Thus, if we denote by \(\langle f, g \rangle\) the \(n \times n\) complex matrix given by \(\langle f, g \rangle_{ij} = \langle f_i, g_j \rangle\), we see that

\[
\langle f_1 \wedge \ldots \wedge f_n, g_1 \wedge \ldots \wedge g_n \rangle = (n!)^{-1} \det((f, g)).
\]

From now on we fix a closed subspace \(K\) of \(H\). We will view \(F_-(K)\) as a closed subspace of \(F_-(H)\) in the natural way.
14.1. Lemma. Let \( e \) denote the orthogonal projection from \( \mathcal{H} \) to \( \mathcal{K} \) and let \( \mathcal{E} \) be the orthogonal projection from \( \mathcal{F}(\mathcal{H}) \) onto \( \mathcal{F}(\mathcal{K}) \). Then

\[
\mathcal{E}(f_1 \wedge \ldots \wedge f_n) = e(f_1) \wedge \ldots \wedge e(f_n).
\]

Proof. Given \( f_1, \ldots, f_n \in \mathcal{H} \) we claim that

\[
y := f_1 \wedge \ldots \wedge f_n - e(f_1) \wedge \ldots \wedge e(f_n) \in \mathcal{F}(\mathcal{K})^\perp.
\]

To see this let \( g_1, \ldots, g_n \in \mathcal{K} \) and notice that

\[
\langle y, g_1 \wedge \ldots \wedge g_n \rangle = \langle f_1 \wedge \ldots \wedge f_n, g_1 \wedge \ldots \wedge g_n \rangle - \langle e(f_1) \wedge \ldots \wedge e(f_n), g_1 \wedge \ldots \wedge g_n \rangle = \det(\langle f, g \rangle) - \det(\langle e(f), g \rangle) = \ldots
\]

Since \( \langle e(f_i), g_j \rangle = \langle f_i, e(g_j) \rangle = \langle f_i, g_j \rangle \), we conclude that the above vanishes. The conclusion then follows easily. \( \square \)

14.2. Lemma. If \( f \in \mathcal{H} \) then \( a(f)\mathcal{E} = \mathcal{E}a(e(f)) = a(e(f))\mathcal{E} \).

Proof. We prove instead that \( \mathcal{E}a^*(f) = a^*(e(f))\mathcal{E} = \mathcal{E}a^*(e(f)) \). Given \( f_1, \ldots, f_n \in \mathcal{H} \) we have

\[
\mathcal{E}a^*(f)(f_1 \wedge \ldots \wedge f_n) = (n + 1)^{1/2}\mathcal{E}(f \wedge f_1 \wedge \ldots \wedge f_n) = (n + 1)^{1/2}e(f) \wedge e(f_1) \wedge \ldots \wedge e(f_n) = a^*(e(f))e(f_1) \wedge \ldots \wedge e(f_n) = a^*(e(f))\mathcal{E}(f_1 \wedge \ldots \wedge f_n),
\]

proving that \( \mathcal{E}a^*(f) = a^*(e(f))\mathcal{E} \). The remaining equality follows from this upon replacing \( f \) by \( e(f) \). \( \square \)

It is interesting to observe that this result implies that \( A(\mathcal{K}) \) commutes with \( \mathcal{E} \).

The following is probably a well known result for specialists in second quantization but we were unable to find a reference for it:

14.3. Proposition. There exists a unique bounded linear map \( E : A(\mathcal{H}) \to A(\mathcal{K}) \) such that

\[
E\left(a^*(f_1) \ldots a^*(f_n)a(g_1) \ldots a(g_m)\right) = a^*(e(f_1)) \ldots a^*(e(f_n))a(e(g_1)) \ldots a(e(g_m)),
\]

for every \( f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathcal{H} \). Moreover \( E \) is a conditional expectation onto \( A(\mathcal{K}) \).
Proof. We have by (14.2) that

$$E a^*(f_1) \cdots a^*(f_n) a(g_1) \cdots a(g_m) E = a^*(e(f_1)) \cdots a^*(e(f_n)) E a(e(g_1)) \cdots a(e(g_m)) =$$

$$= a^*(e(f_1)) \cdots a^*(e(f_n)) a(e(g_1)) \cdots a(e(g_m)) E.$$

Since $A(K)$ is faithfully represented on $F_-(K)$, and since $F_-(K)$ is precisely the range of $E$, we conclude that there exists a unique bounded linear map $E : A(H) \to A(K)$ such that

$$E x E = E(x) E, \quad \forall x \in A(H).$$

It is now easy to verify that $E$ satisfies the required conditions. \qed

Let $s$ be an isometry on $H$. By the universal property of $A(H)$ there exists a unique unital $\ast$-endomorphism $\alpha$ of $A(H)$ such that

$$\alpha(a(f)) = a(s(f)), \quad \forall f \in H.$$  \hspace{1cm} (14.4)

Since $A(H)$ is a simple algebra $\alpha$ is necessarily injective. The range of $\alpha$ obviously coincides with $A(K)$, where $K$ is the range of $s$.

As in [7: 2.6] we may produce a transfer operator $L$ for $\alpha$ by setting

$$L = \alpha^{-1} \circ E,$$  \hspace{1cm} (14.5)

where $E$ is as in (14.3). It is elementary to verify that

$$L\left(a^*(f_1) \cdots a^*(f_n) a(g_1) \cdots a(g_m)\right) = a^*(s^*(f_1)) \cdots a^*(s^*(f_n)) a(s^*(g_1)) \cdots a(s^*(g_m)).$$

Let us now get a bit more concrete and consider $H = \ell^2(\mathbb{N})$. Let $s_1$ and $s_2$ be two commuting isometries on $H$ such that $s_1 s_1^*$ and $s_2 s_2^*$ do not commute. In order to construct these in a concrete way let $s_1$ be the unilateral shift on $\ell^2(\mathbb{N})$ and, identifying $\ell^2(\mathbb{N})$ with the Hardy space $H^2$ as usual, let $s_2 = T_\varphi$ be the Toeplitz operator whose symbol is the Blaschke factor

$$\varphi(z) = \frac{z - a}{1 - \overline{a} z}, \quad \forall z \in \mathbb{C},$$

where $0 < |a| < 1$. We leave it for the reader to prove that $s_1$ and $s_2$ are indeed commuting isometries with noncommuting final projections.

Let $\alpha_1$ and $\alpha_2$ be the endomorphisms of $A(H)$ respectively obtained from $s_1$ and $s_2$ as in (14.4). Also let $L_1$ and $L_2$ be the corresponding transfer operators obtained by (14.5).

Observe that the fact that $s_1$ and $s_2$ commute implies that $\alpha_1$ and $\alpha_2$ commute. Obviously $s_1^*$ and $s_2^*$ commute as well which entails the commutativity of $L_1$ and $L_2$.

For every $(n, m) \in \mathbb{N} \times \mathbb{N}$ set

$$\alpha_{(n, m)} = \alpha_1^n \alpha_2^m, \quad \text{and} \quad \ell_{(n, m)} = L_1^n L_2^m.$$
It is now elementary to check that \((A(\mathcal{H}), \mathbb{N} \times \mathbb{N}, \alpha, \ell)\) is a dynamical system as defined by Larsen in [16: 1]. We shall prove however that it does not satisfy the hypothesis of (13.3). In fact, notice that
\[
\alpha_{(1,0)}\left(\ell_{(1,0)}(a(f))\right) = a(s_1s_1^*(f)),
\]
while
\[
\alpha_{(0,1)}\left(\ell_{(0,1)}(a(f))\right) = a(s_2s_2^*(f)),
\]
from which one immediately sees that \(\alpha_{(1,0)}\ell_{(1,0)}\) and \(\alpha_{(0,1)}\ell_{(0,1)}\) do not commute.

Needless to say, there is no interaction group extending \(\alpha\) and \(\ell\), precisely by Theorem (13.3).

15. Appendix.

In this section we would like to give a detailed proof of the statement made in the introduction, according to which one cannot always find an isometry \(S\) in \(O_A\) such that \(\alpha(a) = SaS^*\), for all \(a\) in \(C(K)\), where \(K\) is Markov’s space and \(\alpha\) is the endomorphism of \(C(K)\) induced by the Markov subshift.

To focus on a single counterexample we will let
\[
A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}
\]
in which case \(O_A\) coincides with \(O_2\). We will denote by \(B\) the standard copy of the CAR algebra inside \(O_2\), and by \(D\) the diagonal subalgebra of \(B\), which is isomorphic to \(C(K)\). It is well known that \(D\) is a maximal abelian subalgebra of \(B\).

A crucial tool in our argument below is the fact that \(D\) is also a maximal abelian subalgebra of \(O_2\) (see [4]). This also follows from [18: Proposition II.4.7] given the groupoid description of \(O_2\) [18: Section III.2].

In order to prove the inexistence of an isometry \(S\) in \(O_2\) satisfying (1.1) let us argue by contradiction and hence suppose that such an \(S\) exists. Since \(\alpha(1) = 1\), one must have that \(S\) is unitary.

Letting \(D^+\) be the necessarily abelian subalgebra of \(O_2\) given by
\[
D^+ = S^*DS,
\]
we claim that \(D \subseteq D^+\). In fact, given \(a \in D\) we have
\[
a = S^*SaS^*S = S^*\alpha(a)S \in S^*DS = D^+.
\]

Since \(D\) is maximal abelian in \(O_2\) we conclude that \(D = D^+\). For every \(a \in D\) we then have that \(b := S^*aS \in D\) and
\[
a = SS^*aSS^* = SbS^* = \alpha(b),
\]
from which one would deduce the absurdity that \(\alpha\) is surjective. Thus, no such \(S\) may exist!
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