Hilbert $C^*$-bimodules over commutative $C^*$-algebras and an isomorphism condition for quantum Heisenberg manifolds.

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Abstract

Abstract: A study of Hilbert $C^*$-bimodules over commutative $C^*$-algebras is carried out and used to establish a sufficient condition for two quantum Heisenberg manifolds to be isomorphic.

Introduction. In [AEE], a theory of crossed products of $C^*$-algebras by Hilbert $C^*$-bimodules was introduced and used to describe certain deformations of Heisenberg manifolds constructed by Rieffel (see [Rf4] and [AEE, 3.3]). This deformation consists of a family of $C^*$-algebras, denoted $D^c_{\mu\nu}$, depending on two real parameters $\mu$ and $\nu$, and a positive integer $c$. In case $\mu = \nu = 0$, $D^c_{\mu\nu}$ turns out to be isomorphic to the algebra of continuous functions on the Heisenberg manifold $M^c$.

For K-theoretical reasons [Ab2], $D^c_{\mu\nu}$ and $D^{c'}_{\mu'\nu'}$ cannot be isomorphic unless $c = c'$. It is the main purpose of this work to show that the $C^*$-algebras $D^c_{\mu\nu}$ and $D^{c'}_{\mu'\nu'}$ are isomorphic when $(\mu, \nu)$ and $(\mu', \nu')$ are in the same orbit under the usual action of $GL_2(\mathbb{Z})$ on the torus $T^2$ (here the parameters are

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viewed as running in $T^2$, since $D^c_{\mu\nu}$ and $D^c_{\mu+n,\nu+m}$ are isomorphic for any integers $m$ and $n$).

As indicated above, the quantum Heisenberg manifold $D^c_{\mu\nu}$ may be described as a crossed product of the commutative $C^*$-algebra $C(T^2)$ by a Hilbert $C^*$-bimodule. Motivated by this, we are led to study some special features of Hilbert $C^*$-bimodules over commutative $C^*$-algebras, which are relevant to our purposes.

In Section 1 we consider, for a commutative $C^*$-algebra $A$, two subgroups of its Picard group $\text{Pic}(A)$: the group of automorphisms of $A$ (embedded in $\text{Pic}(A)$ as in [BGR]), and the classical Picard group $\text{CPic}(A)$ (see, for instance, [DG]) consisting of Hilbert line bundles over the spectrum of $A$. Namely, we prove that $\text{Pic}(A)$ is the semidirect product of $\text{CPic}(A)$ by $\text{Aut}(A)$. This result carries over a slightly more general setting, and a similar statement (see Proposition 1.1) holds for Hilbert $C^*$-bimodules that are not full, partial automorphisms playing then the role of $\text{Aut}(A)$. These results provide a tool that enables us to deal with $\text{Pic}(C(T^2))$ in order to prove our isomorphism theorem for quantum Heisenberg manifolds, which is done in Section 2.

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1 The Picard group and the classical Picard group.

**Notation.** Let $A$ be a $C^*$-algebra. If $M$ is a Hilbert $C^*$-bimodule over $A$ (in the sense of [BMS, 1.8]) we denote by $\langle \cdot,\cdot \rangle^M_L$ and $\langle \cdot,\cdot \rangle^M_R$, respectively, the left and right $A$-valued inner products, and drop the superscript whenever the context is clear enough. If $M$ is a left (resp. right) Hilbert $C^*$-module over $A$, we denote by $K(M_A)$ (resp. $K(A_M)$) the $C^*$-algebra of compact operators on $M$. When $M$ is a Hilbert $C^*$-bimodule over $A$ we will view the elements of $\langle M,M \rangle_R$ (resp. $\langle M,M \rangle_L$) as compact operators on the left (resp. right)
module \( M \), as well as elements of \( A \), via the well-known identity:
\[
\langle m, n \rangle_L p = m \langle n, p \rangle_R,
\]
for \( m, n, p \in M \).

The bimodule denoted by \( \tilde{M} \) is the dual bimodule of \( M \), as defined in [Rf1, 6.17].

By an isomorphism of left (resp. right) Hilbert \( C^* \)-modules we mean an isomorphism of left (resp. right) modules that preserves the left (resp. right) inner product. An isomorphism of Hilbert \( C^* \)-bimodules is an isomorphism of both left and right Hilbert \( C^* \)-modules. We recall from [BGR, 3] that \( \text{Pic}(A) \), the Picard group of \( A \), consists of isomorphism classes of full Hilbert \( C^* \)-bimodules over \( A \) (that is, Hilbert \( C^* \)-bimodules \( M \) such that \( \langle M, M \rangle_L = \langle M, M \rangle_R = A \)), equipped with the tensor product, as defined in [Rf1, 5.9].

It was shown in [BGR, 3.1] that there is an anti-homomorphism from \( \text{Aut}(A) \) to \( \text{Pic}(A) \) such that the sequence
\[
1 \longrightarrow \text{Gin}(A) \longrightarrow \text{Aut}(A) \longrightarrow \text{Pic}(A)
\]
is exact, where \( \text{Gin}(A) \) is the group of generalized inner automorphisms of \( A \). By this correspondence, an automorphism \( \alpha \) is mapped to a bimodule that corresponds to the one we denote by \( A_{\alpha}^{-1} \) (see below), so that \( \alpha \mapsto A_{\alpha} \) is a group homomorphism having \( \text{Gin}(A) \) as its kernel.

Given a partial automorphism \((I, J, \theta)\) of a \( C^* \)-algebra \( A \), we denote by \( J_\theta \) the corresponding ([AEE, 3.2]) Hilbert \( C^* \)-bimodule over \( A \). That is, \( J_\theta \) consists of the vector space \( J \) endowed with the \( A \)-actions:
\[
a \cdot x = ax, \quad x \cdot a = \theta[\theta^{-1}(x)a],
\]
and the inner products
\[
\langle x, y \rangle_L = xy^*,
\]
and
\[
\langle x, y \rangle_R = \theta^{-1}(x^*y),
\]
for \( x, y \in J \), and \( a \in A \). If \( M \) is a Hilbert \( C^* \)-bimodule over \( A \), we denote by \( M_\theta \) the Hilbert \( C^* \)-bimodule obtained by taking the tensor product \( M \otimes_A J_\theta \).
The map $m \otimes j \mapsto mj$, for $m \in M$, $j \in J$, identifies $M_\theta$ with the vector space $MJ$ equipped with the $A$-actions:

$$a \cdot mj = amj, \quad mj \cdot a = m\theta[\theta^{-1}(j)a],$$

and the inner products

$$\langle x, y \rangle_{L}^{M_\theta} = \langle x, y \rangle_{L}^{M},$$

and

$$\langle x, y \rangle_{R}^{M_\theta} = \theta^{-1}(\langle x, y \rangle_{R}^{M}),$$

where $m \in M$, $j \in J$, $x, y \in MJ$, and $a \in A$.

As mentioned above, when $M$ is a $C^*$-algebra $A$, equipped with its usual structure of Hilbert $C^*$-bimodule over $A$, and $\theta \in \text{Aut}(A)$ the bimodule $A_\theta$ corresponds to the element of Pic($A$) denoted by $X_{\theta^{-1}}$ in [BGR, 3], so we have $A_\theta \otimes A_\sigma \sim A_{\theta\sigma}$ and $\tilde{A}_\theta \sim A_{\theta^{-1}}$ for all $\theta, \sigma \in \text{Aut}(A)$.

In this section we discuss the interdependence between the left and the right structure of a Hilbert $C^*$-bimodule. Proposition 1.1 shows that the right structure is determined, up to a partial isomorphism, by the left one. By specializing this result to the case of full Hilbert $C^*$-bimodules over a commutative $C^*$-algebra, we are able to describe Pic($A$) as the semidirect product of the classical Picard group of $A$ by the group of automorphisms of $A$.

**Proposition 1.1** Let $M$ and $N$ be Hilbert $C^*$-bimodules over a $C^*$-algebra $A$. If $\Phi : M \rightarrow N$ is an isomorphism of left $A$-Hilbert $C^*$-modules, then there is a partial automorphism $(I, J, \theta)$ of $A$ such that $\Phi : M_\theta \rightarrow N$ is an isomorphism of $A - A$ Hilbert $C^*$-bimodules. Namely, $I = \langle N, N \rangle_R$, $J = \langle M, M \rangle_R$ and $\theta(\langle \Phi(m_0), \Phi(m_1) \rangle_R) = \langle m_0, m_1 \rangle_R$.

**Proof:** Let $\Phi : M \rightarrow N$ be a left $A$-Hilbert $C^*$-module isomorphism. Notice that, if $m \in M$, and $\|m\| = 1$, then, for all $m_i, m'_i \in M$, and $i = 1, \ldots, n$: 
Therefore:

\[ || \sum \langle m_i, m'_i \rangle_R || = || \sum \langle m, m_i \rangle_L m'_i || \]
\[ = || \Phi(\sum \langle m, m_i \rangle_L m'_i) || \]
\[ = || \sum \langle m, m'_i \rangle_L \Phi(m_i) || \]
\[ = || \sum \Phi(m) \langle \Phi(m_i), \Phi(m'_i) \rangle_R || \]
\[ = || \sum \Phi(m) \langle \Phi(m_i), \Phi(m'_i) \rangle_R || . \]

Set \( I = \langle N, N \rangle_R \), and \( J = \langle M, M \rangle_R \), and let \( \theta : I \rightarrow J \) be the isometry defined by

\[ \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R) = \langle m_1, m_2 \rangle_R, \]

for \( m_1, m_2 \in M \). Then,

\[ \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R^*) = \theta(\langle \Phi(m_2), \Phi(m_1) \rangle_R) \]
\[ = \langle m_2, m_1 \rangle_R \]
\[ = \langle m_1, m_2 \rangle_R^* \]
\[ = \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R)^*, \]

and

\[ \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R \langle \Phi(m'_1), \Phi(m'_2) \rangle_R) = \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R \langle \Phi(m'_1), \Phi(m'_2) \rangle_R || \]
\[ = \theta(\langle \Phi(m_1), \Phi(m_2) \rangle_R \langle \Phi(m'_1), \Phi(m'_2) \rangle_R || \]
\[ = \langle m_1, \langle \Phi(m_2), \Phi(m'_1) \rangle_L m'_2 \rangle_R \]
\[ = \langle m_1, \langle m_2, m'_1 \rangle_L m'_2 \rangle_R \]
\[ = \langle m_1, m_2 \rangle_R \langle m'_1, m'_2 \rangle_R \]
\[ = \theta(\langle m_1, m_2 \rangle_R) \theta(\langle m'_1, m'_2 \rangle_R), \]

which shows that \( \theta \) is an isomorphism.

Besides, \( \Phi : M_\theta \rightarrow N \) is a Hilbert \( C^* \)-bimodule isomorphism:
\[
\Phi(m\langle m_1, m_2 \rangle_R \cdot a) = \Phi(m\theta^{-1}(\langle m_1, m_2 \rangle_R a)]
\]
\[
= \Phi(m\theta(\langle \Phi(m_1), \Phi(m_2) a \rangle_R))
\]
\[
= \Phi(m\langle m_1, \Phi^{-1}(\Phi(m_2) a) \rangle_R)
\]
\[
= \langle m, m_1 \rangle_L \Phi(m_2) a
\]
\[
= \Phi(\langle m, m_1 \rangle_L m_2) a
\]
\[
= \Phi(m\langle m_1, m_2 \rangle_R) a,
\]

and
\[
\langle \Phi(m_1), \Phi(m_2) \rangle_R = \theta^{-1}(\langle m_1, m_2 \rangle^M_R) = \langle m_1, m_2 \rangle^{M_\theta}_R.
\]

Finally, \( \Phi \) is onto because
\[
\Phi(M_\theta) = \Phi(M\langle M, M \rangle_R) = \Phi(M) = N.
\]
Q.E.D.

**Corollary 1.2** Let \( M \) and \( N \) be Hilbert \( C^* \)-bimodules over a \( C^* \)-algebra \( A \), and let \( \Phi : M \longrightarrow N \) be an isomorphism of left Hilbert \( C^* \)-modules. Then \( \Phi \) is an isomorphism of Hilbert \( C^* \)-bimodules if and only if \( \Phi \) preserves either the right inner product or the right \( A \)-action.

**Proof:** Let \( \theta \) be as in Proposition 1.1, so that \( \Phi : M_\theta \longrightarrow N \) is a Hilbert \( C^* \)-bimodule isomorphism. If \( \Phi \) preserves the right inner product, then \( \theta \) is the identity map on \( \langle M, M \rangle_R \) and \( M_\theta = M \).

If \( \Phi \) preserves the right action of \( A \), then, for \( m_0, m_1, m_2 \in M \) we have:
\[
\Phi(m_0)\langle m_1, \Phi(m_2) \rangle_R = \langle \Phi(m_0), \Phi(m_1) \rangle_L \Phi(m_2)
\]
\[
= \langle m_0, m_1 \rangle_L \Phi(m_2)
\]
\[
= \Phi(m_0\langle m_1, m_2 \rangle_R)
\]
\[
= \Phi(m_0)\langle m_1, m_2 \rangle_R,
\]
so \( \Phi \) preserves the right inner product as well. Q.E.D.

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Proposition 1.3 Let $M$ and $N$ be left Hilbert $C^*$-modules over a $C^*$-algebra $A$. If $M$ and $N$ are isomorphic as left $A$-modules, and $K(\mathcal{A}M)$ is unital, then $M$ and $N$ are isomorphic as left Hilbert $C^*$-modules.

Proof: First recall that any $A$-linear map $T : M \rightarrow N$ is adjointable. For if $m_i, m'_i \in M$, $i = 1, \ldots, n$ are such that $\sum \langle m_i, m'_i \rangle_R = 1_{K(\mathcal{A}M)}$, then for any $m \in M$:

$$T(m) = T(\sum \langle m, m_i \rangle_L m'_i) = \sum \langle m, m_i \rangle_L T(m'_i) = (\sum \xi_{m_i}(m'_i))(m),$$

where $\xi_{m,n} : M \rightarrow N$ is the compact operator (see, for instance, [La, 1]) defined by $\xi_{m,n}(m_0) = \langle m_0, m \rangle_L n$, for $m \in M$, and $n \in N$, which is adjointable. Let $T : M \rightarrow N$ be an isomorphism of left modules, and set $S : M \rightarrow N$, $S = T(T^*T)^{-1/2}$. Then $S$ is an $A$-linear map, therefore adjointable. Furthermore, $S$ is a left Hilbert $C^*$-module isomorphism: if $m_0, m_1 \in M$, then

$$\langle S(m_0), S(m_1) \rangle_L = \langle T(T^*T)^{-1/2}m_0, T(T^*T)^{-1/2}m_1 \rangle_L = \langle m_0, (T^*T)^{-1/2}T^*T(T^*T)^{-1/2}m_1 \rangle_L = \langle m_0, m_1 \rangle_L.$$

Q.E.D.

We next discuss the Picard group of a $C^*$-algebra $A$. Proposition 1.1 shows that the left structure of a full Hilbert $C^*$-bimodule over $A$ is determined, up to an isomorphism of $A$, by its left structure.

This suggests describing $\text{Pic}(A)$ in terms of the subgroup $\text{Aut}(A)$ together with a cross-section of the equivalence classes under left Hilbert $C^*$-modules isomorphisms. When $A$ is commutative there is a natural choice for this cross-section: the family of symmetric Hilbert $C^*$-bimodules (see Definition 1.5). That is the reason why we now concentrate on commutative $C^*$-algebras and their symmetric Hilbert $C^*$-bimodules.

Proposition 1.4 Let $A$ be a commutative $C^*$-algebra and $M$ a Hilbert $C^*$-bimodule over $A$. Then $\langle m, n \rangle_{Lp} = \langle p, n \rangle_{Lm}$ for all $m, n, p \in M$. 

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Proof: We first prove the proposition for \( m = n \), the statement will then follow from polarization identities.

Let \( m, p \in M \), then:

\[
\langle \langle m, m \rangle_L p - \langle p, m \rangle_L m, \langle m, m \rangle_L p - \langle p, m \rangle_L m \rangle_L \n=
\langle \langle m, m \rangle_L p, \langle m, m \rangle_L p \rangle_L - \langle \langle m, m \rangle_L p, \langle p, m \rangle_L m \rangle_L \n- \langle \langle p, m \rangle_L m, \langle m, m \rangle_L p \rangle_L + \langle \langle p, m \rangle_L m, \langle p, m \rangle_L m \rangle_L \n= \langle m(p, m)_R (m, p)_R, m \rangle_L - \langle m, m \rangle_L \langle p, m \rangle_L (m, p)_L \n- \langle p, m \rangle_L \langle m, p \rangle_L (m, m)_L + \langle p, m \rangle_L \langle m, m \rangle_L (m, p)_L \n= \langle m(p, m)_R (m, p)_R, m \rangle_L - \langle m, m \rangle_L \langle p, m \rangle_L (m, p)_L \n= \langle (m, p)_L m, \langle m, p \rangle_L m \rangle - \langle m, m \rangle_L \langle p, m \rangle_L (m, p)_L \n= 0.
\]

Now, for \( m, n, p \in M \), we have:

\[
\langle m, n \rangle_L p = \frac{1}{4} \sum_{k=0}^{3} i^k \langle m + i^k n, m + i^k n \rangle_L p
= \frac{1}{4} \sum_{k=0}^{3} i^k \langle p, m + i^k n \rangle_L (m + i^k n)\n= \frac{1}{4} \sum_{k=0}^{3} i^k p \langle m + i^k n, m + i^k n \rangle_R\n= p \langle n, m \rangle_R\n= \langle p, n \rangle_L m.
\]

Definition 1.5 Let \( A \) be a commutative \( C^* \)-algebra. A Hilbert \( C^* \)-bimodule \( M \) over \( A \) is said to be symmetric if \( am = ma \) for all \( m \in M \), and \( a \in A \).
If $M$ is a Hilbert $C^*$-bimodule over $A$, the symmetrization of $M$ is the symmetric Hilbert $C^*$-bimodule $M^*$, whose underlying vector space is $M$ with its given left Hilbert-module structure, and right structure defined by:

$$m \cdot a = am, \quad \langle m_0, m_1 \rangle_{M^*}^{M^*} = \langle m_1, m_0 \rangle^M,$$

for $a \in A$, $m, m_0, m_1 \in M^*$. The commutativity of $A$ guarantees the compatibility of the left and right actions. As for the inner products, we have, in view of Proposition 1.4:

$$\langle m_0, m_1 \rangle_{L}^{M^*} \cdot m_2 = \langle m_0, m_1 \rangle_{L}^{M^*} m_2 = \langle m_2, m_1 \rangle_{L}^{M^*} m_0 = m_0 \cdot \langle m_2, m_1 \rangle_{L}^{M^*} = m_0 \cdot \langle m_1, m_2 \rangle_{R}^{M^*},$$

for all $m_0, m_1, m_2 \in M^*$.

**Remark 1.6** By Corollary 1.2 the bimodule $M^*$ is, up to isomorphism, the only symmetric Hilbert $C^*$-bimodule that is isomorphic to $M$ as a left Hilbert module.

**Remark 1.7** Let $M$ be a symmetric Hilbert $C^*$-bimodule over a commutative $C^*$-algebra $A$ such that $K(\mathcal{A}M)$ is unital. By Remark 1.6 and Proposition 1.3, a symmetric Hilbert $C^*$-bimodule over $A$ is isomorphic to $M$ if and only if it is isomorphic to $M$ as a left module.

**Example 1.8** Let $A = C(X)$ be a commutative unital $C^*$-algebra, and let $M$ be a Hilbert $C^*$-bimodule over $A$ that is, as a left Hilbert $C^*$-module, isomorphic to $A^p p$, for some $p \in \text{Proj}(M_n(A))$. This implies that $p M_n(A) p \cong K(\mathcal{A}M)$ is isomorphic to a $C^*$-subalgebra of $A$ and is, in particular, commutative. By viewing $M_n(A)$ as $C(X, M_n(\mathcal{C}))$ one gets that $p(x) M_n(\mathcal{C}) p(x)$ is a commutative $C^*$-algebra, hence rank $p(x) \leq 1$ for all $x \in X$. 

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Conversely, let $A = C(X)$ be as above, and let $p : X \rightarrow \text{Proj}(M_n(C))$ be a continuous map, such that $\text{rank } p(x) \leq 1$ for all $x \in X$. Then $A^n p$ is a Hilbert $C^*$-bimodule over $A$ for its usual left structure, the right action of $A$ by pointwise multiplication, and right inner product given by:

$$\langle m, r \rangle_L = \tau(m^*r),$$

for $m, r \in A^n p, a \in A$, and where $\tau$ is the usual $A$-valued trace on $M_n(A)$ (that is, $\tau((a_{ij})) = \sum a_{ii}$).

To show the compatibility of the inner products, notice that for any $T \in M_n(A)$, and $x \in X$ we have:

$$(pTp)(x) = p(x)T(x)p(x) = [\text{trace}(p(x)T(x)p(x))]p(x),$$

which implies that $pTp = \tau(pTp)p$. Then, for $m, r, s \in M$:

$$\langle m, r \rangle_L s = mpr^*sp = m\tau(pr^*sp)p = m\tau(r^*s) = m \cdot \langle r, s \rangle_R.$$

Besides, $A^n p$ is symmetric:

$$\langle m, r \rangle_R = \tau(m^*r) = \sum_{i=1}^n m_i^*r_i = \langle r, m \rangle_L,$$

for $m = (m_1, m_2, \ldots, m_n), r = (r_1, r_2, \ldots, r_n) \in M$.

Therefore, by Remark 1.7, if $p, q \in \text{Proj}(M_n(A))$, the Hilbert $C^*$-bimodules $A^n p$ and $A^n q$ described above are isomorphic if and only if $p$ and $q$ are Murray-von Neumann equivalent. Notice that the identity of $K(A^n p)$ is $\tau(p)$, that is, the characteristic function of the set $\{x \in X : \text{rank } p(x) = 1\}$. Therefore $A^n p$ is full as a right module if and only if $\text{rank } p(x) = 1$ for all $x \in X$, which happens in particular when $X$ is connected, and $p \neq 0$.

**Proposition 1.9** Let $A$ be a commutative $C^*$-algebra. For any Hilbert $C^*$-bimodule $M$ over $A$ there is a partial automorphism $(\langle M, M \rangle_R, \langle M, M \rangle_L, \theta)$ of $A$ such that the map $i : (M^*)_\theta \rightarrow M$ defined by $i(m) = m$ is an isomorphism of Hilbert $C^*$-bimodules .
Proof: The map \( i : M^* \rightarrow M \) is a left Hilbert \( C^* \)-modules isomorphism. The existence of \( \theta \), with \( I = \langle M, M \rangle_R \) and \( J = \langle M^*, M^* \rangle_R = \langle M, M \rangle_L \), follows from Proposition 1.1.

Q.E.D.

We now turn to the discussion of the group \( \text{Pic}(A) \) for a commutative \( C^* \)-algebra \( A \). For a full Hilbert \( C^* \)-bimodule \( M \) over \( A \), we denote by \([M]\) its equivalence class in \( \text{Pic}(A) \). For a commutative \( C^* \)-algebra \( A \), the group \( \text{Gin}(A) \) is trivial, so the map \( \alpha \mapsto A_{\alpha} \) is one-to-one. In what follows we identify, via that map, \( \text{Aut}(A) \) with a subgroup of \( \text{Pic}(A) \).

Symmetric full Hilbert \( C^* \)-bimodules over a commutative \( C^* \)-algebra \( A = C(X) \) are known to correspond to line bundles over \( X \). The subgroup of \( \text{Pic}(A) \) consisting of isomorphism classes of symmetric Hilbert \( C^* \)-bimodules is usually called the classical Picard group of \( A \), and will be denoted by \( \text{CPic}(A) \). We next specialize the result above to the case of full bimodules.

Notation 1.10 For \( \alpha \in \text{Aut}(A) \), and \( M \) a Hilbert \( C^* \)-bimodule over \( A \), we denote by \( \alpha(M) \) the Hilbert \( C^* \)-bimodule \( \alpha = A_{\alpha} \otimes M \otimes A_{\alpha^{-1}} \).

Remark 1.11 The map \( a \otimes m \otimes b \mapsto amb \) identifies \( A_{\alpha} \otimes M \otimes A_{\alpha^{-1}} \) with \( M \) equipped with the actions:

\[
    a \cdot m = \alpha^{-1}(a)m, \quad m \cdot a = m\alpha^{-1}(a),
\]

and inner products

\[
    \langle m_0, m_1 \rangle_L = \alpha(\langle m_0, m_1 \rangle^M_L),
\]

and

\[
    \langle m_0, m_1 \rangle_R = \alpha(\langle m_0, m_1 \rangle^M_R),
\]

for \( a \in A \), and \( m, m_0, m_1 \in M \).
Theorem 1.12  Let $A$ be a commutative $C^*$-algebra. Then $\text{CPic}(A)$ is a normal subgroup of $\text{Pic}(A)$ and

$$\text{Pic}(A) = \text{CPic}(A) \rtimes \text{Aut}(A),$$

where the action of $\text{Aut}(A)$ is given by conjugation, that is $\alpha \cdot M = \alpha(M)$.

Proof: Given $[M] \in \text{Pic}(A)$ write, as in Proposition 1.9, $M \cong M^*_\theta$, $\theta$ being an isomorphism from $\langle M, M \rangle_R = A$ onto $\langle M, M \rangle_L = A$.

Therefore $M \cong M^* \otimes A_\theta$, where $[M^*] \in \text{CPic}(A)$ and $\theta \in \text{Aut}(A)$. If $[S] \in \text{CPic}(A)$ and $\alpha \in \text{Aut}(A)$ are such that $M \cong S \otimes A_\alpha$, then $S$ and $M^*$ are symmetric bimodules, and they are both isomorphic to $M$ as left Hilbert $C^*$-modules. This implies, by Remark 1.6, that they are isomorphic. Thus we have:

$M^* \otimes A_\theta \cong M^* \otimes A_\alpha \Rightarrow A_\theta \cong \widetilde{M^*} \otimes M^* \otimes A_\theta \cong \widetilde{M^*} \otimes M^* \otimes A_\alpha \cong A_\alpha$,

which implies ([BGR, 3.1]) that $\theta\alpha^{-1} \in \text{Gin}(A) = \{\text{id}\}$, so $\alpha = \theta$, and the decomposition above is unique.

It only remains to show that $\text{CPic}(A)$ is normal in $\text{Pic}(A)$, and it suffices to prove that $[A_\alpha \otimes S \otimes A_{\alpha^{-1}}] \in \text{CPic}(A)$ for all $[S] \in \text{CPic}(A)$, and $\alpha \in \text{Aut}(A)$, which follows from Remark 1.11.

Q.E.D.

Notation 1.13  If $\alpha \in \text{Aut}(A)$, then for any positive integers $k, l$, we still denote by $\alpha$ the automorphism of $M_{k \times l}(A)$ defined by $\alpha([a_{ij}]) = (\alpha(a_{ij}))$.

Lemma 1.14  Let $A$ be a commutative unital $C^*$-algebra, and $p \in \text{Proj}(M_n(A))$ be such that $A^n p$ is a symmetric Hilbert $C^*$-bimodule over $A$, for the structure described in Example 1.8. If $\alpha \in \text{Aut}(A)$, then $\alpha(A^n p) \cong A^n \alpha(p)$.

Proof: Set $J : \alpha(A^n p) \longrightarrow A^n \alpha(p), \quad J(m \otimes x \otimes r) = m\alpha(xr)$, for $m \in A_\alpha$, $r \in A_{\alpha^{-1}}$, and $x \in A^n p$. Notice that

$\alpha(xr) = \alpha(xpr) = \alpha(xr)\alpha(p) \in A^n \alpha(p)$.
Besides, if \( a \in A \)

\[
J(m \cdot a \otimes x \otimes r) = J(m\alpha(a) \otimes x \otimes r) \\
= m\alpha(axr) \\
= J(m \otimes a \cdot x \otimes r),
\]

and

\[
J(m \otimes x \cdot a \otimes r) = m\alpha(xar) \\
= J(m \otimes x \otimes a \cdot r),
\]

so the definition above makes sense. We now show that \( J \) is a Hilbert \( C^* \)-bimodule isomorphism. For \( m \in A_\alpha, n \in A_{\alpha^{-1}}, x \in A^n p, \) and \( a \in A, \) we have:

\[
J(a \cdot (m \otimes x \otimes r)) = J(am \otimes x \otimes r) \\
= am\alpha(xr) \\
= a \cdot J(m \otimes x \otimes r),
\]

and

\[
J(m \otimes x \otimes r \cdot a) = m\alpha(xr^{-1}(a)) \\
= m\alpha(xr)a \\
= J((m \otimes x \otimes r) \cdot a)
\]

Finally,

\[
(J(m \otimes x \otimes r), J(m' \otimes x' \otimes r'))_L = \langle m\alpha(xr), m'\alpha(x'r') \rangle_L \\
= \langle m \cdot [(xr)(x'r')^*], m' \rangle_L \\
= \langle m \cdot \langle x, r \rangle^A_L, x' \rangle^{A^n p} m' \rangle_L \\
= \langle m \cdot \langle x \otimes r, x' \otimes r' \rangle_L^{A^n \otimes A_{\alpha^{-1}}}, m' \rangle_L \\
= \langle m \otimes x \otimes r, m' \otimes x' \otimes r' \rangle_L,
\]

which shows, by Corollary 1.2, that \( J \) is a Hilbert \( C^* \)-bimodule isomorphism.

Q.E.D.

**Proposition 1.15** Let \( A \) be a commutative unital \( C^* \)-algebra and \( M \) a Hilbert \( C^* \)-bimodule over \( A. \) If \( \alpha \in \text{Aut}(A) \) is homotopic to the identity, then

\[ A_\alpha \otimes M \cong M \otimes A_{\gamma^{-1} \alpha \gamma}, \]

where \( \gamma \in \text{Aut}(A) \) is such that \( M \cong (M^*)_\gamma. \)
Proof: We then have that $K(A^\alpha M)$ is unital so, in view of Proposition 1.3 we can assume that $M^* = A^\alpha p$ with the Hilbert $C^*$-bimodule structure described in Example 1.8, for some positive integer $n$, and $p \in \text{Proj}(M_n(A))$. Since $p$ and $\alpha(p)$ are homotopic, they are Murray-von Neumann equivalent ([Bl, 4]). Then, by Lemma 1.14 and Example 1.8, we have

$$A_\alpha \otimes M \cong A_\alpha \otimes M^* \otimes A_\gamma \cong M^* \otimes A_\alpha \gamma \cong M \otimes A_\gamma^{-1}\alpha_\gamma.$$  

Q.E.D.

We turn now to the discussion of crossed products by Hilbert $C^*$-bimodules, as defined in [AEE]. For a Hilbert $C^*$-bimodule $M$ over a $C^*$-algebra $A$, we denote by $A \rtimes_M \mathbb{Z}$ the crossed product $C^*$-algebra. We next establish some general results that will be used later.

**Notation 1.16** In what follows, for $A - A$ Hilbert $C^*$-bimodules $M$ and $N$ we write $M \overset{cp}{\cong} N$ to denote $A \rtimes_M \mathbb{Z} \cong A \rtimes_N \mathbb{Z}$.

**Proposition 1.17** Let $A$ be a $C^*$-algebra, $M$ an $A - A$ Hilbert $C^*$-bimodule and $\alpha \in \text{Aut}(A)$. Then

i) $M \overset{cp}{\cong} \tilde{M}$.

ii) $M \overset{cp}{\cong} \alpha(M)$.

**Proof:** Let $i_A$ and $i_M$ denote the standard embeddings of $A$ and $M$ in $A \rtimes_M \mathbb{Z}$, respectively.

i) Set

$$i_{\tilde{M}} : \tilde{M} \to A \rtimes_M \mathbb{Z}, \quad i_{\tilde{M}}(\tilde{m}) = i_M(m)^*.$$  

Then $(i_A, i_{\tilde{M}})$ is covariant for $(A, \tilde{M})$:

$$i_{\tilde{M}}(a \cdot \tilde{m}) = i_{\tilde{M}}(\tilde{ma}^*) = [i_M(ma^*)]^* = i_A(a)i_M(m)^* = i_A(a)i_{\tilde{M}}(\tilde{m}),$$

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\[ i_M(m_1)i_M(m_2) = i_A(\langle m_0, m_1 \rangle^M) = i_A(\langle m_0, m_1 \rangle^\tilde{M}), \]

for \( a \in A \) and \( m, m_0, m_1 \in M \). Analogous computations prove covariance on the right. By the universal property of the crossed products there is a homomorphism from \( A\alpha_M\mathbb{Z} \) onto \( A\alpha_M\mathbb{Z} \). Since \( \tilde{M} = M \), by reversing the construction above one gets the inverse of \( J \).

ii) Set
\[ j_A : A \longrightarrow A\alpha_M\mathbb{Z}, \quad j_{\alpha(M)} : M \longrightarrow A\alpha_M\mathbb{Z}, \]
defined by \( j_A = i_A\alpha^{-1} \), \( j_{\alpha(M)}(m) = i_M(m) \), where the sets \( M \) and \( \alpha(M) \) are identified as in Remark 1.11. Then \((j_A, j_{\alpha(M)})\) is covariant for \((A, \alpha(M))\):

\[
\begin{align*}
\quad j_{\alpha(M)}(a \cdot m) &= j_{\alpha(M)}(\alpha^{-1}(a)m) = i_A(\alpha^{-1}(a))i_M(m) = j_A(a)i_{\alpha(M)}(m), \\
\quad j_{\alpha(M)}(m_0)j_{\alpha(M)}(m_1)^* &= i_M(m_0)i_M(m_1)^* = i_A(\langle m_0, m_1 \rangle^M) = \\
&= j_A(\langle m_0, m_1 \rangle^M) = j_A(\langle m_0, m_1 \rangle^\tilde{M}),
\end{align*}
\]

for \( a \in A, m, m_0, m_1 \in M \), and analogously on the right. Therefore there is a homomorphism
\[ J : A\alpha_{\alpha(M)}\mathbb{Z} \longrightarrow A\alpha_M\mathbb{Z}, \]
whose inverse is obtained by applying the construction above to \( \alpha^{-1} \).

Q.E.D.

2 An application: isomorphism classes for quantum Heisenberg manifolds.

For \( \mu, \nu \in \mathbb{R} \) and a positive integer \( c \), the quantum Heisenberg manifold \( D^c_{\mu\nu} \) ([Rf4]) is isomorphic ([AEE, Ex.3.3]) to the crossed product \( C(T^2)\alpha_{(X^\mathbb{C})_{\mu\nu}}\mathbb{Z} \), where \( X^\mathbb{C} \) is the vector space of continuous functions on \( \mathbb{R} \times T \) satisfying \( f(x+1,y) = e(-c(y-\nu))f(x,y) \). The left and right actions of \( C(T^2) \) are defined by pointwise multiplication, the inner products by \( \langle f, g \rangle_L = \int f\overline{g} \), and \( \langle f, g \rangle_R = \int g \), and \( \alpha_{\mu\nu} \in \text{Aut}(C(T^2)) \) is given by \( \alpha_{\mu\nu}(x,y) = (x + 2\mu, y + 2\nu) \), and, for \( t \in \mathbb{R} \), \( e(t) = \exp(2\pi it) \).
Our purpose is to find isomorphisms in the family \( \{ D_{\mu \nu}^c : \mu, \nu \in \mathbb{R}, c \in \mathbb{Z}, c > 0 \} \).

We concentrate in fixed values of \( c \), because \( K_0(D_{\mu \nu}^c) \cong \mathbb{Z}^3 \oplus \mathbb{Z} \alpha([Ab2]) \).

Besides, since \( \alpha_{\mu \nu} = \alpha_{\mu+m,\nu+n} \) for all \( m, n \in \mathbb{Z} \), we view from now on the parameters \( \mu \) and \( \nu \) as running in \( \mathbb{T} \).

Let \( M^c \) denote the set of continuous functions on \( \mathbb{R} \times \mathbb{T} \) satisfying
\[
f(x+1,y) = e(-cy)f(x,y).
\]
Then \( M^c \) is a Hilbert \( C^* \)-bimodule over \( C(\mathbb{T}^2) \), for pointwise action and inner products given by the same formulas as in \( X^c \).

The map \( f \mapsto \tilde{f} \), where \( \tilde{f}(x,y) = f(x, y + \nu) \), is a Hilbert \( C^* \)-bimodule isomorphism between \( (X^c)^{\alpha_{\mu \nu}} \) and \( C(\mathbb{T}^2)^{\sigma} \otimes M^c \otimes C(\mathbb{T}^2)^{\rho} \), where \( \sigma(x, y) = (x, y + \nu) \), and \( \rho(x, y) = (x + 2\mu, y + \nu) \). In view of Proposition 1.17 we have:
\[
D_{\mu \nu}^c \cong C(\mathbb{T}^2)^{\alpha_{(M^c)^{\mu \nu}}} \mathbb{Z} \cong C(\mathbb{T}^2)^{\alpha_{(M^c)^{\mu \nu}}} \mathbb{Z}.
\]

As a left module over \( C(\mathbb{T}^2) \), \( M^c \) corresponds to the module denoted by \( X(1, c) \) in [Rf3, 3.7]. It is shown there that \( M^c \) represents the element \( (1, c) \) of \( K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2 \), where the last correspondence is given by \( [X] \mapsto (a, b) \), \( a \) being the dimension of the vector bundle corresponding to \( X \) and \( -b \) its twist. It is also proven in [Rf3] that any line bundle over \( C(\mathbb{T}^2) \) corresponds to the left module \( M^c \), for exactly one value of the integer \( c \), and that \( M^c \otimes M^d \) and \( M^{c+d} \) are isomorphic as left modules. It follows now, by putting these results together, that the map \( c \mapsto [M^c] \) is a group isomorphism from \( \mathbb{Z} \) to \( CPic(C(\mathbb{T}^2)) \).

**Lemma 2.1**

\[
Pic(C(\mathbb{T}^2)) \cong \mathbb{Z} \rtimes_{\delta} Aut(C(\mathbb{T}^2)),
\]

where \( \delta_{\alpha}(c) = \text{det} \alpha \cdot c \), for \( \alpha \in Aut(C(\mathbb{T}^2)) \), and \( c \in \mathbb{Z} \); \( \alpha \) being the usual automorphism of \( K_0(C(\mathbb{T}^2)) \cong \mathbb{Z}^2 \), viewed as an element of \( GL_2(\mathbb{Z}) \).

**Proof:** By Theorem 1.12 we have:

\[
Pic(C(\mathbb{T}^2)) \cong CPic(C(\mathbb{T}^2)) \rtimes_{\delta} Aut(C(\mathbb{T}^2)).
\]
If we identify $\text{CPic}(C(T^2))$ with $\mathbb{Z}$ as above, it only remains to show that $\alpha(M^c) \cong M^{\det \alpha \cdot c}$. Let us view $\alpha_* \in \text{GL}_2(\mathbb{Z})$ as above. Since $\alpha_*$ preserves the dimension of a bundle, and takes $C(T^2)$ (that is, the element $(1, 0) \in \mathbb{Z}^2$) to itself, we have

$$\alpha_* = \begin{pmatrix} 1 & 0 \\ 0 & \det \alpha_* \end{pmatrix}$$

Now,

$$\alpha_*(M^c) = \alpha_*(1, c) = (1, \det \alpha_* \cdot c) = M^{\det \alpha \cdot c}.$$  

Since there is cancellation in the positive semigroup of finitely generated projective modules over $C(T^2)$ ([Rf3]), the result above implies that $\alpha_*(M^c)$ and $M^{\det \alpha \cdot c}$ are isomorphic as left modules. Therefore, by Remark 1.7, they are isomorphic as Hilbert $C^*$-bimodules.

Q.E.D.

**Theorem 2.2** If $(\mu, \nu)$ and $(\mu', \nu')$ belong to the same orbit under the usual action of $\text{GL}(2, \mathbb{Z})$ on $T^2$, then the quantum Heisenberg manifolds $D_{\mu \nu}^c$ and $D_{\mu' \nu'}^c$ are isomorphic.

**Proof:** If $(\mu, \nu)$ and $(\mu', \nu')$ belong to the same orbit under the action of $\text{GL}(2, \mathbb{Z})$, then $\alpha_{\mu' \nu'} = \sigma \alpha_{\mu \nu} \sigma^{-1}$, for some $\sigma \in \text{GL}(2, \mathbb{Z})$. Therefore, by Lemma 2.1 and Proposition 1.17:

$$M_{\alpha_{\mu \nu}}^c \cong M_{\sigma \alpha_{\mu \nu} \sigma^{-1}}^c \cong M^c \otimes C(T^2)_{\sigma \alpha \sigma^{-1}} \cong C(T^2)_{\sigma} \otimes M^{\det \sigma \cdot c} \cong \sigma(M^{\det \sigma \cdot c}) \cong M^{\det \sigma \cdot c}.$$  

In case $\det \sigma_* = -1$ we have

$$M^{\det \sigma \cdot c}_{\alpha_{\mu \nu}} \cong M^{-c}_{\alpha_{\mu \nu}} \cong M^{-c}_{\alpha_{\mu \nu}} \cong C(T^2)_{\alpha_{\mu \nu}^{-1}} \otimes M^c \cong (M^c)_{\alpha_{\mu \nu}^{-1}},$$

since $\det \alpha_* = 1$, because $\alpha_{\mu \nu}$ is homotopic to the identity.
On the other hand, it was shown in [Ab1, 0.3] that \( M_{\alpha\mu,\nu}^c \cong M_{\alpha\mu\nu}^c \).

Thus, in any case, \( M_{\alpha\mu,\nu}^c \cong M_{\alpha\mu\nu}^c \). Therefore

\[
D_{\mu'\nu'}^c \cong C(T^2) \rtimes_{M_{\alpha'\mu'\nu'}} Z \cong C(T^2) \rtimes_{M_{\alpha\mu\nu}^c} Z \cong D_{\mu\nu}^c.
\]

Q.E.D.

References


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