# Finite-dimensional representations of free product C*-algebras

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**Abstract**

Our main theorem is a characterization of C*-algebras that have a separating family of finite-dimensional representations. This characterization makes possible a solution to a problem posed by Goodearl and Menaul. Specifically, we prove that the free product of such C*-algebras again has this property.

## 1 Introduction

A C*-algebra is called *residually finite-dimensional* (RFD) if it has a separating family of finite-dimensional representations. Clearly a C*-algebra is residually finite-dimensional if it is commutative or finite-dimensional. Also, this property is inherited by subalgebras. We show that a free product of residually finite-dimensional C*-algebras, possibly amalgamated over units, is residually finite-dimensional. Therefore, we have lots of new examples, such as $U_n^{\mathrm{nc}}$, the noncommutative unitary groups. (This is a subalgebra of $M_n *_{\mathbf{C}} C(S^1)$.)

The first consideration of finite-dimensional representations of a free product was by Choi [1]. He starts with a faithful representation

$$C(S^1) *_{\mathbf{C}} C(S^1) \to \mathcal{B}(\mathcal{H})$$

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determined by two unitaries $U$ and $V$. (This free product is isomorphic to the group C*-algebra for the free group on two generators.) Choi shows that there is a sequence of finite-dimensional, unitaries $U_n$ and $V_n$ on subspaces of $\mathcal{H}$ converging star-strongly to $U$ and $V$. This implies that $U_n$ and $V_n$ determine a separating sequence of finite-dimensional representations.

Goodearl and Menaul [4] extend Choi's result. For example, they consider free products of the form $C(S^1) *_{\mathbf{C}} C^*(G)$, where $G$ is a finite group. Again, the starting point is a faithful representation

$$C(S^1) *_{\mathbf{C}} C^*(G) \to \mathcal{B}(\mathcal{H}).$$

They show that given any representations of $C(S^1)$ or $C^*(G)$ on $\mathcal{B}(\mathcal{H})$, there is always a lift to a C*-algebra they call $T(\mathcal{H})$. Therefore, the free product is seen to be a subalgebra of $T(\mathcal{H})$. As we shall see in a moment, $T(\mathcal{H})$ is RFD, so the free product is as well.

Although they do not define it this way, $T(\mathcal{H})$ may be seen to be isomorphic to the following C*-algebra. Suppose $\mathcal{H}$ is separable, with basis $\{e_n\}$. Let $p_n$ denote the projection onto the first $n$ basis vectors. Then $T(\mathcal{H})$ consists of all bounded sequences $(S_n)$ of operators on $\mathcal{H}$ such that $p_n S_n p_n = S_n$ and the star-strong limit of $(S_n)$ exists. The surjection onto $\mathcal{B}(\mathcal{H})$ sends $(S_n)$ to $\lim_{n\to\infty} S_n$.

Described this way, $T(\mathcal{H})$ has a natural generalization. One replaces $\{p_n\}$ by any net of projections, on any Hilbert space, such that $p_\lambda \to I$ strongly. One then considers bounded nets $(S_\lambda)$ that converge star-strongly and satisfy $p_\lambda S_\lambda p_\lambda = S_\lambda$. If all these projections are finite-dimensional then the resulting C*-algebra is RFD. However, we find it simpler to state our results, not as lifting problems to this algebra, but directly in terms of (a modified version of) Fell's topology [2, 3] on $\mathrm{Rep}(A, \mathcal{H})$.

# 2 RFD representations

We begin by describing a topology on representations that is similar to Fell's topology [2, 3].

**Definition 2.1** Let $A$ be a C*-algebra and $\mathcal{H}$ a Hilbert space. We denote by $\mathrm{Rep}(A, \mathcal{H})$ the set of all (possibly degenerate) representations of $A$ on $\mathcal{H}$, equipped with the coarsest topology for which the maps

$$\pi \in \mathrm{Rep}(A, \mathcal{H}) \mapsto \pi(a)\xi \in \mathcal{H}$$

are continuous for all $a \in A$ and $\xi \in \mathcal{H}$.

The difference from Fell's topology is that, for a net $\pi_\alpha$ to converge to $\pi$, we require that

$$\pi_\alpha(a)\xi \to \pi(a)\xi$$

for all vectors $\xi$, whereas in Fell's topology this is required only for vectors in the essential space of $\pi$.

**Definition 2.2** A representation $\pi \in \mathrm{Rep}(A, \mathcal{H})$ is called *finite-dimensional* if its essential space is finite-dimensional. We shall call $\pi$ *residually finite-dimensional* (RFD) if $\pi$ is in the closure, in $\mathrm{Rep}(A, \mathcal{H})$, of the set of finite-dimensional representations.

**Definition 2.3** A state $f$ of $A$ is said to be *finite-dimensional* if the GNS representation determined by $f$ is finite-dimensional.

We can now state our main theorem. Recall from the introduction that a C*-algebra is called residually finite-dimensional if it has a separating family of finite-dimensional representations.

**Theorem 2.4** *Let $A$ be a C*-algebra. The following are equivalent:*

*(a) the finite-dimensional states form a dense subset of the state space $S(A)$*

*(b) every cyclic representation of $A$ is residually finite-dimensional*

*(c) every representation of $A$ is residually finite-dimensional*

*(d) $A$ admits a faithful residually finite-dimensional representation*

*(e) $A$ is residually finite-dimensional*

Before we embark on the proof, we present two lemmas. The first is a reworking of an idea from [2].

**Lemma 2.5** *Let $\mathcal{H}$ be a Hilbert space and $(\mathcal{H}_\alpha)_{\alpha\in\Lambda}$ be a family of Hilbert spaces indexed by a directed set $\Lambda$. Suppose we are given vectors $\xi_1, \ldots, \xi_n$ in $\mathcal{H}$ and that for each $\alpha \in \Lambda$ we choose vectors $\xi_1^\alpha, \ldots, \xi_n^\alpha$ in $\mathcal{H}_\alpha$ such that*

$$\lim_{\alpha\to\infty} \langle \xi_i^\alpha, \xi_j^\alpha \rangle = \langle \xi_i, \xi_j \rangle$$

*for $i, j = 1, \ldots, n$. Then there is an $\alpha_0 \in \Lambda$ and, for each $\alpha \geq \alpha_0$, there is an isometry $u_\alpha$ from the subspace $\mathcal{H}_0$ of $\mathcal{H}$ spanned by $\{\xi_1, \ldots, \xi_n\}$ into $\mathcal{H}_\alpha$ such that*

$$\lim_{\alpha\to\infty} \|u_\alpha(\xi_i) - \xi_i^\alpha\| = 0$$

*for $i = 1, \ldots, n$.*

**Proof.** Let $v : \mathbf{C}^n \to \mathcal{H}_0$ be the linear map sending each $e_i$, the $i$th vector of the canonical basis of $\mathbf{C}^n$, to $\xi_i$. Since $v$ is surjective we can choose a right inverse $w$ to $v$. For each $\alpha \in \Lambda$, let $v_\alpha : \mathbf{C}^n \to \mathcal{H}_\alpha$ be given by $v_\alpha(e_i) = \xi_i^\alpha$, for $i = 1, \ldots, n$. Observe that $v_\alpha^* v_\alpha$, viewed as an element of $M_n(\mathbf{C})$, converges to $v^*v$ since, for all $i$ and $j$,

$$\lim_{\alpha\to\infty} \langle v_\alpha^* v_\alpha(e_i), e_i\rangle = \lim_{\alpha\to\infty} \langle \xi_i^\alpha, \xi_j^\alpha\rangle = \langle \xi_i, \xi_j\rangle = \langle v^*v(e_i), e_j\rangle.$$

Thus, if we let $u'_\alpha : \mathcal{H}_0 \to \mathcal{H}_\alpha$ be defined by $u'_\alpha = v_\alpha w$, we have that

$$\lim_{\alpha\to\infty} {u'_\alpha}^* u'_\alpha = \lim_{\alpha\to\infty} w^* v_\alpha^* v_\alpha w = w^* v^* v w = \mathrm{id}_{\mathcal{H}_0}.$$

Therefore, we can find $\alpha_0$ such that, for $\alpha \geq \alpha_0$, ${u'_\alpha}^* u'_\alpha$ is invertible. For all such $\alpha$, set

$$u_\alpha = u'_\alpha ({u'_\alpha}^* u'_\alpha)^{-1/2}.$$

We then have, for $i = 1, \ldots, n$,

$$\begin{aligned} \lim_{\alpha\to\infty} \|u'_\alpha \xi_i - \xi_i^\alpha\|^2 &= \lim_{\alpha\to\infty} \langle {u'_\alpha}^* u'_\alpha \xi_i, \xi_i\rangle - 2\mathrm{Re}\langle v_\alpha^* v_\alpha w \xi_i, e_i\rangle + \langle \xi_i^\alpha, \xi_i^\alpha\rangle \\ &= 2\langle \xi_i, \xi_i\rangle - 2\langle v^* v w \xi_i, e_i\rangle \\ &= 0. \end{aligned}$$

¿From this, it follows easily that

$$\lim_{\alpha\to\infty} \|u_\alpha \xi_i - \xi_i^\alpha\| = 0.$$

**Lemma 2.6** *Suppose that $\pi$ is a cyclic representation of $A$ on $\mathcal{H}$, with cyclic vector $\xi$, and $(\pi_\gamma)_{\gamma\in\Gamma}$ is a net in $\mathrm{Rep}(A, \mathcal{H})$. If*

$$\lim_{\gamma\to\infty} \pi_\gamma(a)\xi = \pi(a)\xi$$

*for all $a \in A$, then $\pi_\gamma$ converges to $\pi$ in $\mathrm{rep}(A, \mathcal{H})$.*

**Proof.** Notice that if $b \in A$ and $\eta = \pi(b)\xi$ we have

$$\begin{aligned} \|\pi_\gamma(a)\eta - \pi(a)\eta\| &\leq \|\pi_\gamma(a)\pi(b)\xi - \pi_\gamma(a)\pi_\gamma(b)\xi\| + \|\pi_\gamma(ab)\xi - \pi(ab)\xi\| \\ &\leq \|a\|\|\pi(b)\xi - \pi_\gamma(b)\xi\| + \|\pi_\gamma(ab)\xi - \pi(ab)\xi\| \end{aligned}$$

which shows that $\pi_\gamma(a)$ converges to $\pi(a)$ pointwise over the dense set

$$\{\eta \in \mathcal{H} : \eta = \pi(b)\xi, b \in A\}$$

The uniform boundedness of $\{\pi_\gamma(a) \mid a \in A\}$ then tells us that $\pi_\gamma(a)$ converges strongly to $\pi(a)$ for all $a$.

We now give the proof of Theorem 2.4.

(a $\Rightarrow$ b) First, we assume $A$ to be unital. Let $\pi$ be a cyclic representation of $A$ on $\mathcal{H}$ with cyclic vector $\xi$ and state $f$. By assumption, there is a net $(f_\alpha)$ of finite-dimensional states converging to $f$ and let $(\rho_\alpha, \mathcal{H}_\alpha, \xi_\alpha)$ be the corresponding GNS representations.

Given a finite set $\{a_0, a_1, \ldots, a_n\}$ of elements of A, in which $a_0 = 1$, observe that

$$\lim_{\alpha\to\infty} \langle \rho_\alpha(a_i)\xi_\alpha, \rho_\alpha(a_j)\xi_\alpha \rangle = \langle \pi(a_i)\xi, \pi(a_j)\xi \rangle$$

for all $i, j = 1, \ldots, n$. By Lemma 2.5, there exists a net $(u_\alpha)_{\alpha\geq\alpha_0}$ of isometries from

$$\mathcal{H}_0 = \operatorname{span}\{\pi(a_i)\xi \mid i = 0, \ldots, n\}$$

into $\mathcal{H}_\alpha$ such that

$$\lim_{\alpha\to\infty} \|u_\alpha \pi(a_i)\xi - \rho_\alpha(a_i)\xi_\alpha\| = 0. \tag{1}$$

Assuming the non-trivial case of $\mathcal{H}$ infinite-dimensional, we may extend each $u_\alpha$ to a co-isometry onto the finite-dimensional space $\mathcal{H}_\alpha$.

Let $\pi_\alpha$ be the (degenerate) representation of $A$ on $\mathcal{H}$ given by

$$\pi_\alpha(a) = u_\alpha^* \rho_\alpha(a) u_\alpha.$$

We claim that for all $i$ we have

$$\lim_{\alpha\to\infty} \pi_\alpha(a_i)\xi = \pi(a_i)\xi. \tag{2}$$

First, by taking $i = 0$ in (1) we obtain $\lim \|u_\alpha\xi - \xi_\alpha\| = 0$. Therefore,

$$\begin{aligned}
\|\pi_\alpha(a_i)\xi - \pi(a_i)\xi\| &= \|u_\alpha^*\rho_\alpha(a_i)u_\alpha\xi - u_\alpha^* u_\alpha \pi(a_i)\xi\| \\
&\leq \|\rho_\alpha(a_i)u_\alpha\xi - u_\alpha\pi(a_i)\xi\| \\
&\leq \|\rho_\alpha(a_i)u_\alpha\xi - \rho_\alpha(a_i)\xi_\alpha\| + \|\rho_\alpha(a_i)\xi_\alpha - u_\alpha\pi(a_i)\xi\| \\
&\leq \|a_i\|\|u_\alpha\xi - \xi_\alpha\| + \|\rho_\alpha(a_i)\xi_\alpha - u_\alpha\pi(a_i)\xi\|,
\end{aligned}$$

from which (2) follows.

Denote the set $\{a_0, a_1, \ldots, a_n\}$ by $\beta$. For each such $\beta$ and for each $\epsilon > 0$, choose, from among the $\pi_\alpha$, a $\pi_{\epsilon,\beta}$ such that

$$\|\pi_{\epsilon,\beta}(a_i)\xi - \pi(a_i)\xi\| < \epsilon.$$

We thus obtain a net $(\pi_{\epsilon,\beta})$ of finite-dimensional representations such that, for all $a \in A$,

$$\lim \pi_{\epsilon,\beta}(a)\xi = \pi(a)\xi.$$

By Lemma 2.6, this converges to $\pi$ as a net in $\mathrm{Rep}(A, \mathcal{H})$.

In the case that $A$ lacks a unit, let $\tilde{A}$ be the unitization of $A$. The natural inclusion $S(A) \to S(\tilde{A})$ preserves finite-dimensional states so that, if (a) holds for $A$, it will also hold for $\tilde{A}$. Our previous work shows that (b) holds for $\tilde{A}$, and hence also for $A$.

(b $\Rightarrow$ c) Given an arbitrary representation $\pi$ of $A$ on a Hilbert space $\mathcal{H}$, write

$$\pi = \bigoplus_{\lambda \in \Lambda} \pi_\lambda$$

where each $\pi_\lambda$ is a cyclic subrepresentation of $\pi$. For each finite subset $F \subseteq \Lambda$, we let

$$\pi_F = \bigoplus_{\lambda \in F} \pi_\lambda,$$

viewed as a degenerate representation on $\mathcal{H}$. It is clear that the net so obtained converges to $\pi$. A simple argument now shows that each $\pi_F$ is RFD, and hence so is $\pi$.

(c $\Rightarrow$ d) This is obvious.

(d $\Rightarrow$ e) Let $\pi$ be a faithful RFD representation of $A$, so that $\pi = \lim \pi_\alpha$ where each $\pi_\alpha$ is finite-dimensional. If $a \in A$ is nonzero, then $\pi(a) \neq 0$. But since $\pi_\alpha(a)$ converges strongly to $\pi(a)$, some $\pi_\alpha(a)$ must be nonzero.

(e $\Rightarrow$ a) We assume $A$ has a unit, since the non-unital case follows easily from the unital case.

Denote by $F(A)$ the set of finite-dimensional states of $A$. It is a simple matter to verify that $F(A)$ is a convex subset of $S(A)$. In fact, if $f$ and $g$ are in $F(A)$ then the GNS representation of a convex combination $h = (1-t)f + tg$ is equivalent to a sub-representation of the direct sum of the GNS representations for $f$ and $g$.

Arguing by contradiction, assume $g$ is in $S(A)$ but not in the weak*-closure of $F(A)$.

Identify the dual of $A_h'$, the set of self-adjoint continuous linear functionals on $A$ with the weak* topology, and $A_h$, the set of self-adjoint elements of $A$. Now use the Hahn-Banach theorem to obtain an element $a$ in $A_h$ and a real number $r$ such that $g(a) > r$ and $f(a) \leq r$ for all $f$ in $F(A)$.

This implies that for any finite-dimensional representation $\pi$ of $A$ and any unit vector $\xi$, in the space of $\pi$, one has $\langle \pi(a)\xi, \xi \rangle \leq r$. Therefore $\pi(a) \leq r$ and, since the direct sum of all finite-dimensional representations of $A$ is a faithful representation, by hypothesis, we have $a \leq r$. The fact that $g(a) > r$ is then a contradiction.

# 3 Free products

In this section, we give a complete solution to [4, Problem 2.4]. Specifically, we characterize the free product C*-algebras that are residually finite-dimensional.

Our strategy begins with a faithful representation

$$A * B \xrightarrow{\pi} \mathcal{B}(\mathcal{H}).$$

If $A$ and $B$ are RFD, then $\pi|_A$ and $\pi|_B$ can be approximated by nets of finite-dimensional representations. Unfortunately, these will live on different nets of subspaces. To remedy this, we will use the following lemma to increase the essential subspaces for the representations in both nets to accommodate each other.

**Lemma 3.1** *Let $\pi$ be a non-degenerate representation of a C*-algebra $A$ on the Hilbert space $\mathcal{H}$ and suppose that $\pi_\alpha$ is a net in* $\mathrm{Rep}(A, \mathcal{H})$ *that converges to $\pi$. If $\rho_\alpha$ is another net (on the same directed set) in* $\mathrm{Rep}(A, \mathcal{H})$ *such that the restriction of each $\rho_\alpha(a)$ to the essential space $\mathcal{H}_\alpha$ of $\pi_\alpha$ coincides with $\pi_\alpha(a)$ then $\rho_\alpha$ also converges to $\pi$.*

**Proof.** Let $p_\alpha$ denote the orthogonal projection onto $\mathcal{H}_\alpha$. We claim that $p_\alpha$ converges strongly to the identity operator (c.f. [3, page 239]). In fact, for $\xi \in \mathcal{H}$, $a \in A$ and all $\alpha$ we have

$$\begin{aligned} \|\pi(a)\xi - p_\alpha(\pi(a)\xi)\| &= \mathrm{dist}(\pi(a)\xi, \mathcal{H}_\alpha) \\ &\leq \|\pi(a)\xi - p_\alpha(\pi_\alpha(a)\xi)\| \\ &= \|\pi(a)\xi - \pi_\alpha(a)\xi\| \end{aligned}$$

showing that $p_\alpha$ converges pointwise to the indentity over the dense set $\{\pi(a)\xi \mid a \in A, \xi \in \mathcal{H}\}$. Since $(p_\alpha)_\alpha$ is uniformly bounded the claim is proven.

Observing that $\rho_\alpha(a)p_\alpha = \pi_\alpha(a)$, for $a$ in $A$ and $\xi$ in $\mathcal{H}$, we have

$$\begin{aligned} \|\rho_\alpha(a)\xi - \pi(a)\xi\| &\leq \|\rho_\alpha(a)\xi - \rho_\alpha(a)p_\alpha\xi\| + \|\pi_\alpha(a)\xi - \pi(a)\xi\| \\ &\leq \|a\|\|\xi - p_\alpha\xi\| + \|\pi_\alpha(a)\xi - \pi(a)\xi\| \end{aligned}$$

from which we see that $\rho_\alpha(a)\xi$ converges to $\pi(a)\xi$ and hence the conslusion.

**Theorem 3.2** *Let $A_1$ and $A_2$ be C*-algebras. Then $A_1 * A_2$ is RFD if and only if $A_1$ and $A_2$ are RFD. If both $A_1$ and $A_2$ are unital, then $A_1 *_{\mathbf{C}} A_2$ is RFD if and only if $A_1$ and $A_2$ are RFD.*

**Proof.** The RFD property clearly passes to subalgebras, so both forward implications are trivial.

To prove the reverse implication in the unital case, let $\pi$ be a faithful non-degenerate representation of $A_1 *_{\mathbf{C}} A_2$ on a Hilbert space $\mathcal{H}$.

For $i = 1, 2$ let $\pi_i$ be the restriction of $\pi$ to $A_i$ and take a net $(\pi^i_\alpha)_\alpha$ in $\mathrm{Rep}(A, \mathcal{H})$ of finite dimensional-representations converging to $\pi_i$. (We are justified in using a common directed set, as in general we may replace both directed sets by their product.) Let $\mathcal{H}^i_\alpha$ be the essential space of $\pi^i_\alpha$.

For each $\alpha$, choose a finite-dimensional subspace $\mathcal{K}_\alpha$ of $\mathcal{H}$ containing both $\mathcal{H}^1_\alpha$ and $\mathcal{H}^2_\alpha$ with dimension a common multiple of $\dim(\mathcal{H}^1_\alpha)$ and $\dim(\mathcal{H}^2_\alpha)$. Let $\rho^i_\alpha$ be any representation of $A_i$ with $\mathcal{K}_\alpha$ as its essential subspace which restricts to $\pi^i_\alpha$ on $\mathcal{H}^i_\alpha$. For example, one may take an appropriate multiple of $\pi^i_\alpha$.

Both $\pi_1$ and $\pi_2$ are unital, and so nondegenerate. Using Lemma 3.1 we thus obtain $\lim_\alpha \rho^i_\alpha = \pi_\alpha$.

For each $\alpha$, let $\rho_\alpha = \rho^1_\alpha * \rho^2_\alpha$ which is a well-defined, finite-dimensional representation of $A_1 *_{\mathbf{C}} A_2$ since $\rho^1_\alpha(1)$ and $\rho^2_\alpha(1)$ are both equal to the orthogonal projection onto $\mathcal{K}_\alpha$. It now follows that $\lim_\alpha \rho_\alpha = \pi$ which proves that $\pi$, and hence $A_1 *_{\mathbf{C}} A_2$, is RFD.

The proof of the non-unital case is similar, but simpler. As before, take a faithful representation of $A_1 * A_2$ on $\mathcal{H}$. Let $\pi_i$ be the restriction of $\pi$ to $A_i$ and write $\pi_i = \lim_\alpha \pi^i_\alpha$. If one sets $\pi^\alpha = \pi^\alpha_1 * \pi^\alpha_2$ then $(\pi^\alpha)_\alpha$ converges to $\pi$. This completes the proof.

# References


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