Crossed-Products by Finite Index Endomorphisms and KMS states

Ruy Exel*
Departamento de Matemática
Universidade Federal de Santa Catarina
88040-900 Florianópolis SC
BRAZIL
E-mail: exel@mtm.ufsc.br

ABSTRACT. Given a unital C*-algebra $A$, an injective endomorphism $\alpha: A \to A$ preserving the unit, and a conditional expectation $E$ from $A$ to the range of $\alpha$ we consider the crossed-product of $A$ by $\alpha$ relative to the transfer operator $L = \alpha^{-1}E$. When $E$ is of index-finite type we show that there exists a conditional expectation $G$ from the crossed-product to $A$ which is unique under certain hypothesis. We define a “gauge action” on the crossed-product algebra in terms of a central positive element $h$ and study its KMS states. The main result is: if $h > 1$ and $E(ab) = E(ba)$ for all $a,b \in A$ (e.g. when $A$ is commutative) then the KMS $\beta$ states are precisely those of the form $\psi = \phi \circ G$, where $\phi$ is a trace on $A$ satisfying the identity

$$\phi(a) = \phi(L(h^{-\beta \text{ind}(E)}a)),$$

where $\text{ind}(E)$ is the Jones-Kosaki-Watatani index of $E$.

1. Introduction.

In [E2] we have introduced the notion of the crossed-product of a C*-algebra $A$ by a *-endomorphism $\alpha$, a construction which also depends on the choice of a transfer operator, that is a positive continuous linear map $L: A \to A$ such that $L(\alpha(a)b) = aL(b)$, for all $a,b \in A$. In the present work we treat the case in which $\alpha$ is a monomorphism (injective endomorphism) and $L$ is given by $L = \alpha^{-1}E$, where $E$ is a conditional expectation onto the range of $\alpha$.

The first of our main results (Theorem 4.12) is the solution to a problem posed in [E2]: we prove that the canonical mapping of $A$ into $A \rtimes_{\alpha,L} \mathbb{N}$ is injective. The main technique used to accomplish this is based on the celebrated “Jones basic construction” [J], as adapted to the context of C*-algebras by Watatani [W]. In order to briefly describe this technique consider, for each $n \in \mathbb{N}$, the conditional expectation onto the range of $\alpha^n$ given by

$$E_n = \alpha^{n-1}(E\alpha^{-1}) \cdots (E\alpha^{-1}) E, \quad \text{n-1 times}$$

Let $K_n$ be the C*-basic construction [W: Definition 2.1.10] associated to $E_n$ and let $e_n$ be the standard projection as in [W: Section 2.1]. We are then able to find a simultaneous representation of all of the $K_n$ in a fixed C*-algebra. Since $K_0 = A$ we then have that $A$ is also represented there and we find that $e_{n+1} \leq e_n$ for all $n$.

Letting $\mathcal{U}$ be the C*-algebra generated by the union of all the $K_n$ we construct an endomorphism $\beta$ of $\mathcal{U}$ which is not quite an extension of $\alpha$ but which satisfies $\beta(a) = \alpha(a)e_1$ and $\beta(e_n) = e_{n+1}$.

It turns out that the range of $\beta$ is a hereditary subalgebra of $\mathcal{U}$ and hence we may form the crossed-product $\mathcal{U} \rtimes_\beta \mathbb{N}$. We then prove that $A \rtimes_{\alpha,L} \mathbb{N}$ is isomorphic to $\mathcal{U} \rtimes_\beta \mathbb{N}$ (Theorem 6.5).

Crossed-products by endomorphisms with hereditary range are much easier to understand. In particular it is known that $\mathcal{U}$ embeds injectively in $A \rtimes_\beta \mathbb{N}$ and hence we deduce that $A$ is faithfully represented in $A \rtimes_{\alpha,L} \mathbb{N}$ as already mentioned.

* Partially supported by CNPq.
Another consequence of the existence of an isomorphism between $\mathcal{G} \rtimes \beta \mathbb{N}$ and $A \rtimes_{\alpha, L} \mathbb{N}$ is that we get a rather concrete description of the structure of $A \rtimes_{\alpha, L} \mathbb{N}$ and, in particular, of the fixed point subalgebra for the (scalar) gauge action, namely the action of the circle on $A \rtimes_{\alpha, L} \mathbb{N}$ given by

$$\gamma_z(S) = zS, \quad \text{and} \quad \gamma_z(a) = a, \quad \forall a \in A, \quad \forall z \in S^1,$$

where $S$ is the standard isometry in $A \rtimes_{\alpha, L} \mathbb{N}$. That fixed-point algebra, if viewed from the point of view of $\mathcal{G} \rtimes \beta \mathbb{N}$, is well known to be exactly $\mathcal{G}$ (see [M: 4.1]).

Given an action of the circle on a $C^*$-algebra there is a standard way to construct a conditional expectation onto the fixed-point algebra by averaging the action. It is therefore easy to construct a conditional expectation from $A \rtimes_{\alpha, L} \mathbb{N}$ to $\mathcal{G}$. The existence of a conditional expectation onto $A$, however, is an entirely different matter.

Our second main result (Theorem 8.9) is the construction of such a conditional expectation under the special case in which $E$ is of index-finite type [W: 1.2.2]. Precisely we show that there is a (unique under certain circumstances) conditional expectation $G: A \rtimes_{\alpha, L} \mathbb{N} \to A$ such that

$$G(aS^n S^{-m} b) = \delta_{nm} a I_1^{-1} b, \quad \forall a, b \in A, \quad \forall n, m \in \mathbb{N},$$

where $\delta$ is the Kronecker symbol,

$$I_n = \text{ind}(E) \alpha(\text{ind}(E)) \ldots \alpha^{n-1}(\text{ind}(E)),$$

and $\text{ind}(E)$ is the index of $E$ defined by Watatani in [W: 1.2.2], generalizing earlier work of Jones [J] and Kosaki [K].

Our third main result (Theorem 9.6) is related to the KMS states on $A \rtimes_{\alpha, L} \mathbb{N}$ for the one-parameter automorphism group $\sigma$ of $A \rtimes_{\alpha, L} \mathbb{N}$ specified by

$$\sigma_t(S) = h^t S, \quad \text{and} \quad \sigma_t(a) = a, \quad \forall a \in A,$$

where $h$ is any self-adjoint element in the center of $A$ such that $h \geq cI$ for some real number $c > 1$. Under the hypothesis that $E$ is of index-finite type, and hence in the presence of the conditional expectation $G$ above, and also assuming that $E(ab) = E(ba)$ for all $a, b \in A$ (e.g. when $A$ is commutative), we show that all KMS states on $A \rtimes_{\alpha, L} \mathbb{N}$ factor through $G$ and are exactly the states $\psi$ on $A \rtimes_{\alpha, L} \mathbb{N}$ given by $\psi = \phi \circ G$ where $\phi$ is a trace on $A$ such that

$$\phi(a) = \phi(L(h^{-\beta} \text{ind}(E)a))$$

for all $a \in A$. We also show that there are no ground states on $A \rtimes_{\alpha, L} \mathbb{N}$.

We conclude with a brief discussion of the case in which $A$ is commutative and show that the KMS states on $A \rtimes_{\alpha, L} \mathbb{N}$ are related to Ruelle’s work on Statistical Mechanics [R1], [R2].

A word about our notation: most of the time we will be working simultaneously with three closely related algebras, namely the “Toeplitz extension” $\mathcal{T} (A, \alpha, L)$, the crossed-product $A \rtimes_{\alpha, L} \mathbb{N}$, and a concretely realized algebra $\mathcal{G} \rtimes \beta \mathbb{N}$. The features of each of these algebras will most of the time be presented side by side, e.g. each one will contain a distinguished isometry. In order to try to keep our notation simple but easy to understand we have chosen to decorate the notation relative to the first algebra with a “hat”, the one for the third with a “check”, and no decoration at all for $A \rtimes_{\alpha, L} \mathbb{N}$ which is, after all, the algebra that we are most interested in. For example, the three isometries considered will be denoted $\hat{S}$, $S$, and $\check{S}$.

We would like to acknowledge helpful conversations with Marcelo Viana from which some of the intuition for the present work developed.
2. Crossed products.

Throughout this section, and most of this work, we will let $A$ be a unital C*-algebra and $\alpha : A \to A$ be an injective *-endomorphism such that $\alpha(1) = 1$. It is conceivable that some of our results survive without the hypothesis that $\alpha$ be injective but for the sake of simplicity we will stick to the injective case here.

The range of $\alpha$, which will play a predominant role in what follows, will be denoted by $\mathcal{R}$ and we will assume the existence of a non-degenerate\(^1\) conditional expectation

$$E : A \to \mathcal{R}$$

which will be fixed throughout. As in [E2:2.6] it follows that the composition $L := \alpha^{-1}E$ is a transfer operator in the sense of [E2:2.1], meaning a positive linear map $L : A \to A$ such that $L(\alpha(a)b) = aL(b)$, for all $a, b \in A$.

According to Definition 3.1 in [E2] the “Toeplitz extension” $\mathcal{T}(A, \alpha, L)$ is the universal unital C*-algebra generated by a copy of $A$ and an element $\hat{S}$ subject to the relations:

(i) $\hat{S}a = \alpha(a)\hat{S}$, and  
(ii) $\hat{S}a\hat{S} = L(a)$,

for every $a \in A$. As proved in [E2:3.5] the canonical map from $A$ to $\mathcal{T}(A, \alpha, L)$ is injective so we may and will view $A$ as a subalgebra of $\mathcal{T}(A, \alpha, L)$.

Observe that, as a consequence of the fact that $\alpha$ preserves the unit, we have that $1 \in \mathcal{R}$ and hence that $L(1) = \alpha^{-1}(E(1)) = 1$. It follows that

$$\hat{S}^*\hat{S} = \hat{S}^*1\hat{S} = L(1) = 1,$$

and hence we see that $\hat{S}$ is an isometry.

Following [E2:3.6] a redundancy is a pair $(a, k)$ of elements in $\mathcal{T}(A, \alpha, L)$ such that $k$ is in the closure of $\alpha S\hat{S}^*A$, $a$ is in $A$, and

$$ab\hat{S} = kb\hat{S}, \quad \forall b \in A.$$

2.1. Definition. [E2:3.7] The crossed-product of $A$ by $\alpha$ relative to $L$, denoted by $A \rtimes_{\alpha, L} \mathbb{N}$, is defined to be the quotient of $\mathcal{T}(A, \alpha, L)$ by the closed two-sided ideal generated by the set of differences $a - k$, for all\(^2\) redundancies $(a, k)$. We will denote by $q$ the canonical quotient map

$$q : \mathcal{T}(A, \alpha, L) \to A \rtimes_{\alpha, L} \mathbb{N},$$

and by $S$ the image of $\hat{S}$ under $q$.

2.2. Lemma. Given $n, m, j, k \in \mathbb{N}$ and $a, b, c, d \in A$ let $x, y \in \mathcal{T}(A, \alpha, L)$ be given by $x = a\hat{S}^n\hat{S}^*m b$ and $y = c\hat{S}^j\hat{S}^*k d$. Then

$$xy = \begin{cases} a\hat{S}^n(\mathcal{L}^m(bc))\hat{S}^{n-m+j}\hat{S}^*kd, & \text{if } m \leq j, \\ a\hat{S}^n\hat{S}^{*(m-j+k)}\alpha^k(\mathcal{L}^j(bc))d, & \text{if } m \geq j. \end{cases}$$

Proof. If $m \leq j$ one has

$$xy = a\hat{S}^n(\hat{S}^{*(m-j)}(\hat{S}^jbc)\hat{S}^{j-m}\hat{S}^*kd) = a\hat{S}^n\mathcal{L}^m(bc)\hat{S}^{j-m}\hat{S}^*kd = a\alpha^n(\mathcal{L}^m(bc))\hat{S}^{n+j-m}\hat{S}^*kd.$$  

On the other hand, if $m \geq j$ one has

$$xy = a\hat{S}^n\hat{S}^{*(m-j)}(\hat{S}^jbc)\hat{S}^*kd = a\hat{S}^n\hat{S}^{*(m-j)}\mathcal{L}^j(bc)\hat{S}^*kd = a\hat{S}^n\hat{S}^{*(m-j+k)}\alpha^k(\mathcal{L}^j(bc))d. \quad \square$$

As a consequence we have:

---

\(^1\) A conditional expectation $E$ is said to be non-degenerate when $E(a^*a) = 0$ implies that $a = 0$.

\(^2\) We should remark that in Definition 3.7 of [E2] one uses only the redundancies $(a, k)$ such that $a \in \mathbb{R}\mathbb{A}$. But, under the present hypothesis that $\alpha$ preserves the unit, we have that $1 \in \mathcal{R}$ and hence $\mathbb{R}\mathbb{A} = A$.  

---
2.3. Proposition. \( \mathcal{T}(A, \alpha, L) \) is the closed linear span of the set \( X = \{a\hat{S}^n\hat{S}^mb : a, b \in A, n, m \in \mathbb{N} \} \).

Proof. By (2.2) we see that the linear span of \( X \) is an algebra. Since it is also self-adjoint and contains \( A \cup \{\hat{S}\} \) the result follows. \( \square \)

There are may results for \( \mathcal{T}(A, \alpha, L) \) which yield similar results for \( A \rtimes_{\alpha, L} \mathbb{N} \) simply by passage to the quotient, such as (2.2) and (2.3). Most often we will not bother to point these out unless it is relevant to our purposes that we do so.


In this section we will describe certain one-parameter automorphism groups of \( \mathcal{T}(A, \alpha, L) \) and \( A \rtimes_{\alpha, L} \mathbb{N} \) relative to which we will later study KMS states.

3.1. Proposition. Given a unitary element \( u \) in \( \mathcal{Z}(A) \) (the center of \( A \)) there exists a unique automorphism \( \hat{\sigma}_u \) of \( \mathcal{T}(A, \alpha, L) \) such that

\[
\hat{\sigma}_u(\hat{S}) = u\hat{S}, \quad \text{and} \quad \hat{\sigma}_u(a) = a, \quad \forall a \in A.
\]

Moreover \( \hat{\sigma}_u(\text{Ker}(q)) = \text{Ker}(q) \) and hence \( \hat{\sigma}_u \) drops to the quotient providing an automorphism \( \sigma_u \) of \( A \rtimes_{\alpha, L} \mathbb{N} \) such that

\[
\sigma_u(S) = q(u)S, \quad \text{and} \quad \sigma_u(q(a)) = q(a), \quad \forall a \in A.
\]

If \( v \) is another unitary element in \( \mathcal{Z}(A) \) then \( \hat{\sigma}_u \hat{\sigma}_v = \hat{\sigma}_{uv} \).

Proof. Let \( S_u = u\hat{S} \) and observe that for every \( a \) in \( A \) one has

\[
S_u a = u\hat{S} a = u a(a)\hat{S} = a(a) u\hat{S} = a(a) S_u,
\]

and

\[
S_u^* a S_u = \hat{S}^* u^* a u \hat{S} = \hat{S}^* a \hat{S} = L(a).
\]

From the universal property of \( \mathcal{T}(A, \alpha, L) \) it follows that there exists a unique \( * \)-homomorphism \( \hat{\sigma}_u : \mathcal{T}(A, \alpha, L) \to \mathcal{T}(A, \alpha, L) \) such that \( \hat{\sigma}_u(a) = a, \) for all \( a \in A, \) and \( \hat{\sigma}_u(\hat{S}) = S_u \). Given \( v \) as above notice that

\[
\hat{\sigma}_u \hat{\sigma}_v(\hat{S}) = \hat{\sigma}_u(v\hat{S}) = \hat{\sigma}_u(v)\hat{\sigma}_u(\hat{S}) = vu\hat{S} = uv\hat{S} = \hat{\sigma}_{uv}(\hat{S}),
\]

and that \( \hat{\sigma}_u \hat{\sigma}_v(a) = a, \) for all \( a \in A. \) Thus \( \hat{\sigma}_u \hat{\sigma}_v = \hat{\sigma}_{uv}. \) It follows that \( \hat{\sigma}_{u^{-1}} \) is the inverse of \( \hat{\sigma}_u \) and hence \( \hat{\sigma}_u \) is an automorphism.

Let \( (a, k) \) be a reducency. Then

\[
\hat{\sigma}_u(k) \in \hat{\sigma}_u(A \hat{S} \hat{S}^* A) = A u \hat{S} \hat{S}^* u^* A = A \hat{S} \hat{S}^* A.
\]

For every \( b \) in \( A \) we have

\[
\hat{\sigma}_u(k) b \hat{S} = \hat{\sigma}_u(k) b u^{-1} u \hat{S} = \hat{\sigma}_u(k b u^{-1} \hat{S}) = \hat{\sigma}_u(ab u^{-1} \hat{S}) = ab \hat{S},
\]

so \( (a, \hat{\sigma}_u(k)) \) is also a reducency and it follows that \( \hat{\sigma}_u(a - k) \in \text{Ker}(q) \) and hence that \( \hat{\sigma}_u(\text{Ker}(q)) \subseteq \text{Ker}(q) \). Since the same holds for \( \hat{\sigma}_{u^{-1}} \) we have that \( \text{Ker}(q) \subseteq \hat{\sigma}_u(\text{Ker}(q)) \). \( \square \)

Let \( h \in \mathcal{Z}(A) \) be a self-adjoint element such that \( h \geq cI \) for some real number \( c > 0 \). For every \( t \in \mathbb{R} \) we have that \( h^t \) is a unitary in \( \mathcal{T}(A) \) and hence defines an automorphism \( \hat{\sigma}_{ht} \) by (3.1) which we will denote by \( \hat{\sigma}^h \). Again by (3.1) we have that \( \hat{\sigma}^h \hat{\sigma}^k = \hat{\sigma}^{h+k} \), so that \( \hat{\sigma}^h \) is a one-parameter automorphism group of \( \mathcal{T}(A, \alpha, L) \) which is clearly strongly continuous.

3.2. Definition. Both the action \( \hat{\sigma}^h \) defined above and the action \( \sigma^h \) of \( \mathbb{R} \) on \( A \rtimes_{\alpha, L} \mathbb{N} \) obtained by passing \( \hat{\sigma}^h \) to the quotient will be called the *gauge action associated to \( h \).*
It is well known that the map \( \gamma \) with period 2 and the \( R^n \) used in later sections. For every \( x \in A \), \( x_{\gamma} = a \) for all \( a \in A \), and hence defines an action \( \gamma \) of the unit circle on \( \mathcal{F}(A, \alpha, \mathcal{L}) \) such that
\[
\gamma_z(\tilde{S}) = z\tilde{S}, \quad \text{and} \quad \gamma_z(a) = a, \quad \forall a \in A, \quad \forall z \in S^1.
\]

3.3. Definition. Both the action \( \gamma \) defined above and the action \( \gamma \) of the circle group on \( A \otimes \alpha, \mathcal{L} \otimes N \) obtained by passing \( \gamma \) to the quotient will be called the scalar gauge action.

We will later be interested in the fixed point algebra for the scalar gauge action so the following result will be useful:

3.4. Proposition. Let \( B \) be a C*-algebra with a strongly continuous action \( \gamma \) of the circle group. Suppose that \( B \) is the closed linear span of a set \( \{x_i : i \in I\} \) such that for every \( i \in I \) there exists \( n_i \in \mathbb{Z} \) such that \( \gamma_z(x_i) = z^{n_i}x_i \) for all \( z \in \mathbb{C} \). Then the fixed point algebra for \( \gamma \) is the closed linear span of \( \{x_i : n_i = 0\} \).

Proof. It is well known that the map \( P : B \to B \) given by
\[
P(a) = \int_{S^1} \gamma_z(a) \, dz
\]
is a conditional expectation onto the fixed point algebra for \( \gamma \). By direct computation it is easy to see that \( P(x_i) = 0 \) when \( n_i \neq 0 \) and \( P(x_i) = x_i \) when \( n_i = 0 \).

Given a fixed point \( b \) and \( \varepsilon > 0 \) let \( \{\lambda_i\}_i \) be a family of scalars with finitely many nonzero elements such that \( \|b - \sum_{i \in I} \lambda_i x_i\| < \varepsilon \). It follows that
\[
\left\| b - \sum_{i \in I} \lambda_i x_i \right\| = \left\| P\left( b - \sum_{i \in I} \lambda_i x_i \right) \right\| \leq \left\| b - \sum_{i \in I} \lambda_i x_i \right\| < \varepsilon.
\]
Therefore \( b \in \overline{\text{Span}}\{x_i : n_i = 0\} \). \( \square \)

3.5. Corollary. The fixed point subalgebra of \( \mathcal{F}(A, \alpha, \mathcal{L}) \) (resp. \( A \otimes \alpha, \mathcal{L} \otimes N \)) for the scalar gauge action \( \gamma \) (resp. \( \gamma \)) is the closed linear span of the set of elements \( a\tilde{S}^n\tilde{S}^{*n}b \) (resp. \( a\tilde{S}^n\tilde{S}^{*n}b \)) for all \( a, b \in A \) and \( n \in \mathbb{N} \).


In this section we will describe certain conditional expectations and certain Hilbert modules which will be used in later sections. For every \( n \in \mathbb{N} \) we shall let \( \mathcal{R}_n \) denote the range of \( \alpha^n \). Therefore \( \mathcal{R}_0 = A, \mathcal{R}_1 = \mathcal{R} \), and the \( \mathcal{R}_n \) form a descending chain of closed *-subalgebras of \( A \)
\[
A = \mathcal{R}_0 \supseteq \mathcal{R}_1 \supseteq \mathcal{R}_2 \supseteq \ldots.
\]
Clearly each \( \mathcal{R}_n \) is isomorphic to \( A \) under \( \alpha^n \). For each \( n \in \mathbb{N} \) consider the map
\[
E_n : \mathcal{R}_n \to \mathcal{R}_{n+1}
\]
given by \( E_n = \alpha^n E \alpha^{-n} \). It is elementary to verify that each \( E_n \) is a non-degenerate conditional expectation. Likewise, for each \( n \in \mathbb{N} \), the composition
\[
A \xrightarrow{E_0} \mathcal{R}_1 \xrightarrow{E_1} \mathcal{R}_2 \xrightarrow{E_2} \cdots \xrightarrow{E_{n-1}} \mathcal{R}_n
\]
is a non-degenerate conditional expectation onto \( \mathcal{R}_n \), which we denote by \( \mathcal{E}_n \). By default we let \( \mathcal{E}_0 \) be the identity map on \( A \) and it is clear that \( \mathcal{E}_1 = E_0 = E \).

For future use it is convenient to record the following elementary facts:
4.1. Proposition. For every \( n \in \mathbb{N} \) one has that

(i) \( \mathcal{E}_{n+1} = \mathcal{E}_n \mathcal{E} = \alpha \mathcal{E}_n \alpha^{-1} \mathcal{E} \),

(ii) \( \mathcal{E}_{n+1} \mathcal{E}_n = \mathcal{E}_n \mathcal{E}_{n+1} = \mathcal{E}_{n+1} \).

We now need to use a simple construction from the theory of Hilbert modules: let \( B \) be any C*-algebra and let \( C \subseteq B \) be a sub-C*-algebra. Also let \( E : B \to C \) be a non-degenerate conditional expectation. Given a right Hilbert \( B \)-module \( M \) (with inner–product \( \langle \cdot, \cdot \rangle \)) one gets a \( C \)-valued inner–product on \( M \) defining

\[
\langle x, y \rangle_C = E(\langle x, y \rangle), \quad \forall x, y \in M.
\]

We shall denote the Hilbert \( C \)-module obtained by completing \( M \) under the norm \( \| x \|_C = \| \langle x, x \rangle_C \|^{1/2} \) by \( MC \).

We plan to apply this construction in order to obtain a sequence \( \{ M_n \}_{n \in \mathbb{N}} \), where each \( M_n \) is a Hilbert \( \mathcal{R}_n \)-module as follows: let \( M_0 = A \) viewed as a right Hilbert \( A \)-module under the obvious right module structure and inner–product given by \( \langle a, b \rangle = a^* b \), for all \( a \) and \( b \) in \( A \).

Once \( M_n \) is constructed let \( M_{n+1} \) be the \( \mathcal{R}_{n+1} \)-module obtained by applying the procedure described above to \( M_n \) and the conditional expectation \( E_n \). For simplicity we let \( \langle \cdot, \cdot \rangle_n \) denote the inner–product on \( M_n \) and by \( \| \cdot \|_n \) the associated norm. By construction we have that

\[
A = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots
\]

where the inclusion maps are continuous and each \( M_n \) is a dense subset of \( M_{n+1} \) (with respect to \( \| \cdot \|_{n+1} \)). It follows that \( A \) is dense in each \( M_n \) and it is convenient to observe that

\[
\langle a, b \rangle_n = E_{n-1} \cdots E_0(a^* b) = \mathcal{E}_n(a^* b), \quad \forall a, b \in A.
\]

4.2. Proposition. For every \( n \in \mathbb{N} \) there exists an isometric complex-linear map \( \tilde{\alpha}_n : M_n \to M_{n+1} \) such that \( \tilde{\alpha}_n(a) = \alpha(a) \) for all \( a \in A \).

Proof. Given \( a \in A \) we have

\[
\langle \alpha(a), \alpha(a) \rangle_{n+1} = \mathcal{E}_{n+1}(\alpha(a^* a)) = \alpha \mathcal{E}_n \alpha^{-1} E(\alpha(a^* a)) = \alpha \mathcal{E}_n (a^* a) = \alpha \langle a, a \rangle_n.
\]

This implies that \( \| \alpha(a) \|_{n+1} = \| a \|_n \) from where the conclusion easily follows. \( \square \)

4.3. Proposition. For every \( n \in \mathbb{N} \) there exists a contractive complex-linear map \( \tilde{\mathcal{L}}_n : M_{n+1} \to M_n \) such that \( \tilde{\mathcal{L}}_n(a) = \mathcal{L}(a) \) for all \( a \in A \).

Proof. Using the well known fact that \( E(a^*) E(a) \leq E(a^* a) \) (plug \( x := a - E(a) \) in “\( E(x^* x) \geq 0 \)” in order to prove it) we have that

\[
\langle \mathcal{L}(a), \mathcal{L}(a) \rangle_n = \mathcal{E}_n(\alpha^{-1}(E(a)^* E(a))) \leq \mathcal{E}_n \alpha^{-1} E(a^* a) = \\
= \alpha^{-1} \alpha \mathcal{E}_n \alpha^{-1} E(a^* a) = \mathcal{E}_{n+1}(a^* a) = \alpha^{-1} \langle a, a \rangle_{n+1}.
\]

We then have that \( \| \mathcal{L}(a) \|_n \leq \| a \|_{n+1} \) from where one easily deduces the existence of \( \tilde{\mathcal{L}}_n \). \( \square \)

From now on we will denote by \( \mathcal{L}(M_n) \) the C*-algebra of all adjointable operators on \( M_n \).

4.4. Proposition. For every \( n \in \mathbb{N} \) there exists a self-adjoint idempotent \( \tilde{e}_n \in \mathcal{L}(M_n) \) such that \( \tilde{e}_n(a) = \mathcal{E}_n(a) \) for all \( a \in A \).
Proof. For $a \in A$ we have
\[ \langle \mathcal{E}_n(a), \mathcal{E}_n(a) \rangle_n = \mathcal{E}_n(\mathcal{E}_n(a^*) \mathcal{E}_n(a)) = \mathcal{E}_n(a^*) \mathcal{E}_n(a) \leq \mathcal{E}_n(a^*a) = \langle a, a \rangle_n. \]
Therefore $\|\mathcal{E}_n(a)\|_n \leq \|a\|_n$ and hence the correspondence $a \mapsto \mathcal{E}_n(a)$ extends to a bounded linear map $\hat{e}_n : M_n \rightarrow M_n$. For $a, b \in A$ we have that
\[ \langle \hat{e}_n(a), b \rangle_n = \mathcal{E}_n(\mathcal{E}_n(a^*) b) = \mathcal{E}_n(a^*) \mathcal{E}_n(b) = \mathcal{E}_n(a^* \mathcal{E}_n(b)) = \langle a, \hat{e}_n(b) \rangle_n. \]
By continuity it follows that $\langle \hat{e}_n(\xi), \eta \rangle = \langle \xi, \hat{e}_n(\eta) \rangle$ for all $\xi, \eta \in M_n$ so that $\hat{e}_n$ is in fact an adjointable operator on $M_n$. The remaining assertions are now easy to prove. \qed

It should be remarked that $\hat{e}_n$ is precisely the projection introduced in [W: Section 2.1] relative to the conditional expectation $\mathcal{E}_n$. Therefore $\mathcal{E}_n\mathcal{A}$ is the associated reduced $\mathcal{C}^*$-basic construction [W: Definition 2.1.2]. We will soon have more to say about this.

4.5. Proposition. For every $n \in \mathbb{N}$
(i) $\hat{\mathcal{L}}_n \hat{\alpha}_n$ is the identity on $M_n$,
(ii) $\langle \hat{\alpha}_n(\xi), \eta \rangle_{n+1} = \alpha(\langle \xi, \hat{\mathcal{L}}_n(\eta) \rangle_n)$, for all $\xi \in M_n$ and $\eta \in M_{n+1}$, and
(iii) $\hat{\alpha}_n \hat{e}_n \hat{\mathcal{L}}_n = \hat{e}_{n+1}$.

Proof. With respect to (i) we have for all $a \in A$
\[ \hat{\mathcal{L}}_n \hat{\alpha}_n(a) = \mathcal{L}\alpha(a) = \alpha^{-1} \mathcal{E}\alpha(a) = a, \]
so the conclusion follows by continuity. As for (ii) one has for all $a, b \in A$
\[ \langle \alpha(a), b \rangle_{n+1} = \mathcal{E}_{n+1}(\alpha(a^*) b) = \alpha \mathcal{E}_n \alpha^{-1} \mathcal{E}(\alpha(a^*) b) = \alpha \mathcal{E}_n \alpha^{-1}(\alpha(a^*) \mathcal{E}(b)) = \alpha \mathcal{E}_n(\alpha(a^* \mathcal{L}(b))) = \alpha(\langle a, \mathcal{L}(b) \rangle_n). \]

Speaking of (iii), fix $a \in A$ and notice that
\[ \hat{\alpha}_n \hat{e}_n \hat{\mathcal{L}}_n(a) = \alpha \mathcal{E}_n \mathcal{L}(a) = \alpha \mathcal{E}_n \alpha^{-1} \mathcal{E}(a) = \mathcal{E}_{n+1}(a) = \hat{e}_{n+1}(a). \]

4.6. Proposition. For each $n \in \mathbb{N}$ the map
\[ \beta_n : T \in \mathcal{L}(M_n) \mapsto \hat{\alpha}_n T \hat{\mathcal{L}}_n \in \mathcal{L}(M_{n+1}) \]
is a well defined *-monomorphism of $\mathcal{C}^*$-algebras.

Proof. For each $T \in \mathcal{L}(M_n)$ it is clear that $\hat{\alpha}_n T \hat{\mathcal{L}}_n$ is a bounded complex-linear map on $M_n$. Given $\xi, \eta \in M_{n+1}$ notice that
\[ \langle \beta_n(T) \xi, \eta \rangle_{n+1} = \langle \hat{\alpha}_n T \hat{\mathcal{L}}_n(\xi), \eta \rangle_{n+1} = \alpha(\langle T \hat{\mathcal{L}}_n(\xi), \hat{\mathcal{L}}_n(\eta) \rangle_n) = \alpha(\langle \hat{\mathcal{L}}_n(\xi), T^* \hat{\mathcal{L}}_n(\eta) \rangle_n) = \langle \xi, \hat{\alpha}_n T^* \hat{\mathcal{L}}_n(\eta) \rangle_n = \langle \xi, \beta_n(T^*) \eta \rangle_{n+1}. \]
This proves that $\beta_n(T)$ is an adjointable operator on $M_{n+1}$ with $\beta_n(T)^* = \beta_n(T^*)$. So $\beta_n$ is a well defined linear map from $\mathcal{L}(M_n)$ to $\mathcal{L}(M_{n+1})$ which moreover respects the involution. Given $T, S \in \mathcal{L}(M_n)$ we have that
\[ \beta_n(T) \beta_n(S) = \hat{\alpha}_n T \hat{\mathcal{L}}_n \hat{\alpha}_n S \hat{\mathcal{L}}_n = \hat{\alpha}_n T S \hat{\mathcal{L}}_n = \beta_n(T S), \]
proving that $\beta$ is a *-homomorphism. Suppose that $T \in \mathcal{L}(M_n)$ is such that $\beta_n(T) = 0$. Then
\[ 0 = \mathcal{L}_n \beta_n(T) \hat{\alpha}_n = \hat{\alpha}_n T \hat{\mathcal{L}}_n \hat{\alpha}_n = T. \]
Therefore $\beta_n$ is injective. \qed
We now need another result from the theory of Hilbert modules.

**4.7. Lemma.** Under the assumption that $E : B \to C$ is a non-degenerate conditional expectation, and $M$ is a right Hilbert $B$–module, there exists an injective *-homomorphism

$$
\Phi : \mathcal{L}_B(M) \to \mathcal{L}_C(M_C),
$$

such that $\Phi(T)(\xi) = T(\xi)$, for all $T \in \mathcal{L}_B(M)$, and all $\xi \in M$.

**Proof.** Let $T \in \mathcal{L}_B(M)$. Since $T^* T \leq \|T\|^2$ one has for all $\xi \in M$

$$
(T(\xi), T(\xi)) = (T^* T(\xi), \xi) \leq \|T\|^2 \langle \xi, \xi \rangle.
$$

Applying $E$ to the above inequality yields

$$
\langle T(\xi), T(\xi) \rangle_C \leq \|T\|^2 \langle \xi, \xi \rangle_C,
$$

and hence we conclude that $\|T(\xi)\|_C \leq \|T\| \|\xi\|_C$, so that $T$ is bounded with respect to $\| \cdot \|_C$ and hence extends to a bounded linear map $\Phi(T)$ on $M_C$. We leave it for the reader to verify that $\Phi(T)$ indeed belongs to $\mathcal{L}_C(M_C)$ and that the correspondence $T \mapsto \Phi(T)$ is a *-homomorphism.

Given that $\Phi(T)$ is an extension of $T$ it is clear that $\Phi(T) \neq 0$ when $T \neq 0$, so that $\Phi$ is injective. \hfill $\Box$

Applying the above result to the present situation we obtain an inductive sequence of $C^*$-algebras

$$
A = \mathcal{L}(M_0) \xrightarrow{\Phi_0} \mathcal{L}(M_1) \xrightarrow{\Phi_1} \cdots \mathcal{L}(M_n) \xrightarrow{\Phi_n} \cdots
$$

We temporarily denote the inductive limit of this sequence (in the category of $C^*$-algebras) by $B$.

**4.8. Definition.** We will denote by $\mathcal{U}$ the sub-$C^*$-algebra of $B$ generated by $A \cup \{\hat{e}_n : n \in \mathbb{N}\}$.

The following provides some useful information on the algebraic structure of $\mathcal{U}$ (see also [J], [W]):

**4.9. Proposition.** For every $n \in \mathbb{N}$ one has

(i) $\hat{e}_{n+1} \leq \hat{e}_n$,

(ii) $\hat{e}_n a \hat{e}_n = \mathcal{E}_n(a) \hat{e}_n = \hat{e}_n \mathcal{E}_n(a)$,

(iii) the linear span of the set $\{a \hat{e}_n b : n \in \mathbb{N}, \ a, b \in A\}$ is dense in $\mathcal{U}$.

**Proof.** In order to prove (i) we need to verify that $\hat{e}_{n+1} \hat{e}_n = \hat{e}_{n+1}$ or, more precisely, that $\hat{e}_{n+1} \Phi_n(\hat{e}_n) = \hat{e}_{n+1}$ as operators on $M_{n+1}$. Given any $a \in A \subseteq M_{n+1}$ we have

$$
\hat{e}_{n+1} \Phi_n(\hat{e}_n)(a) = \hat{e}_{n+1} \hat{e}_n(a) = \mathcal{E}_{n+1} \mathcal{E}_n(a) = \mathcal{E}_{n+1}(a) = \hat{e}_{n+1}(a).
$$

For all $b \in A \subseteq M_n$ we have

$$
\hat{e}_n a \hat{e}_n(b) = \mathcal{E}_n(a \mathcal{E}_n(b)) = \mathcal{E}_n(a)\mathcal{E}_n(b) = \mathcal{E}_n(a) \hat{e}_n(b).
$$

This proves that $\hat{e}_n a \hat{e}_n = \mathcal{E}_n(a) \hat{e}_n$. Taking adjoints it follows that $\hat{e}_n a \hat{e}_n = \hat{e}_n \mathcal{E}_n(a)$ also. Now let $\mathcal{U}_0$ be the linear span of the set described in (iii). We claim that it is a *-subalgebra of $\mathcal{U}$. Clearly $\mathcal{U}_0$ is self-adjoint so we are left with the task of checking it to be closed under multiplication. In order to see it let $a, b, c, d \in A$, and $n, m \in \mathbb{N}$. We then have

$$(a \hat{e}_n b)(c \hat{e}_m d) = a(\hat{e}_n bc \hat{e}_n) \hat{e}_m d = \cdots$$

where we are assuming, without loss of generality, that $m \geq n$ and hence that $\hat{e}_m \leq \hat{e}_n$ by (i). Using (ii) we conclude that the above equals

$$
\cdots = a \mathcal{E}_n(bc) \hat{e}_n \hat{e}_m d = a \mathcal{E}_n(bc) \hat{e}_m d,
$$

which is seen to belong to $\mathcal{U}_0$. This proves our claim. It is evident that $A \subseteq \mathcal{U}_0$ (because $\hat{e}_0 = 1$), and that $\hat{e}_n \in \mathcal{U}_0$ for all $n$. Since $\mathcal{U}$ is generated by $A \cup \{\hat{e}_n : n \in \mathbb{N}\}$ it follows that $\mathcal{U}_0$ is dense in $\mathcal{U}$. \hfill $\Box$
Recalling the maps $\beta_n$ constructed in (4.6) we have:

4.10. Proposition. For each $n \in \mathbb{N}$ the diagram

\[
\begin{array}{ccc}
\mathcal{L}(M_n) & \xrightarrow{\Phi_n} & \mathcal{L}(M_{n+1}) \\
\downarrow \beta_n & & \downarrow \beta_{n+1} \\
\mathcal{L}(M_{n+1}) & \xrightarrow{\Phi_{n+1}} & \mathcal{L}(M_{n+2})
\end{array}
\]

commutes and hence there exists a unique injective $*$-endomorphism $\beta : B \to B$ of the inductive limit C*-algebra $B$, which coincides with $\beta_n$ on each $\mathcal{L}(M_n)$. Moreover

(i) $\beta(\hat{e}_n) = \hat{e}_{n+1}$, for all $n \in \mathbb{N}$,
(ii) $\beta(a) = \alpha(a) \hat{e}_1 = \hat{e}_1 \alpha(a)$, for all $a \in A$,
(iii) $\mathcal{U}$ is invariant under $\beta$, and
(iv) $\beta(\mathcal{U}) = \hat{e}_1 \mathcal{U} \hat{e}_1$.

Proof. Given $T \in \mathcal{L}(M_n)$ we have for all $a \in A$ that

$\Phi_{n+1}(\beta_n(T))a = \beta_n(T)a = \hat{\alpha}_n T \hat{\mathcal{L}}_n(a) = \hat{\alpha}_n T \mathcal{L}(a)$.

On the other hand

$\beta_{n+1}(\Phi_n(T))a = \alpha_{n+1} \Phi_n(T) \hat{\mathcal{L}}_{n+1}(a) = \alpha_{n+1} \Phi_n(T) \mathcal{L}(a) = \alpha_{n+1} T \mathcal{L}(a)$.

By checking first on the dense set $A \subseteq M_n$ it is easy to see that the following diagram commutes

\[
\begin{array}{ccc}
M_n & \xrightarrow{\hat{\alpha}_n} & M_{n+1} \\
\downarrow \alpha_{n+1} & & \downarrow \\
M_{n+1} & \xrightarrow{\hat{\alpha}_{n+1}} & M_{n+2}
\end{array}
\]

where the vertical arrows are the standard embeddings. In other words $\hat{\alpha}_n(\xi) = \hat{\alpha}_{n+1}(\xi)$ for all $\xi \in M_n$. If we now plug $\xi := T \mathcal{L}(a)$ in this identity we conclude that $\Phi_{n+1}(\beta_n(T)) = \beta_{n+1}(\Phi_n(T))$ as desired. Considering (i) we have

$\beta(\hat{e}_n) = \beta_n(\hat{e}_n) = \hat{\alpha}_n \hat{e}_n \hat{\mathcal{L}}_n = \hat{e}_{n+1}$,

by (4.5.iii). Given $a \in A = \mathcal{L}(M_0)$, and $b \in A \subseteq M_1$, we have

$\beta(a)(b) = \beta_0(a)(b) = \hat{\alpha}_0 a \hat{\mathcal{L}}_0(b) = \alpha(a \mathcal{L}(b)) = \alpha(a) \alpha(\mathcal{L}(b)) = \alpha(a) \alpha(\mathcal{L}(b)) = \alpha(a) E(b) = \alpha(\hat{e}_1 b)$,

so we see that $\beta(a) = \alpha(a) \hat{e}_1$. Taking adjoints we also prove that $\beta(a) = \hat{e}_1 \alpha(a)$, hence proving (ii). It is clear that (iii) follows from (i) and (ii). In order to prove (iv) observe that by (ii) we have $\beta(1) = \hat{e}_1$ so it must be that $\beta(\mathcal{U}) \subseteq \hat{e}_1 \mathcal{U} \hat{e}_1$.

To prove the reverse inclusion it suffices, by (4.9.iii), to show that for all $a, b \in A$ and $n \in \mathbb{N}$ one has that $\hat{e}_1 (a \hat{e}_n b) \hat{e}_1 \in \beta(\mathcal{U})$. Assuming initially that $n \geq 1$ notice that

$\hat{e}_1 (a \hat{e}_n b) \hat{e}_1 = (\hat{e}_1 a \hat{e}_1) \hat{e}_n (\hat{e}_1 b \hat{e}_1) = E(a) \hat{e}_1 \hat{e}_n \hat{e}_1 E(b) = \alpha(\mathcal{L}(a)) \hat{e}_1 \hat{e}_n \hat{e}_1 \alpha(\mathcal{L}(b)) = \beta(\mathcal{L}(a) \hat{e}_{n-1} \mathcal{L}(b))$.

On the other hand if $n = 0$ we have

$\hat{e}_1 (ab) \hat{e}_1 = E(ab) \hat{e}_1 = \alpha(\mathcal{L}(ab)) \hat{e}_1 = \beta(\mathcal{L}(ab))$. \hfill \Box

Recall ([C], [S], [M], [E2: 4.4 and 4.7]) that, since $\beta(\mathcal{U})$ is a hereditary subalgebra of $\mathcal{U}$ by (4.10.iv), the crossed product $\mathcal{U} \rtimes_\beta \mathbb{N}$ is the universal unital C*-algebra generated by a copy of $\mathcal{U}$ and an isometry $\hat{S}$ subject to the relation that

$S x S^* = \beta(x), \quad \forall x \in \mathcal{U}$.

It is well known that $\mathcal{U}$ embeds injectively in $\mathcal{U} \rtimes_\beta \mathbb{N}$ (see the remark after Definition 4.4 in [E2], [S: Section 2], or [M: Section 2]), so we will view $\mathcal{U}$ as a subalgebra of $\mathcal{U} \rtimes_\beta \mathbb{N}$. Since $A$ is a subalgebra of $\mathcal{U}$ we also have that $A \subseteq \mathcal{U} \rtimes_\beta \mathbb{N}$. 


4.11. Proposition. There exists a \(*\)-homomorphism $\phi : A \times_{\alpha, \mathcal{L}} \mathbb{N} \to \mathcal{W} \times_{\beta} \mathbb{N}$ such that $\phi(S) = \hat{S}$, and $\phi(q(a)) = a$, for all $a \in A$, where $q$ is the canonical quotient map from $\mathcal{T}(A, \alpha, \mathcal{L})$ to $A \times_{\alpha, \mathcal{L}} \mathbb{N}$.

Proof. It will be useful to keep in mind that $\hat{e}_1 = \beta(1) = \hat{S} \hat{S}^*$, and hence that $\hat{e}_1 \hat{S} = \hat{S}$. Considering the natural inclusion of $A$ in $\mathcal{W}$ notice that for all $a \in A$ one has

$$\hat{S}a \hat{S}^* \hat{S} = \beta(a) \hat{S} = \alpha(a) \hat{e}_1 \hat{S} = \alpha(a) \hat{S}.$$

Also

$$\hat{S}^* a \hat{S} = \hat{S}^* \hat{e}_1 a \hat{e}_1 \hat{S} = \hat{S}^* E(a) \hat{e}_1 \hat{S} = \hat{S}^* E(a) \hat{S} = \hat{S}^* \alpha(\mathcal{L}(a)) \hat{S} = \hat{S}^* \hat{S} \mathcal{L}(a) = \mathcal{L}(a).$$

It follows from the universal property of $\mathcal{T}(A, \alpha, \mathcal{L})$ that there exists a \(*\)-homomorphism $\phi : \mathcal{T}(A, \alpha, \mathcal{L}) \to \mathcal{W} \times_{\beta} \mathbb{N}$ which is the identity on $A$ and such that $\phi(\hat{S}) = \hat{S}$.

Now let $(a, k)$ be a redundancy in $\mathcal{T}(A, \alpha, \mathcal{L})$. We claim that $a = \phi(k)$. In order to see this note that since $k \in \mathcal{ASS}^* A$ one has that $\phi(k) \in \mathcal{ASS}^* A = \mathcal{A}e_1 A$.

For all $b \in A$ it is assumed that $ab \hat{S} = kb \hat{S}$ so that $ab \hat{S} \hat{S}^* = kb \hat{S} \hat{S}^*$ and hence $ab \hat{e}_1 = \phi(k)b \hat{e}_1$. Observe that all terms occurring in this last identity lie in the algebra generated by $A$ and $\hat{e}_1$, which consists of operators on the Hilbert module $M_1$. In particular, considering $1$ as an element of $M_1$ we have that

$$ab = ab \hat{e}_1(1) = \phi(k)b \hat{e}_1(1) = \phi(k)b.$$

If follows that $a$ and $\phi(k)$ coincide on $A$, which is a dense subspace of $M_1$, and hence that $a = \phi(k)$ as claimed. Therefore $\phi$ vanishes on the ideal generated by the differences $a - k$ for all redundancies $(a, k)$ and hence factors through the quotient yielding the desired map.

We may now answer a question raised in [E2]:

4.12. Theorem. Let $\alpha$ be an injective endomorphism of a unital C*-algebra $A$ with $\alpha(1) = 1$ and let $\mathcal{L}$ be a transfer operator of the form $\mathcal{L} = \alpha^{-1} \cdot E$, where $E$ is a non-degenerate conditional expectation from $A$ to the range of $\alpha$. Then the natural map from $A$ to $A \times_{\alpha, \mathcal{L}} \mathbb{N}$ is injective.

Proof. Given that $\phi(q(a)) = a$, we see that $q$ is injective when restricted to $A$.

From now on we are therefore allowed to identify $A$ and its image $q(A)$ within $A \times_{\alpha, \mathcal{L}} \mathbb{N}$.

5. Consequences of Watatani’s work.

Our next main goal will be to prove that the map $\phi$ of (4.11) is in fact an isomorphism. But before we are able to attack this question we need to do some more work. We begin by introducing some notation. Recalling that $\hat{S}$ is the standard isometry in $\mathcal{T}(A, \alpha, \mathcal{L})$ we will let

$$\hat{e}_n = \hat{S}^n \hat{S}^{*n},$$

and $\hat{K}_n$ be the closed linear span of $A \hat{e}_n A$, so that each $\hat{K}_n$ is a sub-C*-algebra of $\mathcal{T}(A, \alpha, \mathcal{L})$, as well as an $A$–bimodule. Some elementary properties of the $\hat{e}_n$ and the $\hat{K}_n$ are recorded in the next:

5.1. Proposition. For all $n, m \in \mathbb{N}$ with $n \leq m$ one has that

(i) $\hat{e}_m \leq \hat{e}_n$,

(ii) $\overline{\hat{K}_n \hat{K}_m} = \hat{K}_m \hat{K}_n = \hat{K}_m$,

(iii) $\hat{e}_n a \hat{e}_n = \mathcal{E}_n(a) \hat{e}_n = \hat{e}_n \mathcal{E}_n(a)$, for all $a \in A$,

(iv) $\hat{S}^* \hat{K}_{n+1} \hat{S} \subseteq \hat{K}_n$, and

(v) $\hat{S} \hat{K}_n \subseteq \hat{K}_{n+1} \hat{S}$.
Proof. The first point is trivial and hence we omit it. Regarding (iii) we have by induction that
\[ \hat{e}_{n+1}a\hat{e}_n = \hat{S}\hat{e}_n\hat{S}^*a\hat{S}\hat{e}_n\hat{S}^* = \hat{S}\hat{e}_n\mathcal{L}(a)\hat{e}_n\hat{S}^* = \hat{S}\mathcal{E}_n(\mathcal{L}(a))\hat{e}_n\hat{S}^* = \alpha\mathcal{E}_n(\mathcal{L}(a))\hat{e}_{n+1} = \alpha\mathcal{E}_n(a)\hat{e}_{n+1} = \mathcal{E}_n(a)\hat{e}_{n+1}. \]
This proves that \( \hat{e}_n a \hat{e}_n = \mathcal{E}_n(a) \hat{e}_n \) and hence also that \( \hat{e}_n a \hat{e}_n = \mathcal{E}_n(a) \) by taking adjoints. Speaking of (ii) notice that
\[ (A\hat{e}_n A)(A\hat{e}_n A) = A(\hat{e}_n A\hat{e}_n A) \subseteq A\mathcal{E}_n(A)\hat{e}_n A\hat{e}_n A = A\mathcal{E}_n(A)\hat{e}_m A \subseteq \hat{K}_m. \]
proving that \( \hat{K}_n \hat{K}_m \subseteq \hat{K}_m \) so that \( \hat{K}_n \hat{K}_m \subseteq \hat{K}_m \). In order to prove the reverse inclusion notice that for all \( a, b \in A \) one has
\[ a\hat{e}_m b = (a\hat{e}_1)(\hat{e}_m b) \in \hat{K}_m \hat{K}_m. \]
The equality \( \hat{K}_n \hat{K}_m = \hat{K}_m \) follows by taking adjoints. Turning now to the proof of (iv) we have for all \( a, b \in A \)
\[ \hat{S}^* a\hat{e}_{n+1} b\hat{S} = \hat{S}^* a\hat{S}\hat{e}_n\hat{S}^* b\hat{S} = \mathcal{L}(a)\hat{e}_n \mathcal{L}(b) \in \hat{K}_n. \]
As for (v) notice that
\[ \hat{S} a\hat{e}_n b = \alpha(a)\hat{S}\hat{e}_n\hat{S}^* b = \alpha(a)\hat{e}_{n+1} a(b)\hat{S} \in \hat{K}_{n+1} \hat{S}. \]

5.2. Lemma. For each \( n \in \mathbb{N} \) let \( \mathcal{K}_n = q(\hat{K}_n) \), \( \hat{K}_n = \phi(\mathcal{K}_n) \), and \( e_n = q(\hat{e}_n) \). Then
(i) \( \phi(e_n) = \hat{e}_n \),
(ii) \( \mathcal{K}_n = \overline{\mathcal{A}e_n A} \), and
(iii) \( \hat{K}_n = \overline{\mathcal{A}e_n A} \).
Proof. To prove (i) notice that
\[ \phi(e_n) = \phi(q(\hat{e}_n)) = \phi(q(\hat{S}^n \hat{e}^n)) = \hat{S}^n \hat{e}^n = \beta^n(1) = \hat{e}_n. \]
As for (ii) \( \mathcal{K}_n = q(\overline{\mathcal{A}e_n A}) = \overline{\mathcal{A}e_n A} \). Finally \( \hat{K}_n = \phi(\mathcal{K}_n) = \phi(\overline{\mathcal{A}e_n A}) = \overline{\mathcal{A}e_n A} \). \( \square \)

Observe that by (5.2.iii) \( \hat{K}_n \) is precisely the reduced \( C^* \)-basic construction [W: 2.1.2] relative to the conditional expectation \( \mathcal{E}_n : A \to \mathcal{R}_n \).

In trying to prove that the map \( \phi \) of (4.11) is injective a crucial step will be taken by the following important consequence of [W].

5.3. Proposition. The following \( * \)-homomorphisms are in fact \( * \)-isomorphisms
(i) \( q : \hat{K}_n \to \mathcal{K}_n \),
(ii) \( \phi : \mathcal{K}_n \to \hat{K}_n \),
(iii) \( \phi \circ q : \hat{K}_n \to \hat{K}_n \).
Proof. By [W: 2.2.9] we have that \( \hat{K}_n \) is canonically isomorphic to the unreduced \( C^* \)-basic construction relative to \( \mathcal{E}_n \) and thus possesses the universal property described in [W: 2.2.7].

Supposing that \( \mathcal{T}(A, \alpha, \mathcal{L}) \) is faithfully represented on a Hilbert space \( \mathcal{H} \) observe that by (5.1.iii) the triple \( (id_A, \hat{e}_n, \mathcal{H}) \) is a covariant representation of the conditional expectation \( \mathcal{E}_n \), according to Definition 2.2.6 in [W]. It follows that there exists a \( * \)-representation \( \rho \) of \( \hat{K}_n \) on \( \mathcal{H} \) such that \( \rho(a\hat{e}_n b) = a\hat{e}_n b \) for all \( a, b \in A \). Since \( \phi \circ q \) maps \( a\hat{e}_n b \) to \( a\hat{e}_n b \) we see that \( \phi \circ q \) and \( \rho \) are each others inverse, hence proving (iii). This implies that \( q \) is injective on \( \hat{K}_n \) and since \( q \) is obviously also surjective (i) is proven. Clearly (ii) follows from (i) and (iii). \( \square \)
The following elementary properties should also be noted:

5.4. Proposition. Let \( n, m \in \mathbb{N} \) with \( n \leq m \). Then

(i) \( \bar{\mathcal{K}}_n \mathcal{K}_m = \mathcal{K}_m \mathcal{K}_n = \mathcal{K}_m \).

(ii) \( \bar{\mathcal{K}}_n \mathcal{K}_m = \mathcal{K}_m \bar{\mathcal{K}}_n = \bar{\mathcal{K}}_m \).

(iii) Denote by \( \psi_n : X_n \to Y_n \) any one of the isomorphisms in (5.3.i–iii). Then for every \( x_n \in X_n \) and \( x_m \in X_m \) one has that \( \psi_m(x_n x_m) = \psi_n(x_n) \psi_m(x_m) \) and \( \psi_m(x_m x_n) = \psi_m(x_m) \psi_n(x_n) \).

Taking \( n = 0 \) in (5.4.iii), in which case \( \bar{\mathcal{K}}_n = \mathcal{K}_n = \bar{\mathcal{K}}_n = A \), we see that all of the isomorphisms in (5.3.i–iii) are also \( A \)-bimodule maps.


We now wish to study a generalization of the notion of redundancy. For this we need the following fact for which we have found no reference in the literature.

6.1. Lemma. Let \( \mathcal{B} \) and \( \mathcal{J} \) be closed *-subalgebras of some C*-algebra \( \mathcal{C} \) such that \( \mathcal{J} \mathcal{B} = \mathcal{J} \). If \( x \in \mathcal{J} \) and \( xb \subseteq \mathcal{B} \) then \( x \in \mathcal{B} \).

Proof. Viewing \( \mathcal{J} \) as a right Banach \( \mathcal{B} \)-module we have by the Cohen-Hewitt factorization theorem [HR: 32.22] that \( x = ya \) for some \( y \in \mathcal{J} \) and \( a \in \mathcal{B} \). Choosing an approximate unit \( \{u_i\}_i \) for \( \mathcal{B} \) we have that

\[
\sum_{i=0}^{n} u_i = \lim_{i \to \infty} a_{n} = \lim_{i \to \infty} x_{u_i} \in \mathcal{B} \subseteq \mathcal{B}.
\]

\( \square \)

6.2. Definition. Let \( n \geq 1 \) be an integer. A redundancy of order \( n \), or an \( n \)-redundancy, is a finite sequence \( (a_0, a_1, \ldots, a_n) \in \prod_{i=0}^{n} \bar{\mathcal{K}}_n \) such that \( \sum_{i=0}^{n} a_i x = 0 \), for all \( x \in \bar{\mathcal{K}}_n \).

Up to a minus sign the above notion generalizes the notion of redundancy introduced in [E2]. In fact it is easy to see that the pair \((a, k)\) is a redundancy according to [E2] if and only if \((a, -k)\) is a 1-redundancy.

We now come to a main technical result:

6.3. Proposition. Let \( n \geq 1 \) and let \( (a_0, a_1, \ldots, a_n) \) be a redundancy of order \( n \). Then \( \sum_{i=0}^{n} q(a_i) = 0 \).

Proof. We proceed by induction observing that the case \( n = 1 \) follows easily from the observation already made that 1-redundancies are simply redundancies. So let \( n > 1 \) and let \((a_0, a_1, \ldots, a_n)\) be an \( n \)-redundancy. Given \( b \in \mathcal{A} \) let \( a'_i = \tilde{S}^* b^* a_i b \tilde{S} \) for all \( i = 0, 1, \ldots, n \) and observe that by (5.1.v) one has that

\[
\left( \sum_{i=0}^{n} a'_i \right) \bar{\mathcal{K}}_{n-1} = \tilde{S}^* b^* \left( \sum_{i=0}^{n} a_i \right) b \tilde{S} \bar{\mathcal{K}}_{n-1} \subseteq \tilde{S}^* b^* \left( \sum_{i=0}^{n} a_i \right) b \bar{\mathcal{K}}_{n} \tilde{S} = \{0\}.
\]

Since \( a'_0 = \mathcal{L}(b^* a_0 b) \in \mathcal{A} \), and \( a'_i \in \bar{\mathcal{K}}_{n-1} \) for \( i \geq 1 \), by (5.1.iv), we have that \((a'_0 + a'_1, a'_2, \ldots, a'_n)\) is a redundancy of order \( n - 1 \). So \( \sum_{i=0}^{n} q(a'_i) = 0 \) by induction. Equivalently \( \sum_{i=0}^{n} a'_i \in \text{Ker}(q) \). Assume first that \( \sum_{i=0}^{n} a_i \) is positive. We then have that

\[
\text{Ker}(q) \ni \sum_{i=0}^{n} a'_i = \tilde{S}^* b^* \left( \sum_{i=0}^{n} a_i \right) b \tilde{S} = \tilde{S}^* b^* \left( \sum_{i=0}^{n} a_i \right)^{1/2} \left( \sum_{i=0}^{n} a_i \right)^{1/2} b \tilde{S}.
\]

It follows that \( \left( \sum_{i=0}^{n} a_i \right)^{1/2} b \tilde{S} \in \text{Ker}(q) \) and hence also \( \left( \sum_{i=0}^{n} a_i \right) b \tilde{S} \in \text{Ker}(q) \). Multiplying this on the right by \( \tilde{S}^{n-2}(\tilde{S}^*)^{n-1}c \), for \( c \in \mathcal{A} \), we see that

\[
\left( \sum_{i=0}^{n} a_i \right) b \tilde{e}_{n-1} c \in \text{Ker}(q), \quad \forall b, c \in \mathcal{A},
\]
and hence that \( \left( \sum_{i=0}^{n} a_i \right) \hat{K}_{n-1} \subseteq \text{Ker}(q) \). For all \( y \in q(\hat{K}_{n-1}) = \mathcal{K}_{n-1} \) it follows that

\[
q(a_n)y = -\sum_{i=0}^{n-1} q(a_i)y,
\]

from where we deduce that \( q(a_n) \mathcal{K}_{n-1} \subseteq \mathcal{K}_{n-1} \). By (6.1) with \( B = \mathcal{K}_{n-1} \) and \( J = \mathcal{K}_n \) we have that \( q(a_n) \in \mathcal{K}_{n-1} \) and hence there exists \( b_n \in \hat{K}_{n-1} \) such that \( q(b_n) = q(a_n) \). Observe that for every \( x \in \hat{K}_{n-1} \) we then have

\[
q \left( \sum_{i=0}^{n-1} a_i x + b_n x \right) = \sum_{i=0}^{n-1} q(a_i) q(x) + q(a_n) q(x) = 0,
\]

by (6.4). Observing that the term within the big parenthesis above lies in \( \hat{K}_{n-1} \), we have by (5.3.i) that

\[
\sum_{i=0}^{n-1} a_i x + b_n x = 0, \quad \forall x \in \hat{K}_{n-1},
\]

and hence \( (a_0, a_1, \ldots, a_{n-2}, a_{n-1} + b_n) \) is a redundancy of order \( n-1 \). Once again by the induction hypothesis it follows that

\[
0 = \sum_{i=0}^{n-1} q(a_i) + q(b_n) = \sum_{i=0}^{n} q(a_i).
\]

Without assuming that \( \sum_{i=0}^{n} a_i \) be positive one can expand the expression \( \left( \sum_{i=0}^{n} a_i \right)^* \left( \sum_{i=0}^{n} a_i \right) \) and rearrange its terms in order to form a redundancy \( (b_0, b_1, \ldots, b_n) \) such that

\[
\sum_{i=0}^{n} b_i = \left( \sum_{i=0}^{n} a_i \right)^* \left( \sum_{i=0}^{n} a_i \right)
\]

and the conclusion will follow easily.

6.5. Theorem. The map \( \phi : A \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \to \hat{U} \rtimes_{\beta} \mathbb{N} \) of (4.11) is an isomorphism.

Proof. We begin by proving that \( \phi \) is surjective. Since \( \phi \) is the identity on \( A \) and since \( \phi(S^n S^{*n}) = S^n 1 S^{*n} = \beta^n(1) = \epsilon_n \) we have that \( \hat{U} \), which is generated by \( A \cup \{ \epsilon_n : n \in \mathbb{N} \} \), is contained in the range of \( \phi \). On the other hand \( \phi(S) = \hat{S} \) and \( \hat{U} \rtimes_{\beta} \mathbb{N} \) is generated by \( \hat{U} \cup \{ \hat{S} \} \). So we see that \( \phi \) is indeed surjective.

Using the universal property of \( \hat{U} \rtimes_{\beta} \mathbb{N} \) it is easy to see that there exists a circle action \( \hat{\gamma} \) on \( \hat{U} \rtimes_{\beta} \mathbb{N} \) such that

\[
\hat{\gamma}_z(\hat{S}) = z\hat{S}, \quad \text{and} \quad \hat{\gamma}_z(f) = f, \quad \forall f \in \hat{U}, \quad \forall z \in S^1.
\]

Since \( \phi \) is clearly covariant with respect to \( \gamma \) and \( \hat{\gamma} \), if we prove that \( \phi \) is injective on the fixed point subalgebra for \( \gamma \), which we denote by \( F \), then by [El1: 2.9] we would have proven that \( \phi \) is injective. Recall from (3.5) that \( F = \overline{\text{span}}\{ a S^n S^{*n} b : a, b \in A, \ n \in \mathbb{N} \} \). If we further observe that \( S^n S^{*n} = q(\hat{S} S^{*n}) = q(\epsilon_n) = \epsilon_n \) we see that

\[
F = \sum_{n \in \mathbb{N}} \mathcal{K}_n.
\]

In order to prove that \( \phi \) is injective it is thus enough to show it to be injective, and hence isometric, on each subalgebra of the form

\[
F_n = \sum_{0 \leq i \leq n} \mathcal{K}_i,
\]

for \( n \in \mathbb{N} \). Applying [P: 1.5.8] repeatedly it is easy to see that \( \sum_{0 \leq i \leq n} \mathcal{K}_i \) is closed so that, in fact, \( F_n = \sum_{0 \leq i \leq n} \mathcal{K}_i \).
Let $a \in F_n$ be such that $\phi(a) = 0$ and write $a = \sum_{i=0}^n a_i$, with $a_i \in K_i$. For every $i = 0,\ldots,n$ choose $b_i \in K_i$ with $q(b_i) = a_i$ (such $b_i$ exists and is unique by (5.3.i)). We now claim that $(b_0,b_1,\ldots,b_n)$ is a redundancy of order $n$. In fact, for every $x \in K_n$ one has that
\[ \phi \circ q \left( \sum_{i=0}^n b_i x \right) = \phi \left( \sum_{i=0}^n a_i \right) \phi(q(x)) = \phi(a) \phi(q(x)) = 0. \]
By (5.3.iii) we have that $\sum_{i=0}^n b_i x = 0$ hence proving our claim. Employing (6.3) we then conclude that $a = \sum_{i=0}^n q(b_i) = 0$. \hfill \Box

7. A bit of cohomology.

We will set this section aside to list certain elementary definitions and facts about an ingredient of cohomological flavor which will be recurrent in our development.

7.1. Definition. Given $a \in A$ and $n \in \mathbb{N}$ we will let
\[ a^{[n]} := a \alpha(a) \cdots \alpha^{n-1}(a) \]
with the convention that $a^{[0]} = 1$.

It is elementary to prove that:

7.2. Proposition. For all $a \in A$ and $n,m \in \mathbb{N}$ one has that $a^{[n+m]} = a^{[n]} \alpha^n(a^{[m]})$.

Although $\mathcal{Z}(A)$ is not necessarily invariant by $\alpha$ observe that given $a,b \in \mathcal{Z}(A)$ and $n,m \in \mathbb{N}$ one has that $\alpha^n(a)$ and $\alpha^m(b)$ commute. In fact, supposing without loss of generality that $n \leq m$, observe that
\[ \alpha^n(a)\alpha^m(b) = \alpha^n(\alpha^m(b) a) = \alpha^n(b)\alpha^n(a). \]

This shows that, when $a \in \mathcal{Z}(A)$, the order of the factors in the definition of $a^{[n]}$ above is irrelevant. It is also easy to conclude that:

7.3. Proposition. For all $a,b \in \mathcal{Z}(A)$ and $n \in \mathbb{N}$ one has that
\begin{enumerate}[(i)]
  \item $a^{[n]} b^{[m]} = (ab)^{[n]}$,
  \item if $a$ is invertible then $(a^{[n]})^{-1} = (a^{-1})^{[n]}$,
  \item if $a$ is a self-adjoint and $0 \leq a \leq c$, where $c \in \mathbb{R}$, then $0 \leq a^{[n]} \leq c^n$.
\end{enumerate}

7.4. Definition. From now on we will denote by $C_A(\mathcal{R}_n)$ the commutant of $\mathcal{R}_n$ in $A$, that is, the set of elements in $A$ which commute with $\mathcal{R}_n$.

Since the $\mathcal{R}_n$ are decreasing it is clear that the $C_A(\mathcal{R}_n)$ are increasing. Moreover, if $a \in \mathcal{Z}(A)$ it is clear that $\alpha^n(a) \in \mathcal{Z}(\mathcal{R}_k) \subseteq C_A(\mathcal{R}_k)$. From this one immediately has:

7.5. Proposition. For all $a \in \mathcal{Z}(A)$ and $n \geq 1$ one has that $a^{[n]} \in C_A(\mathcal{R}_n^{-1})$.

In connection with our transfer operator $\mathcal{L}$ we will later need the following fact:

7.6. Proposition. For every invertible element $\lambda \in \mathcal{Z}(A)$, $a \in A$, and $n,m,p \in \mathbb{N}$ one has
\begin{enumerate}[(i)]
  \item $\lambda^{-[m+p]} a^m \mathcal{L}^n (\lambda^{[n+p]} a) = \lambda^{-[m]} a^m \mathcal{L}^n (\lambda^{[n]} a)$,
  \item $\alpha^m \mathcal{L}^n (a \lambda^{[n+p]}) \lambda^{-[m+p]} = \alpha^m \mathcal{L}^n (a \lambda^{[n]}) \lambda^{-[m]}$,
\end{enumerate}
where by $\lambda^{-[m]}$ we mean $(\lambda^{-1})^{[m]}$.

Proof. Observing that for all $x,y \in A$ we have
\[ \alpha^m \mathcal{L}^n (\alpha^m(x) y) = \alpha^m (x \mathcal{L}^n (y)) = \alpha^m (x) \alpha^m \mathcal{L}^n (y), \]
and using (7.2) to compute $\lambda^{-[m+p]}$ and $\lambda^{-[n+p]}$ we have
\[ \lambda^{-[m+p]} \alpha^m \mathcal{L}^n (\lambda^{[n+p]} a) = \lambda^{-[m]} \alpha^m (\lambda^{-[p]} a) \alpha^m \mathcal{L}^n (\lambda^{[n]} a) \alpha^n (\lambda^{[p]} a) = \lambda^{-[m]} a^m \mathcal{L}^n (\lambda^{[n]} a). \]
This proves (i) and the proof of (ii) is similar. \hfill \Box
8. Finite Index automorphisms.

Given a C*-algebra $B$ and a closed *-subalgebra $C$ recall from [W: 1.2.2 and 2.1.6] that a conditional expectation $E : B \to C$ is said to be of index-finite type if there exists a quasi-basis for $E$, i.e. a finite sequence $\{u_1, \ldots, u_m\} \subseteq B$ such that

$$a = \sum_{i=1}^{m} u_i E(u_i^*a), \quad \forall a \in B.$$  

In this case one defines the index of $E$ by

$$\text{ind}(E) = \sum_{i=1}^{m} u_i u_i^*.$$  

It is well known that $\text{ind}(E)$ does not depend on the choice of the $u_i$’s, that it belongs to the center of $B$ [W: 1.2.8] and is invertible [W: 2.3.1].

8.1. Definition. We shall say that a pair $(\alpha, E)$ is a finite index endomorphism of the C*-algebra $A$ if $\alpha$ is a *-endomorphism of $A$ and $E$ is a conditional expectation of index-finite type from $A$ to the range of $\alpha$.

Throughout this section we will fix a finite index endomorphism $(\alpha, E)$ and a quasi-basis $\{u_1, \ldots, u_m\}$ for $E$. As before we will let $L$ be the transfer operator given by $L = \alpha^{-1}\circ E$.

8.2. Proposition. For every $n \in \mathbb{N}$ one has that $\bar{e}_n = \sum_{i=1}^{m} \alpha^n(u_i) \bar{e}_{n+1} \alpha^n(u_i^*)$.

Proof. Observe that for all $a \in A \subseteq M_1$ one has that

$$\sum_{i=1}^{m} u_i \bar{e}_1 u_i^*(a) = \sum_{i=1}^{m} u_i E(u_i^*a) = a,$$

so that $1 = \sum_{i=1}^{m} u_i \bar{e}_1 u_i^*$, hence proving the statement for $n = 0$. Assuming that $n \geq 1$ apply the endomorphism $\beta$ of (4.10) to both sides of the expression $\bar{e}_{n-1} = \sum_{i=1}^{m} \alpha^{n-1}(u_i) \bar{e}_n \alpha^{n-1}(u_i^*)$ to conclude that

$$\bar{e}_n = \beta(\bar{e}_{n-1}) = \sum_{i=1}^{m} \beta(\alpha^{n-1}(u_i)) \bar{e}_{n+1} \beta(\alpha^{n-1}(u_i^*)) =$$

$$= \sum_{i=1}^{m} \alpha^n(u_i) \bar{e}_1 \bar{e}_{n+1} \alpha^n(u_i^*) = \sum_{i=1}^{m} \alpha^n(u_i) \bar{e}_{n+1} \alpha^n(u_i^*).$$

As a consequence we have:

8.3. Proposition. For every $n \in \mathbb{N}$ one has that $\mathcal{K}_n \subseteq \mathcal{K}_{n+1}$.

Proof. Given $a, b \in A$ observe that by (8.2) one has that

$$a \bar{e}_n b = \sum_{i=1}^{m} a \alpha^n(u_i) \bar{e}_{n+1} \alpha^n(u_i^*) b,$$

and hence $A \bar{e}_n A \subseteq A \bar{e}_{n+1} A$. The conclusion then follows from (5.2.iii).

It is our next major goal to define a conditional expectation $G : A \times_{\alpha, L} \mathbb{N} \to A$. In order to do this we will first define conditional expectations $G_n : \mathcal{K}_n \to A$. Considering that any such conditional expectation is an $A$–bimodule map we will begin with a brief study of $A$–bimodule maps. So let $f : \mathcal{K}_n \to A$ be any $A$–bimodule map. Observing that $\bar{e}_n$ commutes with $R_n$ by (4.9.ii) notice that for all $x \in R_n$ we have that

$$xf(\bar{e}_n) = f(x \bar{e}_n) = f(\bar{e}_n x) = f(\bar{e}_n) x,$$

so $f(\bar{e}_n) \in C_A(R_n)$.
8.4. Proposition. For every \( \lambda \in C_A(\mathcal{R}_n) \) there is one and only one \( A \)-bimodule map \( f : \hat{K}_n \to A \) such that
\[
 f(\alpha \hat{e}_n b) = a \lambda b, \quad \forall a, b \in A.
\]

If \( \lambda \geq 0 \) then \( f \) is a positive map and vice-versa.

Proof. Recall from [W:2.2.2] that \( \hat{K}_n \), being the C*-basic construction relative to the conditional expectation \( \mathcal{E}_n : A \to \mathcal{R}_n \), is isomorphic to the algebraic tensor product (no completion) \( A \otimes_{\mathcal{R}_n} A \) under the map
\[
 \psi : a \otimes b \in A \otimes_{\mathcal{R}_n} A \mapsto a \hat{e}_n b \in \hat{K}_n.
\]

Given \( \lambda \in C_A(\mathcal{R}_n) \) we have that the map
\[
 (a, b) \in A \times A \mapsto a \lambda b \in A
\]
is \( \mathcal{R}_n \)-balanced, which in turn defines a linear map \( \hat{f} : A \otimes_{\mathcal{R}_n} A \to A \) such that \( \hat{f}(a \otimes b) = a \lambda b \), for all \( a \) and \( b \) in \( A \). Composing this with the inverse of the isomorphism \( \psi \) defined above yields a linear map \( f : \hat{K}_n \to A \) such that \( f(a \hat{e}_n b) = a \lambda b \). It is now easy to see that \( f \) is an \( A \)-bimodule map and that it is uniquely determined in terms of \( \lambda \).

If \( f \) is positive it is clear that \( \lambda = f(\hat{e}_n) \geq 0 \). Conversely suppose that \( \lambda \geq 0 \). Given \( x \in \hat{K}_n \) of the form \( x = \sum_{i=1}^{m} a_i \hat{e}_n b_i \) notice that
\[
 x^* x = \sum_{i, j=1}^{m} b_i^* \hat{e}_n a_i^* a_j \hat{e}_n b_j = \sum_{i, j=1}^{m} b_i^* \mathcal{E}_n(a_i^* a_j) \hat{e}_n b_j,
\]
by (4.9.ii). Since conditional expectations are completely positive by [T:IV.3.4], we have that \( \{ \mathcal{E}_n(a_i^* a_j) \}_{i,j} \) is a positive matrix and hence there exists an \( m \times m \) matrix \( c = \{ c_{ij} \}_{ij} \) over \( \mathcal{R}_n \) such that
\[
 \mathcal{E}_n(a_i^* a_j) = \sum_{k=1}^{m} c_{ki} c_{kj}, \quad \forall i, j = 1, \ldots, m.
\]
Therefore
\[
 x^* x = \sum_{i, j, k=1}^{m} b_i^* c_{kj} \hat{e}_n c_{kj} b_j = \sum_{k=1}^{m} d_k^* \hat{e}_n d_k,
\]
where \( d_k = \sum_{j=1}^{m} c_{kj} b_j \). If follows that
\[
f(x^* x) = \sum_{k=1}^{m} d_k^* \lambda d_k \geq 0.
\]

As mentioned above \( \text{ind}(E) \in \mathcal{Z}(A) \) so we have by (7.5) that
\[
 I_n := \left( \text{ind}(E) \right)^{[n]} \in C_A(\mathcal{R}_{n-1}) \subseteq C_A(\mathcal{R}_n), \quad \forall n \geq 1.
\] (8.5)

8.6. Proposition. For each \( n \in \mathbb{N} \) let \( G_n : \hat{K}_n \to A \) be the unique \( A \)-bimodule map such that
\[
 G_n(a \hat{e}_n b) = a I_n^{-1} b, \quad \forall a, b \in A,
\]
given by (8.4). Then for every \( n \in \mathbb{N} \) one has that \( G_n \) is a positive contractive conditional expectation from \( \hat{K}_n \) to \( A \). Moreover the restriction of \( G_{n+1} \) to \( \hat{K}_n \) coincides with \( G_n \).
Observing that $I_{n+1} = C_A(R_n)$ by (8.5) we see that the above equals

$$\cdots = \sum_{i=1}^{m} \alpha^n(u_i) \alpha^n(u_i^*) I_{n+1}^{-1} = \alpha^n(\text{ind}(E)) I_{n+1}^{-1} = I_{n+1} = G_n(e_n).$$

This proves that $G_{n+1}(e_n) = G_n(e_n)$ from where one easily deduces that $G_{n+1}|\mathcal{K}_n = G_n$. It follows that each $G_n$ coincides with $G_0$ on $\mathcal{K}_n$. In other words $G_n$ is the identity on $A$ and, being an $A$-bimodule map, we see that it is in fact a conditional expectation onto $A$. Since $I_n$ is positive by (7.3.iii) we have by (8.4) that $G_n$ is a positive map. To conclude observe that a positive conditional expectation is always contractive. \qed

8.7. Remark. We should remark that $G_1$ is precisely the dual conditional expectation defined in [W: 2.3.2]. For $n \geq 2$, even though each $\mathcal{E}_n$ is a conditional expectation of index-finite type and $\mathcal{K}_n$ is the C*-basic construction relative to $\mathcal{E}_n$, $G_n$ may not be the dual conditional expectation if $A$ is non-commutative. This is due to the fact that $\text{ind}(\mathcal{E}_n)$ may differ from $I_n$. See also [W: 1.7.1].

Proposition (8.6) says that the $G_n$ are compatible with each other and hence may be put together in the following way:

8.8. Proposition. There exists a conditional expectation $\tilde{F}: \mathcal{W} \to A$ such that

$$\tilde{F}(a e_n b) = a I_n^{-1} b, \quad \forall a, b \in A, \quad \forall n \in \mathbb{N}.$$ 

If $A$ is commutative there is no other conditional expectation from $\mathcal{W}$ to $A$.

Proof. From (4.9.iii) it follows that $\mathcal{W} = \sum_{n \in \mathbb{N}} A e_n A = \sum_{n \in \mathbb{N}} \mathcal{K}_n$, but since the $\mathcal{K}_n$ are increasing by (8.3), we see that $\mathcal{W}$ is in fact the inductive limit of the $\mathcal{K}_n$. The existence of $\tilde{F}$ then follows easily from (8.6).

Since $\mathcal{K}_n$ is the C*-basic construction relative to $\mathcal{E}_n$, it follows from [W: 1.6.4] that there exists a unique conditional expectation from $\mathcal{K}_n$ to $A$, under the hypothesis that $A$ is commutative. Therefore any conditional expectation $F'$ from $\mathcal{W}$ to $A$ must coincide with $G_n$ on each $\mathcal{K}_n$ and hence $F' = \tilde{F}$. \qed

We now come to one of our main results:

8.9. Theorem. Let $A$ be a unital C*-algebra and let $(\alpha, E)$ be a finite index endomorphism of $A$ such that $\alpha$ is injective and preserves the unit. Then there exists a conditional expectation $G: A \rtimes_{\alpha, E} \mathbb{N} \to A$ such that

$$G(a S^n S^m b) = \delta_{nm} a I_n^{-1} b, \quad \forall a, b \in A, \quad \forall n, m \in \mathbb{N},$$

where $\delta$ is the Kronecker symbol. If $A$ is commutative any conditional expectation from $A \rtimes_{\alpha, E} \mathbb{N}$ to $A$ which is invariant under the scalar gauge action $\gamma$ (see (3.3)) coincides with $G$.

Proof. Identifying $A \rtimes_{\alpha, E} \mathbb{N}$ and $\mathcal{W} \rtimes_{\beta} \mathbb{N}$ via the isomorphism $\phi$ of (6.5) it is enough to prove the corresponding result for $\mathcal{W} \rtimes_{\beta} \mathbb{N}$, with $S$ replacing $S$, and the action $\gamma$ described in the proof of (6.5) replacing $\gamma$. Consider the operator $\tilde{P}$ on $\mathcal{W} \rtimes_{\beta} \mathbb{N}$ given by

$$\tilde{P}(a) = \int_{\mathcal{W}} \gamma_z(a) dz, \quad \forall a \in \mathcal{W} \rtimes_{\beta} \mathbb{N}.$$ 

As mentioned in the proof of (3.4) $\tilde{P}$ is a conditional expectation onto the fixed point algebra for $\gamma$. Considering that $\mathcal{W} \rtimes_{\beta} \mathbb{N}$ is the closed linear span of the set $\{ a S^n S^m b : a, b \in A, \ n, m \in \mathbb{N} \}$ one may use (3.4) in order to prove that the fixed point algebra for $\gamma$ coincides with $\mathcal{W}$. The composition $G := \tilde{F} \circ \tilde{P}$ is therefore the conditional expectation sought. Now suppose that $A$ is commutative and let $G'$ be any conditional expectation from $\mathcal{W} \rtimes_{\beta} \mathbb{N}$ to $A$. By (8.8) we have that $G'|\mathcal{W} = \tilde{F}$. If $G'$ is moreover invariant under $\gamma$ we have for all $a \in \mathcal{W} \rtimes_{\beta} \mathbb{N}$ that

$$G'(a) = G'(\int_{\mathcal{W}} \gamma_z(a) dz) = G'(\tilde{P}(a)) = \tilde{F}(\tilde{P}(a)) = G(a).$$

As an immediate consequence we have:
8.10. Corollary. There exists a conditional expectation $\hat{G} : \mathcal{T}(A, \alpha, \mathcal{L}) \to A$ such that

$$\hat{G}(aS^n\hat{S}^mb) = \delta_{nm}aI_n^{-1}b, \quad \forall a, b \in A, \quad \forall n, m \in \mathbb{N}.$$ 

Proof. It is enough to put $\hat{G} = G \circ q$. \qed

9. KMS states.

Throughout this section and until further notice we will assume the following:


(i) $A$ is a unital C*-algebra,
(ii) $\alpha$ is an injective endomorphism of $A$ such that $\alpha(1) = 1$,
(iii) $E$ is a conditional expectation from $A$ to the range of $\alpha$,
(iv) $E$ is of index-finite type,
(v) $\mathcal{L}$ is the transfer operator given by $\mathcal{L} = \alpha^{-1} \circ E$,
(vi) $h$ is a fixed self-adjoint element in the center of $A$ such that $h \geq cI$ for some real number $c > 0$ (later we will actually require that $c > 1$),
(vii) $\hat{\sigma}$ and $\sigma$ will denote the gauge actions referred to in (3.2) as $\hat{\sigma}^h$ and $\sigma^h$, respectively.

The purpose of this section will be to study the KMS states on $\mathcal{T}(A, \alpha, \mathcal{L})$ and $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ relative to $\hat{\sigma}$ and $\sigma$. Whenever we say that a state is a KMS state on $\mathcal{T}(A, \alpha, \mathcal{L})$ (resp. $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$) it will be with respect to $\hat{\sigma}$ (resp. $\sigma$).

Observe that the canonical quotient map $q$ from $\mathcal{T}(A, \alpha, \mathcal{L})$ to $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ is covariant for $\hat{\sigma}$ and $\sigma$. Therefore any KMS state $\psi$ on $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ yields the KMS state $\psi \circ q$ on $\mathcal{T}(A, \alpha, \mathcal{L})$. Conversely, any KMS state on $\mathcal{T}(A, \alpha, \mathcal{L})$ which vanish on $\text{Ker}(q)$ gives a KMS state on $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ by passage to the quotient.

Observe that the element $S \in A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ is analytic for the gauge action and that

$$\sigma_z(S) = h^{iz}S, \quad \forall z \in \mathbb{C}.$$ 

We then have for every $n \in \mathbb{N}$ that

$$\sigma_z(S^n) = \underbrace{h^{iz}S \cdots h^{iz}S}_{n \text{ times}} = h^{iz} \alpha(h^{iz}) \cdots \alpha^{n-1}(h^{iz})S^n = h^{iz[n]}S^n,$$

where by $h^{iz[n]}$ we of course mean $(h^{iz})^{[n]}$. Since $\sigma_t(S^*) = S^*h^{-iz}$, for $t \in \mathbb{R}$, we have that $\sigma_z(S^*) = S^*h^{-iz}$, for $z \in \mathbb{C}$, so that for $m \in \mathbb{N}$

$$\sigma_z(S^m) = \underbrace{(S^*h^{-iz}) \cdots (S^*h^{-iz})}_{m \text{ times}} = S^m \alpha^{m-1}(h^{iz}) \cdots \alpha(h^{iz})h^{iz} = S^mh^{-iz[m]}.$$ 

It is therefore clear that any element of the form $aS^nS^mb$, with $a, b \in A$, is analytic and that

$$\sigma_z(aS^nS^mb) = ah^{iz[n]}S^nS^mh^{-iz[m]}b, \quad \forall z \in \mathbb{C}.$$ 

Obviously the same holds for $\hat{\sigma}$ and $\hat{S}$.

Our next goal will be the characterization of the states $\hat{\phi}$ of $A$ such that the composition $\hat{\phi} \circ G$ is a KMS state on $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ (and hence $\hat{\phi} \circ G \circ q$ is a KMS state on $\mathcal{T}(A, \alpha, \mathcal{L})$).
9.2. Proposition. Let \( \phi \) be a state on \( A \) and let \( \beta > 0 \) be a real number. Then the state \( \psi \) on \( A \times_{\alpha,L} N \) given by \( \psi = \phi \circ G \) is a KMS\( \beta \) state if and only if \( \phi \) is a trace and

\[
\phi(a) = \phi(L(\Lambda a))
\]

for all \( a \in A \), where \( \Lambda = h^{-\beta} \text{ind}(E) \).

Proof. Suppose that \( \phi \) is a trace satisfying the condition in the statement. In view of (2.3), in order to prove \( \psi \) to be a KMS\( \beta \) state it is enough to show that

\[
\psi(x\sigma_{i\beta}(y)) = \psi(yx) \tag{9.3}
\]

whenever \( x \) and \( y \) have the form \( x = aS^nS^{-m}b \) and \( y = cS^jS^kd \), where \( n, m, j, k \in \mathbb{N} \) and \( a, b, c, d \in A \). We have

\[
x\sigma_{i\beta}(y) = aS^nS^{-m}b \sigma_{i\beta}(cS^jS^kd) = aS^nS^{-m}bch^{-\beta[j]}S^jS^kd =
\]

by (2.2) as long as we assume that \( m \leq j \). We therefore see that \( G(x\sigma_{i\beta}(y)) = 0 \) if \( n - m + j - k \neq 0 \). Under the latter condition it is also easy to see that \( G(yx) = 0 \), in which case (9.3) is verified. We thus assume that \( n - m + j - k = 0 \).

Setting \( p = j - m \) we have that \( j = m + p \) and \( k = n + p \). So

\[
\psi(x\sigma_{i\beta}(y)) = \phi \circ G(ac^nL^m(bch^{-\beta[m+p]})S^{m+p}S^{n+p}h^{\beta[n+p]}d) =
\]

where \( u := bch^{-\beta[m+p]} \) and \( v := I^{-1}_{m+p}bch^{-\beta[n+p]}da \). On the other hand,

\[
\psi(yx) = \psi(cS^{m+p}S^{n+p}bS^nS^{-m}b) = \phi \circ G(cS^{m+p}S^{n+p}S^{-m}S^{-n}b) =
\]

We thus see that (9.3) holds under the present hypotheses that \( m \leq j \), if and only if

\[
\phi(a^nL^m(u)v) = \phi(u\Lambda^{-[m]}\alpha^nL^m(\Lambda^n v)), \quad \forall u, v \in A, \quad \forall n, m \in \mathbb{N}. \tag{\dagger}
\]

Consider the linear maps \( \tilde{\alpha}, \tilde{L} : A \to A \) given respectively by \( \tilde{\alpha}(a) = \Lambda^{-1}\alpha(a) \), and \( \tilde{L}(a) = L(\Lambda a) \), for all \( a \in A \). It is then easy to see that \( \tilde{\alpha}^m(a) = \Lambda^{-[m]}\alpha^m(a) \), and that \( \tilde{L}^n(a) = L^n(\Lambda^n a) \). The equation in (\dagger) is then expressed as

\[
\phi(a^nL^m(u)v) = \phi(u\tilde{\alpha}^m\tilde{L}^n(v)). \tag{\dagger}\]

Observe that by hypotheses we have for all \( a, b \in A \) that

\[
\phi(a\alpha(b)) = \phi(\Lambda(\Lambda \alpha(a)b)) = \phi(\alpha(L(b))) = \phi(\alpha\tilde{L}(b)),
\]

and

\[
\phi(L(ab)) = \phi(L(\alpha(a)b)) = \phi(L(\Lambda^{-1}\alpha(a)b)) = \phi(\Lambda^{-1}\alpha(b)) = \phi(a\tilde{\alpha}(b)).
\]

This may be interpreted as saying that with respect to the inner–product \( \langle a, b \rangle = \phi(a^*b) \) one has that the adjoint of \( \alpha \) is \( \tilde{L} \) and the adjoint of \( L \) is \( \tilde{\alpha} \). It is now evident that (\dagger) holds, hence completing the proof of (9.3) in the case that \( m \leq j \).

When \( m \geq j \) it is also true that both sides of (9.3) vanish unless \( n - m + j - k = 0 \). In this case, letting \( p = m - j \), we have that \( m = j + p \) and \( n = k + p \), so that

\[
\psi(x\sigma_{i\beta}(y)) = \psi(aS^{k+p}S^{j+p}bch^{-\beta[j]}S^jS^kh^{\beta[k]}d) = \phi \circ G(aS^{k+p}S^{j+p}\alpha^kL^j(bch^{-\beta[j]}h^{\beta[k]}d) =
\]


where \( u := \text{bch}^{-\beta[j]} \) and \( v := h^{\beta[k]} \text{da}_{k+p}^{-1} \). On the other hand

\[
\psi(yz) = \psi(cS^k \Sigma^k \text{da}_{k+p} S^{(j+p)} b) = \psi(c \alpha \Sigma^k (\text{da}) S^{(j+p)} b) = \phi(\alpha \Sigma^k (\text{da}) I_{k+p}^{-1} b) = \phi(\alpha \Sigma^k (\text{bch}^{-\beta[k]} v I_{k+p}^{-1}) I^{-1}_{j+p} u h^{\beta[j]}),
\]

where \( \alpha \) and \( \hat{\Sigma} \) are defined respectively by \( \hat{\alpha}(a) = h^\alpha \alpha(a) \text{ ind}(E)^{-1} \) and \( \hat{\Sigma}(a) = \Sigma(h^{-\beta} a \text{ ind}(E)) \). However, since both \( \text{ind}(E) \) and \( h \) belong to the center of \( A \) we have that \( \hat{\alpha} = \alpha \) and \( \hat{\Sigma} = \Sigma \), so that under the hypotheses that \( m \geq j \) we see that (9.3) is equivalent to

\[
\phi(\alpha \Sigma^k (u \Sigma^j (v))) = \phi(\alpha \Sigma^j (v)),
\]

which follows as above.

Conversely, suppose that \( \psi \) is a KMS\(\beta \) state on \( A \times_{\alpha, \Sigma} N \) we have that (9.3) holds for all analytic elements \( x \) and \( y \). Given \( a, b \in A \) plug \( x = a \) and \( y = b \) in (9.3) to conclude that \( \phi(ab) = \phi(ba) \) so that \( \phi \) must be a trace on \( A \). On the other hand, plugging \( x = S^* \) and \( y = a \text{ ind}(E) S \) in (9.3) gives

\[
\phi(\Sigma(a \text{ ind}(E) h^{-\beta})) = \phi(a),
\]

hence completing the proof. □

The KMS states provided by the above result necessarily vanish on elements of the form \( aS^n S^* m b \) with \( n \neq m \) since so does \( G \). We shall see next that this is necessarily the case for all KMS states when \( h \geq cI \) for some real number \( c > 1 \) (as opposed to \( c > 0 \) which we have been assuming so far). We will in fact prove a slightly stronger result by considering KMS states on \( \Sigma(A, \alpha, \Sigma) \), which include the KMS states on \( A \times_{\alpha, \Sigma} N \) as already mentioned.

9.4. Proposition. Suppose that \( h \geq cI \) for some real number \( c > 1 \) and let \( \psi \) be a KMS\(\beta \) state on \( \Sigma(A, \alpha, \Sigma) \), where \( \beta > 0 \). Then for every \( a, b \in A \) and every \( n, m \in N \) with \( n \neq m \) one has that \( \psi(a \hat{S}^n \hat{S}^m b) = 0 \).

Proof. Taking adjoints we may assume that \( n > m \). So write \( n = m + p \) with \( p > 0 \). We have

\[
\psi(a \hat{S}^n \hat{S}^m b) = \psi(a \hat{S}^m \hat{S}^p \hat{S}^m b) = \psi(\hat{S}^p \hat{S}^m b \hat{S}_\beta(a \hat{S}^m)) = \psi(\hat{S}^p \hat{S}^m b h^{-\beta[m]} \hat{S}^m) = \psi(\hat{S}^p \Sigma^m (b h^{-\beta[m]} \hat{S}^m)) = \psi(\alpha \Sigma^m (b h^{-\beta[m]} \hat{S}^m)),
\]

so it suffices to prove that \( \psi(a \hat{S}^p) = 0 \) for all \( a \in A \) and \( p > 0 \). In order to accomplish this notice that

\[
\psi(a \hat{S}^p) = \psi(\hat{S}^p \alpha \hat{S}_\beta(a)) = \psi(\hat{S}^p \alpha) = \psi(a \hat{S}^p),
\]

so that

\[
\psi(ak \hat{S}^p) = 0, \quad \forall a \in A, \quad (\dagger)
\]

where \( k = 1 - h^{-\beta[p]} \). Since \( h \geq c \) we have that \( h^{-\beta} \leq c^{-\beta} \) and hence \( h^{-\beta[p]} \leq c^{-\beta p} \) by (7.3.iii). This implies that \( k \geq 1 - c^{-\beta p} > 0 \) and hence that \( k \) is invertible. The conclusion then follows upon replacing \( a \) with \( ak^{-1} \) in \( (\dagger) \). □

Observe that we haven’t used that \( E \) is of index-finite type in the above proof. Also notice that it follows from the above result that any KMS\(\beta \) state on \( A \times_{\alpha, \Sigma} N \) must vanish on elements of the form \( aS^n S^* m b \) with \( n \neq m \).

We would now like to address the question of whether all KMS state on \( A \times_{\alpha, \Sigma} N \) are given by (9.2). Should there exist more than one conditional expectation from \( A \times_{\alpha, \Sigma} N \) to \( A \) it would probably be unreasonable to expect this to be true. In view of (8.8) and (8.9) one is led to believe that the question posed above is easier to be dealt with under the hypothesis that \( A \) is commutative.

After having proved the result below for commutative algebras I noticed that the commutativity hypothesis was used only very slightly and could be replaced by the weaker requirement that \( E(ab) = E(ba) \) for all \( a, b \in A \). In the hope that a relevant example might be found under this circumstances we will restrict ourselves to this weaker hypothesis whenever possible.
9.5. Proposition. Suppose that $h \geq cI$ for some real number $c > 1$ and let $\psi$ be a KMS$_3$ state on $A \rtimes \alpha, \mathcal{L} \mathbb{N}$, where $\beta > 0$. Suppose moreover that $E(ab) = E(ba)$ for all $a, b \in A$ (e.g. when $A$ is commutative). Then $\psi = \psi \circ G$. Therefore $\psi$ is given as in (9.2) for $\phi = \psi|_A$.

Proof. We shall prove the equivalent statement that all KMS$_3$ states $\psi$ on $\mathcal{F}(A, \alpha, \mathcal{L})$ which vanish on $\text{Ker}(q)$ satisfy $\psi = \psi \circ G$.

Let $(u_1, \ldots, u_m)$ be a quasi-basis for $E$ as in the beginning of section (8). Setting $k = \sum_{j=1}^m u_j \hat{S}^* u_j^*$ observe that for all $b \in A$ one has

$$kb \hat{S} = \sum_{j=1}^m u_j \hat{S} \hat{S}^* u_j^* b \hat{S} = \sum_{j=1}^m u_j \hat{S} \mathcal{L}(u_j^* b) \hat{S} = \sum_{j=1}^m u_j E(u_j^* b) \hat{S} = b \hat{S},$$

showing that the pair $(1, k)$ is a redundancy. It follows that $1 - k \in \text{Ker}(q)$ and hence for all $a \in A$

$$\psi(a) = \psi(ak) = \psi\left(\sum_{j=1}^m au_j \hat{S} \hat{S}^* u_j^*\right) = \sum_{j=1}^m \psi(\hat{S}^* u_j^* \mathcal{L}(au_j) \hat{S}) = \sum_{j=1}^m \psi(\hat{S}^* u_j^* au_j h^{-\beta} \hat{S}) =$$

$$= \sum_{j=1}^m \psi(\mathcal{L}(au_j h^{-\beta})) = \sum_{j=1}^m \psi(\mathcal{L}(ah^{-\beta} u_j u_j^*)) = \psi(\mathcal{L}(ah^{-\beta} \text{ind}(E))) = \psi(\mathcal{L}(a \Lambda)), $$

where, as before, $\Lambda = h^{-\beta} \text{ind}(E)$. Replacing $a$ by $a \Lambda^{-1}$ above leads to $\psi(\mathcal{L}(a)) = \psi(\Lambda^{-1} a)$. It is then easy to prove by induction that

$$\psi(\mathcal{L}^n(a)) = \psi(\Lambda^{-[n]} a),$$

for all $a \in A$ and $n \in \mathbb{N}$. Given $n, m \in \mathbb{N}$ and $a, b \in A$ we claim that

$$\psi(a \hat{S}^n \hat{S}^* b) = \psi(\mathcal{G}(a \hat{S}^n \hat{S}^* b)).$$

Observe that the case in which $n \neq m$ follows immediately from (9.4). So we assume that $n = m$. We then have that

$$\psi(a \hat{S}^n \hat{S}^* b) = \psi(\hat{S}^n b \mathcal{L}(a \hat{S}^n) = \psi(\hat{S}^n b \mathcal{L}(a \hat{S}^n) = \psi(\mathcal{L}^n(b a h^{-\beta} [n] b) = \psi(\mathcal{L}^n(b a \text{ind}(E)^{-[n]} b) = \psi(\mathcal{G}(a \hat{S}^n \hat{S}^* b)),$$

where we have used in (⋆) the fact that the restriction of a KMS state to the algebra of fixed points is a trace. This proves our claim and the result follows from (2.3). \hfill \Box

Summarizing we have:

9.6. Theorem. Let $\alpha$ be an injective endomorphism of a unital $C^*$-algebra $A$ with $\alpha(1) = 1$. Let $E$ be a conditional expectation of index-finite type from $A$ onto the range of $\alpha$ such that $E(ab) = E(ba)$ for all $a, b \in A$ (e.g. when $A$ is commutative). Let $\mathcal{L} = \alpha^{-1} \ast E$ be the corresponding transfer operator. Given a self-adjoint element $h \in \mathcal{F}(A)$ with $h \geq cI$ for some real number $c > 1$, consider the unique one-parameter automorphism group $\sigma$ of $A \rtimes \alpha, \mathcal{L} \mathbb{N}$ given for $t \in \mathbb{R}$ by $\sigma_t(S) = h^{it} S$ and $\sigma_t(a) = a$ for all $a \in A$. Then, for all $\beta > 0$ the correspondence

$$\psi \mapsto \phi = \psi|_A$$

is a bijection from the set of KMS$_3$ states $\psi$ on $A \rtimes \alpha, \mathcal{L} \mathbb{N}$ and the set of states $\phi$ on $A$ such that $\phi(a) = \phi(\mathcal{L}(a \Lambda))$ for all $a \in A$, where $\Lambda = h^{-\beta} \text{ind}(E)$. The inverse of the above correspondence is given by $\phi \mapsto \psi = \phi \circ G$, where $G$ is the conditional expectation given in (8.9).
10. Ground states.

In this section we retain the standing assumptions made in (9.1) but we will drop (9.1.iv) at a certain point below. Our goal is to treat the case of ground states on $A \rtimes_{\alpha,L} \mathbb{N}$ for the gauge action $\sigma^h$. Recall that a state $\psi$ on $A \rtimes_{\alpha,L} \mathbb{N}$ is a ground state if

$$\sup_{\text{Im} z \geq 0} |\psi(x\sigma_z(y))| < \infty$$

for every analytic elements $x,y \in A \rtimes_{\alpha,L} \mathbb{N}$. Let $(u_1,\ldots,u_m)$ be a quasi-basis for $E$ as in the beginning of section (8). As seen in the proof of (9.5) the pair $(1,k)$ is a redundancy, where $k = \sum_{j=1}^m u_j S^* u_j^*$. Therefore one has that

$$1 = \sum_{j=1}^m u_j S^* u_j^*$$

in $A \rtimes_{\alpha,L} \mathbb{N}$. Assuming that $\psi$ is a ground state on $A \rtimes_{\alpha,L} \mathbb{N}$ one has that the following is bounded for $z$ in the upper half plane:

$$\sum_{j=1}^m \psi(u_j S \sigma_z(S^* u_j^*)) = \sum_{j=1}^m \psi(u_j S S^* h^{-iz} u_j^*) = \psi(h^{-iz}),$$

say by a constant $K > 0$. With $z = i\beta$ we conclude that $\psi(h^\beta) \leq K$ for all $\beta > 0$. Suppose that $h \geq cI$ for some real number $c > 1$ as before. Then $h^\beta \geq c^\beta$ and

$$K \geq \psi(h^\beta) \geq c^\beta.$$

Observing that the term in right hand side above converges to infinity as $\beta \to \infty$ we arrive at a contradiction thus proving:

10.1. Proposition. Suppose that $E$ is of index-finite type and that $h \geq cI$ for some real number $c > 1$. Then there are no ground states on $A \rtimes_{\alpha,L} \mathbb{N}$.

In the remainder of this section we will discuss the ground states on $\mathcal{F}(A,\alpha,L)$. Our results in this direction will no longer depend on the fact that $E$ is of index-finite type.

10.2. Proposition. Suppose that $h \geq cI$ for some real number $c > 1$. Then a state $\psi$ on $\mathcal{F}(A,\alpha,L)$ is a ground state if and only if $\psi$ vanishes on any element of the form $a \hat{S}^n \hat{S}^* m b$ if $(n,m) \neq (0,0)$.

Proof. Let $a,b \in A$ and $n,m \in \mathbb{N}$ with $(n,m) \neq (0,0)$ and let $\psi$ be a ground state on $\mathcal{F}(A,\alpha,L)$. By taking adjoints it suffices to prove the result in the case that $m \neq 0$. Letting $x = a \hat{S}^n$ and $y = \hat{S}^* m b$ we have that

$$\psi(x\sigma_z(y)) = \psi(a \hat{S}^n \hat{S}^* m h^{-iz} |m| b) \quad (\dagger)$$

is bounded as a function of $z$ on the upper half plane. For $z = x + iy$ we have

$$||h^{-iz}|| = ||h^{y-iz}|| = ||h^y||.$$

If $z$ is in the lower half plane, that is if $y \leq 0$, then since $h \geq cI$ we have that $h^y \leq c^y < 1$ so that $(\dagger)$ is actually bounded everywhere. By Liouville’s Theorem $(\dagger)$ is constant and that constant must be zero since zero is the limit of $(\dagger)$ as $z$ tends to infinity over the negative imaginary axis. Plugging $z = 0$ in $(\dagger)$ gives the desired conclusion. We leave the proof of the converse statement to the reader. \hfill $\square$

We now need some insight on the structure of the fixed-point algebra for the scalar gauge action $\hat{\gamma}$ on $\mathcal{F}(A,\alpha,L)$.

10.3. Proposition. Let $\mathcal{W}$ be the subalgebra of $\mathcal{F}(A,\alpha,L)$ consisting of the fixed-points for $\hat{\gamma}$. Then there exists a $^*$-homomorphism $\pi : \mathcal{W} \to A$ such that $\pi(a) = a$, for all $a \in A$, and $\pi(S^n \hat{S}^m) = 0$, for all $n > 0$. 

Proof. Consider the representation \( \rho : \mathcal{T}(A, \alpha, \mathcal{L}) \to \mathcal{L}(M_\infty) \) described in the proof of [E2: Theorem 3.4]. It is easy to see that \( \rho \) maps \( \hat{\mathcal{U}} \) into the set of diagonal operators with respect to the decomposition \( M_\infty = \bigoplus_{n=0}^{\infty} M_{L^n} \). Therefore, letting \( e \) be the projection onto \( M_{L^0} \), we have that the map

\[
\pi : x \in \hat{\mathcal{U}} \mapsto e \rho(x)e \in \mathcal{L}(M_{L^0})
\]

is a *-homomorphism. It is evident that \( \pi \) maps each \( a \in A \) to the same \( a \) in the canonical copy of \( A \) within \( \mathcal{L}(M_0) \) while \( \pi(S^n S^m) = 0 \) for all \( n > 0 \).

10.4. Proposition. Suppose that \( h \geq cI \) for some real number \( c > 1 \). Then (regardless of \( E \) being of index-finite type or not) the ground states on \( \mathcal{T}(A, \alpha, \mathcal{L}) \) are precisely the states of the form \( \phi \circ \pi \circ \hat{P} \) where \( \hat{P} \) is the conditional expectation onto \( \mathcal{U} \) given by

\[
\hat{P}(x) = \int_{\mathbb{R}^1} \hat{\chi}_z(x) \, dz, \quad \forall x \in \mathcal{T}(A, \alpha, \mathcal{L}).
\]

and \( \phi \) is any state whatsoever on \( A \).

Proof. Let \( \psi \) be a ground state on \( A \). Then as a special case of (10.2) we see that \( \psi \) vanishes on \( a \hat{S}^n \hat{S}^m b \) whenever \( n \neq m \). By checking first on the generators of \( \mathcal{T}(A, \alpha, \mathcal{L}) \) provided by (2.3) it is easy to see that \( \psi = \phi \circ \hat{P} \). Letting \( \chi \) denote the restriction of \( \psi \) to \( \mathcal{U} \) we then evidently have that \( \psi = \chi \circ \hat{P} \).

Let now \( \phi \) be the restriction of \( \chi \) (and hence also of \( \psi \)) to \( A \). Then one may prove that \( \chi = \phi \circ \pi \) by checking on the generators of \( \mathcal{U} \) given by (3.5). So \( \psi = \chi \circ \hat{P} = \phi \circ \pi \circ \hat{P} \) as desired.

Conversely, given any state \( \phi \) on \( A \) it is easy to see that \( \psi = \phi \circ \pi \circ \hat{P} \) is a ground state by (10.2). □

11. The commutative case.

Let us now discuss the case of a commutative \( A \). Rather than employ Gelfand’s Theorem and view \( A \) as the algebra of continuous functions on its spectrum we will let \( A \) be any closed unital *-subalgebra of the \( \mathcal{C}^* \)-algebra \( B(X) \) of all bounded functions on a set \( X \) (with the sup norm). Examples are:

(i) if \( X \) is a measure space take \( A \) to be the set of all bounded measurable functions on \( X \),

(ii) if \( X \) is a topological space choose a subset \( \{x_1, x_2, \ldots \} \subseteq X \) and let \( A \) be the set of all bounded functions which are continuous at all points of \( X \) except, perhaps, at the points of the set above.

Let us also fix a surjective mapping

\[
\theta : X \to X
\]

such that \( f \circ \theta \in A \) for all \( f \in A \). Clearly one gets a unital *-monomorphism \( \alpha : A \to A \) by letting

\[
\alpha(f) = f \circ \theta, \quad \forall f \in A.
\]

Assume that there exists a finite subset \( \{v_1, \ldots, v_m\} \subseteq A \) such that for all \( i = 1, \ldots, m \):

(i) \( \theta \) is injective when restricted to the set \( \{x \in X : v_i(x) \neq 0\} \),

(ii) \( v_i \geq 0 \),

(iii) \( \sum_{i=1}^{m} v_i = 1 \).

For each \( x \in X \) define

\[
N(x) = \# \{t \in X : \theta(t) = x\}
\]

and observe that the existence of the \( v_i \)'s above implies that \( N(x) \leq m \). For \( f \in A \) consider the function \( T(f) \) on \( X \) given by

\[
T(f)|_x = \sum_{\theta(t) = x} f(t),
\]

for each \( x \in X \).
If we assume that $T(f) \in A$ for all $f \in A$ and moreover that $N$, seen as a bounded function on $X$, belongs to $A$ then the operator $\mathcal{L} : A \to A$ given by $\mathcal{L}(f) = N^{-1} T(f)$ is a transfer operator. In addition the composition $E = \alpha \circ \mathcal{L}$ is a conditional expectation from $A$ to the range of $\alpha$, which may be expressed as

$$E(f)|_x = \frac{1}{\mu(x)} \sum_{t \in X \atop \theta(t)=\theta(x)} f(t)$$

where $\mu = N \circ \theta$. Setting $u_i = (\mu v_i)^{1/2}$ observe that for all $f \in A$ and $x \in X$ one has that

$$\sum_{i=1}^m u_i E(u_i f)|_x = \sum_{i=1}^m u_i(x) \frac{1}{\mu(x)} \sum_{t \in X \atop \theta(t)=\theta(x)} u_i(t) f(t) = \sum_{i=1}^m u_i(x) \frac{1}{\mu(x)} u_i(x) f(x) = \sum_{i=1}^m v_i(x) f(x) = f(x).$$

Therefore $\{u_1, \ldots, u_m\}$ is a quasi-basis for $E$, which says that $E$ is of index-finite type, and

$$\text{ind}(E) = \sum_{i=1}^m u_i^2 = \sum_{i=1}^m \mu v_i = \mu.$$  \hfill (1)

Fix a positive element $h \in A$ with $h \geq c I$ for some real number $c > 1$ and consider the gauge action $\sigma^h$ on $A \rtimes_{\alpha,L} \mathbb{N}$. By (9.6) we have that the KMS$_\beta$ states on $A \rtimes_{\alpha,L} \mathbb{N}$ for the gauge action $\sigma^h$ correspond to the states $\phi$ on $A$ such that

$$\phi(f) = \phi(\mathcal{L}(h^{-\beta \text{ind}(E)} f))$$

for all $f \in A$. In the present context we have that

$$\mathcal{L}(h^{-\beta \text{ind}(E)} f)|_x = \frac{1}{N(x)} \sum_{t \in X \atop \theta(t)=x} h(t)^{-\beta} \mu(t) f(t) = \sum_{t \in X \atop \theta(t)=x} h(t)^{-\beta} f(t).$$

The operator $f \mapsto \mathcal{L}(h^{-\beta \text{ind}(E)} f)$ therefore coincides with the operator $L_{h^{-\beta}}$ introduced by Ruelle in [R1], [R2].

**References**


