THE CROSSED-PRODUCT OF A C*-ALGEBRA BY A SEMIGROUP OF ENDOMORPHISMS

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This talk is based on:

R. Exel, "A new look at the crossed-product of a C*-algebra by an endomorphism", *Ergodic Theory Dynam. Systems*, **23** (2003), 1733–1750, [arXiv:math.OA/0012084].

R. Exel, "A new look at the crossed-product of a C*-algebra by a semigroup of endomorphisms", *Ergodic Theory Dynam. Systems*, to appear, [arXiv:math.OA/0511061].

R. Exel and J. Renault, "Semigroups of local homeomorphisms and interaction groups", *Ergodic Theory Dynam. Systems*, to appear, [arXiv:math.OA/0608589].

• HISTORY

(1978) Arzumanian and Vershik introduce a concrete crossed-product construction using the Koopmann operator.

(1978) Cuntz states that $\mathcal{O}_2$ is the crossed product of UHF_{2^∞} by the "shift" endomorphism

$$a_1 \otimes a_2 \otimes \ldots \mapsto e \otimes a_1 \otimes a_2 \otimes \ldots,$$

where e is a minimal projection in $M_2(\mathbb{C})$.

(1980) Paschke develops some of Cuntz's ideas without actually introducing a formal notion of crossed product by endomorphisms.

(1993) Stacey introduces a general theory of crossed products by endomorphism as universal C*-algebras for the "covariance condition"

$$\sigma(x) = S_1 x S_1^* + \ldots S_n x S_n^*,$$

where the S_i's are isometries, but gives no recipe to determine the number n of summands.

(1993) Boyd, Keswani, and Raeburn study faithful representations of such crossed products by endomorphisms.

(1994) Adji, Laca, May, and Raeburn study Toeplitz algebras of ordered groups using semigroup crossed product.

(1996) (Took a while to be published) Murphy studied abstract notion of endomorphism crossed products already observing that the case in which the range of σ is hereditary works better.

Other names: Doplicher and Roberts, Deaconu, Fowler, Hirshberg, Khoshkam and Skandalis, Larsen, Muhly and Solel, ...

• PROGRAM FOR THIS TALK

(1) Classical notion of crossed products by semigroups of endomorphisms.

(2) Crossed products by single endomorphisms with transfer operators.

(3) Interaction groups and crossed products.

(4) Extension problem.

(5) Examples.

(6) Counter Examples.

• CLASSICAL NOTION OF CROSSED PRODUCTS

One is given an action σ of a semigroup P on a C*-algebra A, i.e. a semigroup homomorphism

$$\sigma : P \to \mathrm{End}(A).$$

The crossed product of A by P under σ, denoted $A \rtimes_\sigma P$, is defined to be the universal C*-algebra generated by a copy of A, together with a collection of isometries $\{S_x\}_{x\in P}$, satisfying

$$S_x S_y = S_{xy}, \quad \text{and} \quad S_x a S_x^* = \sigma_x(a), \quad \forall\, x, y \in P, \quad \forall\, a \in A.$$

The problem with this definition is that one has no control over

$$S_x^* a S_x = ???$$

If the range of σ_x is hereditary and $1 \in A$, Murphy observed that

$$S_x^* a S_x = S_x^* S_x S_x^* a S_x S_x^* S_x = S_x^* \sigma_x(1) a \sigma_x(1) S_x = \ldots$$

Since the range of σ_x is hereditary, then

$$\sigma_x(1) a \sigma_x(1) = \sigma_x(b),$$

for some $b \in A$, and hence the above equals

$$\ldots = S_x^* \sigma_x(b) S_x = S_x^* S_x b S_x^* S_x = b.$$

It follows that

$$S_x^* A S_x \subseteq A.$$

But we want to work with general endomorphisms which do not have hereditary range, so this cannot be used.

• CROSSED PRODUCTS VIA TRANSFER OPERATORS

Let σ be an action of $\mathbb{N}$ on a C*-algebra A.

If $\alpha = \sigma_1$, then necessarily $\sigma_n = \alpha^n$, so we are actually given a single endomorphisms α.

We want to construct a crossed product algebra $A \rtimes_\alpha \mathbb{N}$ as being the universal C*-algebra generated by a copy of A and an isometry S such that

$$Sa = \alpha(a)S, \quad \text{and} \quad S^* A S \subseteq A.$$

We must therefore specify in advance what should $\underline{S^* a S}$ be, for every $a \in A$!!!

Definition. A *transfer operator* for α is a positive linear map

$$L : A \to A,$$

such that

(i) $L(1) = 1$, and

(ii) $L(a\alpha(b)) = L(a)b$, for all $a, b \in A$.

Given a transfer operator L, we then consider the universal C*-algebra generated by a copy of A and an isometry S subject to the relations

(i) $Sa = \alpha(a)S$,

(ii) $S^* a S = L(a)$,

for every $a \in A$.

This algebra is denoted $\mathcal{T}(A, \alpha, L)$ and is called the *Toeplitz algebra* for the system (A, α, L). The crossed product $A \rtimes_{\alpha,L} \mathbb{N}$ is a quotient of $\mathcal{T}(A, \alpha, L)$ by an ideal called the *redundancy ideal*.

$A \rtimes_{\alpha,L} \mathbb{N}$ may also be defined as a Cuntz-Pimsner algebra. Just consider $X := A$, as a Hilbert bimodule (correspondence) over itself with the following operations

$$a \cdot x = ax,$$
$$x \cdot a = x\alpha(a),$$
$$\langle x, y \rangle = L(x^* y), \quad \forall\, x, y \in X, \quad \forall\, a \in A.$$

Then $\mathcal{T}(A, \alpha, L)$ coincides with the Toeplitz-Cuntz-Pimsner algebra $\mathcal{TO}_X$, while

$$A \rtimes_{\alpha,L} \mathbb{N} = \mathcal{O}_X.$$

• INTERACTION GROUPS

The goal here is to generalize the above construction for a larger semigroup of endomorphisms.

Initially suppose that we are just given an endomorphism α of a C*-algebra A as well as a transfer operator L, as above. Define for every $n \in \mathbb{Z}$

$$V_n = \begin{cases} \alpha^n, & \text{if } n \geq 0, \\ \\ L^{-n}, & \text{if } n < 0. \end{cases}$$

One can then prove that

$$V_{-n}\ \underline{V_n V_m} = V_{-n}\ \underline{V_{n+m}},$$

and

$$\underline{V_n V_m}\, V_{-m} = \underline{V_{n+m}}\, V_{-m},$$

for all $n, m \in \mathbb{Z}$.

Definition. A partial representation of a group G on a Banach space X is a map

$$V : G \to B(X) \quad \text{(bounded operators on } X\text{)},$$

such that

(i) $V_1 = id$,

(ii) $V_{g^{-1}}\, V_g V_h = V_{g^{-1}}\, V_{gh}$,

(iii) $V_g V_h\, V_{h^{-1}} = V_{gh}\, V_{h^{-1}}$,

for all $g, h \in G$.

Definition. An *interaction group* is a triple (A, G, V), where A is a unital C*-algebra, G is a group, and

$$V : G \to B(A)$$

is a *partial representation* such that, for every g in G,

(i) V_g is a positive map,

(ii) $V_g(1) = 1$,

(iii) $V_g(ab) = V_g(a)V_g(b)$, for every $a, b \in A$, such that <u>either</u> a or b belongs to the range of $V_{g^{-1}}$.

Definition. The *Toeplitz algebra* $\mathcal{T}(A, G, V)$ is the universal C*-algebra generated by a copy of A and a collection of partial isometries $\{S_x\}_{x \in G}$, satisfying

(i) $S_1 = 1_A$,

(ii) $S_{g^{-1}} = S_g^*$,

(iii) $S_{g^{-1}}\, S_g S_h = S_{g^{-1}}\, S_{gh}$,

(iv) $S_g a S_g^* = V_g(a) S_g S_g^* \quad \big(= S_g S_g^* V_g(a)\big)$.

(Rough) Definition. The crossed product $A \rtimes_V G$ is the quotient of $\mathcal{T}(A, G, V)$ by a certain ideal called the *redundancy ideal*.

Why is this a sensible definition? Suppose that ϕ is a faithful V–invariant state on A, and consider

$$A \subseteq B(H)$$

via the GNS representation of ϕ. Using invariance it is easy to show that there is a partial representation

$$v : G \to B(H)$$

such that

$$v_g(a\xi) = V_g(a)\xi, \quad \forall\, a \in A,$$

where ξ is the cyclic vector. One may then prove that $A \rtimes_V G$ is isomorphic to the algebra of operators on $H \otimes \ell_2(G)$ generated by

$$\{a \otimes 1 : a \in A\} \cup \{v_g \otimes \lambda_g : g \in G\},$$

where λ is the regular representation, provided G is amenable.

• EXTENSION PROBLEM

Suppose we are given a group G, a subsemigroup $P \subseteq G$, and an action by endomorphisms

$$\sigma : P \to \mathrm{End}(A).$$

Question. Is there an interaction group (A, G, V) such that

$$V_g = \sigma_g, \quad \forall\, g \in P.$$

If the answer is affirmative one may form the crossed product $A \rtimes_V G$, which will perhaps depend on the extension chosen, but not always.

This is a very delicate problem which we will revisit shortly.

• **EXAMPLES**

Given a compact topological space X let

$$\mathrm{End}(X)$$

denote the semigroup of all surjective local homeomorphisms $T : X \to X$.

Let G be a group, P be a subsemigroup of G, and

$$\theta : P \to \mathrm{End}(X)$$

be a right action, meaning that $\theta_n \theta_m = \theta_{mn}$, for all $n, m \in P$.

Define

$$\sigma_n : f \in C(X) \mapsto f \circ \theta_n \in C(X),$$

so σ becomes a (left) action of P on A.

For example, let $S, T \in \mathrm{End}(X)$ be commuting elements, let $G = \mathbb{Z} \times \mathbb{Z}$, let $P = \mathbb{N} \times \mathbb{N}$, and define

$$\theta_{(n,m)} = S^n T^m, \quad \forall\, (n, m) \in \mathbb{N} \times \mathbb{N}.$$

Question. Can the extension problem be solved?

Back to the situation of a general semigroup action θ, how would we find an extension?

We first attempt to define $V_{n^{-1}}$, for $n \in P$. The axioms imply that this must be a transfer operator for σ_n, and further, that it must be of the form

$$V_{n^{-1}}(f)\big|_y = \sum_{x \in \theta_n^{-1}(y)} \omega(n, x) f(x), \quad \forall\, f \in C(X), \quad \forall\, y \in X,$$

where $\omega : P \times X \to \mathbb{R}_+$ is continuous in the second variable, and normalized in the sense that

$$\sum_{x \in \theta_n^{-1}(y)} \omega(n, x) = 1, \quad \forall\, n \in P, \quad \forall\, y \in X.$$

Therefore $V_{n^{-1}}(f)\big|_y$ is a weighted average of f on $\theta_n^{-1}(y)$, and hence $V_{n^{-1}}(f)$ is a decent single-valued function to replace the multi-valued $f \circ \theta_n^{-1}$.

The whole problem boils down to choosing the right function ω. A necessary condition for the extension problem to be solved affirmatively is that

$$\omega(nm, x) = \omega(n, x)\, \omega(m, \theta_n(x)), \quad \forall\, m, n \in P, \quad \forall\, x \in X.$$

In other words ω must be a *cocycle* for the semigroup action.

Another necessary condition is that

$$\sum_{y \in C_x^m \cap C_z^n} \omega(n, y)\, \omega(m, x) = \sum_{y \in C_x^n \cap C_z^m} \omega(m, y)\, \omega(n, x). \tag{$\dagger$}$$

for all $m, n \in P$, and $x, z \in X$, where

$$C_x^m = \big\{ y \in X : \theta_m(y) = \theta_m(x) \big\}.$$

This is a bit of a mystery, but we have been able to verify it in concrete examples, such as the following:

Suppose that $S, T \in \mathrm{End}(X)$ are commuting elements as above. Arzumanian and Renault say that S and T *star-commute* if, whenever $T(x) = S(y)$, there is a unique $z \in X$, such that $S(z) = x$, and $T(z) = y$.

$$\begin{tikzcd} & z \arrow[dl, "S"'] \arrow[dr, "T"] & \\ x \arrow[dr, "T"'] & & y \arrow[dl, "S"] \\ & \bullet & \end{tikzcd}$$

Lemma. (E., Renault) Let S and T be star-commuting endomorphisms of X. Then there exists a normalized cocycle

$$\omega : (\mathbb{N} \times \mathbb{N}) \times X \to \mathbb{R}_+$$

satisfying condition ($\dagger$).

Theorem. (E., Renault) Let

(i) G be a group and X be a compact space,

(ii) P be a subsemigroup of G such that $G = P^{-1}P$,

(iii) $\theta : P \to \mathrm{End}(X)$ be a right action, and

(iv) ω be a normalized cocycle satisfying condition (†).

Then there exists an interaction group $V = \{V_g\}_{g\in G}$ on $C(X)$, given by

$$V_g(f)\big|_y = \sum_{\theta_n(x)=y} \omega(n,x) f(\theta_m(x)), \quad \forall\, f \in C(X), \quad \forall\, y \in X,$$

if $g = n^{-1}m$, with $n, m \in P$.

Notice that $V_g(f)\big|_y$ is a weighted average of f on $\theta_m(\theta_n^{-1}(y))$, so $V_g(f)$ is a single-valued function replacing the multi-valued $f \circ \theta_m \circ \theta_n^{-1}$.

Let us now study the corresponding crossed product C*-algebra.

Given the right action

$$\sigma : P \subseteq G \to \mathrm{End}(X),$$

and supposing that $P^{-1}P \subseteq PP^{-1}$, one proves that

$$\mathcal{G} = \big\{(x,g,y) \in X \times G \times X : \exists\, n,m \in P,\ g = nm^{-1},\ \theta_n(x) = \theta_m(y)\big\}.$$

is a locally compact groupoid under the operations

$$(x,g,y)(y,h,z) = (x,gh,z), \quad \text{and} \quad (x,g,y)^{-1} = (y,g^{-1},x).$$

Theorem. (E., Renault) Under the hypothesis of the above Theorem (existence of the interaction group), if moreover $G = P^{-1}P = PP^{-1}$, then $C(X) \rtimes_V G$ is naturally isomorphic to the C*-algebra of the above groupoid.

Remarks.

(1) The groupoid C*-algebra can only be defined if $P^{-1}P \subseteq PP^{-1}$. So the interaction group point of view may be used to study more general dynamical systems, such as that given by a pair of noncommuting maps $S, T : X \to X$, in which case the natural semigroup will be subsemigroup $\mathbb{F}_2^+ \subseteq \mathbb{F}_2$.

(2) Back to the case of commuting maps $S, T : X \to X$, the above groupoid may be defined regardless of the existence of cocycles. However neither this groupoid, nor its C*-algebra have been well understood. Perhaps the existence of such cocycles singles out cases in which the situation is nicer in some sense.

• COUTER-EXAMPLES

Hese is an example of a pair of commuting maps which does not admit a strictly positive cocycle with the above properties, and hence the semigroup action of $N \times \mathbb{N}$ does not extend to an interaction group of $\mathbb{Z} \times \mathbb{Z}$.

Let $\Omega = \{0,1\}^N$ be Bernoulli's space and let

$$S(x_1, x_2, \ldots) = (x_2, x_3, \ldots)$$

be the left shift. Let us find another map T commuting with S (any continuous map on Ω commuting with S is called a *cellular automaton*).

It was proved by Hedlund that T must be given in the following way: choose $p \in \mathbb{N}$ and choose a subset $D \subseteq \{0,1\}^p$, called the *dictionary*.

Define $T(x) = y$, where $y_n = 0$ or 1, according to whether or not the boxed word

$$x = (x_1, \ldots, \boxed{x_n, x_{n+1}, \ldots, x_{n+p-1}}, x_{n+p}, \ldots)$$

belongs to the dictionary or not.

If $p = 3$ and $D = \{000, 100, 010, 111\}$, then T is a surjective local homeomorphism commuting with S, for which there is no strictly positive cocycle ω satisfying the required condition (†).

For example, let us compute $T(x)$, where $x = (1,0,0,1,1,1,0,\ldots)$.

Recall that $D = \{000, 100, 010, 111\}$

$x = (\boxed{1,0,0}, 1, 1, 1, 0, \ldots)$ yes! $y = (1,$

$x = (1, \boxed{0,0,1}, 1, 1, 0, \ldots)$ no! $y = (1, 0,$

$x = (1, 0, \boxed{0,1,1}, 1, 0, \ldots)$ no! $y = (1, 0, 0,$

$x = (1, 0, 0, \boxed{1,1,1}, 0, \ldots)$ yes! $y = (1, 0, 0, 1,$

$x = (1, 0, 0, 1, \boxed{1,1,0}, \ldots)$ no! $y = (1, 0, 0, 1, 0,$

etc...