INTERACTION GROUPS

Ruy Exel
Universidade Federal de Santa Catarina
Florianópolis – Brazil

GOALS:

(1) To study the crossed-product of a C*-algebra by a semigroup of endomorphisms.

(2) To justify the need to reformulate the notion of semigroups of endomorphisms in order to accomplish (1).

HISTORY

(1978) Arzumanian and Vershik introduce a concrete crossed-product construction using the Koopmann operator.

(1978) Cuntz states that $\mathcal{O}_2$ is the crossed product of UHF_{2^∞} by the "shift" endomorphism

$$a_1 \otimes a_2 \otimes \ldots \mapsto e \otimes a_1 \otimes a_2 \otimes \ldots,$$

where e is a minimal projection in $M_2(\mathbb{C})$.

(1980) Paschke develops some of Cuntz's ideas without actually introducing a formal notion of crossed product by endomorphisms.

(1993) Stacey introduces a general theory of crossed products by endomorphism as universal C*-algebras for the "covariance condition"

$$\alpha(x) = S_1 x S_1^* + \ldots S_n x S_n^*,$$

where the S_i's are isometries, but gives no recipe to determine the number n of summands.

(1993) Boyd, Keswani, and Raeburn study faithful representations of crossed products by endomorphisms.

(1994) Adji, Laca, May, and Raeburn study Toeplitz algebras of ordered groups using semigroup crossed product.

(1996) (Took a while to be published) Murphy studied abstract notion of endomorphism crossed products already observing that the case in which the range of α is hereditary works better.

Other names: Doplicher and Roberts, Deaconu, Fowler, Hirshberg, Khoshkam and Skandalis, Larsen, Muhly and Solel, ...

Main idea of these constructions is to start with a C*-algebra A, a semigroup Γ, and an action

$$\alpha : \Gamma \to \mathrm{End}(A).$$

One wants to form a bigger C*-algebra containing A and a semigroup of isometries $\{S_g\}_{g\in\Gamma}$, such that

$$\alpha_g(a) = S_g a S_g^*, \quad \forall\, g \in \Gamma, \quad \forall\, a \in A.$$

This works well when the range of each α_g is a hereditary subalgebra, but does not give good results in other cases.

IRREVERSIBLE SYSTEMS

Consider a *classical irreversible system*, that is, X is a compact space and

$$T : X \to X$$

is a surjective (perhaps non-injective) continuous map.

We imagine that the points of X describe the possible states of a physical system and that T represents time evolution:

$$\begin{array}{ccl} x & \to & \text{state of the system today,} \\ T(x) & \to & \text{state of the system tomorrow.} \end{array}$$

Let f be a continuous function (observable)

$$f : X \to \mathbb{C}.$$

We imagine that f describes some measurement made on the system, so that $f(x)$ is the number measured when the system is in state x. Thus

$$\begin{array}{ccl} f(x) & \to & \text{measurement today,} \\ f(T(x)) & \to & \text{measurement tomorrow.} \end{array}$$

Irreversibility means that if the system is now in state x, one does not know in which state the system was yesterday (T is not injective). Any element in

$$T^{-1}(x) = \{y \in X : T(y) = x\}$$

is a possibility.

Since there is no certainty about yesterday's state one could introduce a probability measure μ_x on $T^{-1}(x)$ describing the likelihood of each alternative.

Lef f be an observable. The *average measurement yesterday*, given that the system is in state x today, is then given by

$$\mathcal{L}(f)\big|_x := \int_{T^{-1}(x)} f(y)\, d\mu_x(y).$$

We then have two operators defined on the algebra of all observables

$$\begin{array}{lll} f & \mapsto & \alpha(f) = f \circ T \quad \text{(deterministic future evolution)}, \\ f & \mapsto & \mathcal{L}(f) \qquad\qquad \text{(probabilistic past evolution)}. \end{array}$$

Observe that α is an *endomorphism* of the algebra $A = C(X)$, and $\mathcal{L}$ is a *transfer operator* for α, in the sense that

(1) $\mathcal{L}$ is a positive operator on A,

(2) $\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b, \quad \forall\, a, b \in A$.

In the paper *"A New Look at The Crossed-Product of a C*-algebra by an Endomorphism"* I argued that, in order to define the *crossed-product* of A by an endomorphism α, one needs to provide a transfer operator beforehand. That is, the endomorphism itself is not enough information. The probabilistic past evolution $\mathcal{L}$ is as important as the deterministic future evolution α!

Given a C*-algebra, an endomorphism α, and a transfer operator $\mathcal{L}$, the crossed-product

$$A \rtimes_{\alpha,\mathcal{L}} \mathbb{Z}$$

is the universal C*-algebra generated by A and an isometry S subject to

$$Sa = \alpha(a)S, \quad \text{and} \quad S^*aS = \mathcal{L}(a), \quad \forall\, a \in A,$$

plus some other relations called "redundancies".

How do we generalize this to groups other than $\mathbb{Z}$?

Back to the above situation let us consider, for every $n \in \mathbb{Z}$, the map

$$V_n : C(X) \to C(X)$$

given by

$$V_n(f) = \begin{cases} \alpha^n(f), & \text{if } n \geq 0, \\ \\ \mathcal{L}^{-n}(f), & \text{if } n < 0. \end{cases}$$

So $V_n(f)$ describes both the past and future evolution of f, depending on whether n is positive or negative. The whole story is thus told by the collection $\{V_n\}_{n\in\mathbb{Z}}$.

Before attempting to generalize, let us list the crucial properties of V:

Proposition. For all $g, h \in \mathbb{Z}$ one has that

(a) $V_0 = id_{C(X)}$,

(b) $V_g V_h \quad V_{h^{-1}} = V_{g+h} \quad V_{h^{-1}}$,

(c) $V_{g^{-1}} \quad V_g V_h = V_{g^{-1}} \quad V_{g+h}$.

(d) $V_g(1) = 1$,

(e) $V_g(A_+) \subseteq A_+$,

(f) $V_g(ab) = V_g(a)V_g(b)$, for every a in the range of $V_{g^{-1}}$, and for every b in A.

Observe that (a), (b), and (c) say that V is a partial representation of G in the algebra $\mathscr{B}(A)$ of all bounded operators on A.

Definition. Let G be a group and let A be a C*-algebra. An *interaction group* is a collection of bounded linear operators $V_g : A \to A$, for all g in G, satisfying (a)-(f) above.

In case the group operation is denoted multiplicatively, with 1 denoting the unit group element, we must replace (a) by "$V_1 = id_{C(X)}$", and in (b) and (c) we must replace "$g + h$" by "gh".

For those of you who were at my talk on "Interactions" last year, notice that the above axioms imply that $(V_g, V_{g^{-1}})$ is an *interaction* for every g.

Remark. Given an interaction V, as above, let P be a subsemigroup of G. Suppose that for every $p \in P$ one has that V_p is *injective*, then

(a) $\{V_p\}_{p\in P}$ is a *semigroup of endomorphisms* of A,

(b) $V_{p^{-1}}$ is a transfer operator for V_p, for every $p \in P$.

It is in this sense that the notion of interaction groups relate to the usual notion of semigroups of endomorphisms.

Our main point is that a semigroup of endomorphisms is not enough information to describe the whole system. One needs to know how do the other group elements relate to A, and this is given by all of the V_g.

Extension Problem. Given a semigroup $\{\alpha_g\}_{g\in P}$ of endomorphisms, find an interaction group V such that $V_p = \alpha_p$, for all $p \in P$.

In order to solve this one must pick $V_{p^{-1}}$ to be a transfer operator for V_p, thus extending things to $P \cup P^{-1}$. But notice that this is not the end of the story in case $G \neq P \cup P^{-1}$.

We'll say more about this later.

COVARIANT REPRESENTATIONS

From now on we fix an interaction group V.

Definition. A *covariant representation* of (A, G, V) in a C*-algebra B is a pair (π, s), where

$$\pi : A \to B$$

is a *-homomorphism and

$$s : G \to B$$

is a *-partial representation such that

$$s_g \pi(a) s_{g^{-1}} = \pi(V_g(a)) s_g s_{g^{-1}},$$

for all $g \in G$, and $a \in A$.

Recall that s is a *-partial representation if

(i) $s_1 = 1_B$,

(ii) $s_g^* = s_{g^{-1}}$,

(iii) $s_g s_h \quad s_{h^{-1}} = s_{gh} \quad s_{h^{-1}}$,

(iv) $s_{g^{-1}} \quad s_g s_h = s_{g^{-1}} \quad s_{gh}$,

This implies, in particular, that each s_g is a partial isometry.

Example. Suppose that φ is a state on A which is V-invariant, meaning that

$$\varphi(V_g(a)) = \varphi(a).$$

Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS representation of A. It is easy to show that, for every $g \in G$, there exists a bounded linear operator $s_\varphi(g)$ on H such that

$$s_\varphi(g) : \pi_\varphi(a)\xi_\varphi \mapsto \pi_\varphi(V_g(a))\xi_\varphi.$$

Then (π_φ, s_φ) is a covariant representation!

AMPLIFICATION

Denote the left regular representation of G by

$$\lambda : G \to \mathcal{U}(\ell_2(G)).$$

Given any covariant representation (π, s) of (A, G, V) in the C*-algebra B, we can create another one by putting

$$\pi \otimes 1 : A \to B \otimes \mathscr{B}(\ell_2(G)),$$

$$s \otimes \lambda : G \to B \otimes \mathscr{B}(\ell_2(G)).$$

We say that $(\pi \otimes 1, s \otimes \lambda)$ is the *amplification* of (π, s).

In particular, given an invariant state φ, we can consider the amplification $(\pi_\varphi \otimes 1, s_\varphi \otimes \lambda)$.

Theorem / Ad Hoc Definition. Assuming that G is amenable, let φ be a faithfull invariant state. The closed *-subalgebra of operators on $H_\varphi \otimes \ell_2(G)$ generated by

$$\big\{\pi_\varphi(a) \otimes 1 : a \in A\big\} \cup \big\{s_\varphi(g) \otimes \lambda(g) : g \in G\big\}$$

does not depend on the choice of φ and will be called the *crossed-product* of A by G under V. Notation: $A \rtimes_V G$.

We now wish to give an alternate definition of the crossed-product which does not depend on invariant states.

THE TOEPLITZ ALGEBRA

Let $\mathcal{T}(A, G, V)$ denote the universal C*-algebra generated by

$$A \,\dot{\cup}\, \{s_g : g \in G\}$$

(here the s_g are just symbols) subject to the relations which make the pair (i, s) a covariant representation, where i is the natural inclusion of A.

Notice that among these relations we have included

$$s_g a s_{g^{-1}} = V_g(a) s_g s_{g^{-1}}.$$

One can use a well known paper by Blackadar on Shape Theory to prove that such an algebra exists.

Since partial representations do not perfectly obey the group law we often need to work with products of the form

$$s_{g_1} s_{g_2} \ldots s_{g_n}$$

for each "word"

$$\alpha = (g_1, g_2, \ldots, g_n)$$

of group elements.

Definition. Given α as above let

(i) $\mathcal{M}_\alpha = A s_{g_1} s_{g_2} \ldots s_{g_n} A$, and

(ii) $\mathcal{Z}_\alpha = A s_{g_1} A s_{g_2} A \ldots A s_{g_n} A$ (closed linear span).

Observe that the covariance relation

$$s_g a s_{g^{-1}} = V_g(a) s_g s_{g^{-1}}$$

provides no help to deal with $\mathcal{Z}_\alpha$! But at least we have:

Theorem. If $\alpha = (g_1, \ldots, g_n)$ and $\beta = (h_1, \ldots, h_m)$ are words in G then

$$\mathcal{Z}_\alpha \mathcal{M}_\beta \subseteq \mathcal{M}_{\alpha\beta},$$

provided that $\mu(\alpha^{-1}) \subseteq \mu(\beta)$.

Explanation:

$\alpha\beta$ means concatenation of words,

$\alpha^{-1} := (g_n^{-1}, \ldots, g_1^{-1})$, and

$\mu(\beta) := \{1,\ h_1,\ h_1h_2,\ h_1h_2h_3,\ \ldots,\ h_1h_2h_3\ldots h_n\}$.

Corollary. Let α be a word in G and set

$$\mathcal{K}_\alpha = \overline{\sum \mathcal{Z}_\beta}$$

where $\beta = (h_1, \ldots, h_m)$ range in the set of all words such that $\mu(\beta^{-1}) \subseteq \mu(\alpha)$, and $h_1 \ldots h_m = 1$. Then $\mathcal{K}_\alpha$ is a closed *-subalgebra of $\mathcal{T}(A, G, V)$ and

$$\mathcal{K}_\alpha \mathcal{M}_\alpha \subseteq \mathcal{M}_\alpha.$$

Definition.

(i) A *redundancy* is an element $x \in \mathcal{K}_\alpha$ such that

$$x\mathcal{M}_\alpha = 0.$$

(ii) The *redundancy ideal* is the closed two sided ideal of $\mathcal{T}(A, G, V)$ generated by the set of all redundancies.

(iii) The crossed-product of A by G under V is the C*-algebra $A \rtimes_V G$ obtained as the quotient of $\mathcal{T}(A, G, V)$ by the redundancy ideal.

Theorem. If there exists a faithfull invariant state then the two definitions of the crossed-product coincide.

SEMIGROUPS OF ENDOMORPHISMS

In this section we will let A be a unital C*-algebra, P be a subsemigroup of a group G and

$$\alpha : P \to \mathrm{End}(A)$$

be a semigroup action by means of unital endomorphisms. We will also let φ be a faithfull α-invariant state on A.

Let $(\pi_\varphi, H_\varphi, \xi_\varphi)$ be the GNS representation of A. Again it is easy to show that, for every $p \in P$, there exists an isometry s_p on H such that

$$s_p : \pi_\varphi(a)\xi_\varphi \mapsto \pi_\varphi(\alpha_p(a))\xi_\varphi.$$

In fact $\{s_p\}_{p \in P}$ is a semigroup of isometries.

Another Ad Hoc Definition. The (reduced) crossed-product $A \rtimes_\alpha P$ will be defined as the closed *-subalgebra of operators on $H_\varphi \otimes \ell_2(G)$ generated by

$$\{\pi_\varphi(a) \otimes 1 : a \in A\} \ \cup \ \{s_p \otimes \lambda_p : p \in P\},$$

where λ is the left-regular representation of G.

This definition is modeled on Arzumanian and Vershik's original 1978 definition. Observe that it depends on φ!

How do all of this relate to interaction groups?

Theorem. If G is amenable and if $G = P^{-1}P$ then there exists at most one interaction group V extending α and leaving φ invariant, in which case we have

$$A \rtimes_{\alpha} P \simeq A \rtimes_{V} G.$$

In particular this says that, if such an interaction group exists, it may somehow be constructed from the semigroup action and the invariant state!

Also notice that the left-hand side cannot be defined without using φ, while the right-hand side depends only on V!

LARSEN'S CONSTRUCTION

Larsen has proposed to consider essentially the following situation. Let, as before,

$$\alpha : P \to \mathrm{End}(A)$$

be a semigroup action by unital endomorphisms. Also for each $p \in P$, let $\mathcal{L}_p$ be a transfer operator relative to α_p such that $\mathcal{L}_p(1) = 1$, and

$$\mathcal{L}_p \mathcal{L}_q = \mathcal{L}_{qp}, \quad \forall\, p, q \in P.$$

The extension question may be modified as follows:

Question. Is there an interaction group V such that $\alpha_g = V_g$, and $\mathcal{L}_g = V_{g^{-1}}$, for all g in P?

The following is a partial answer:

Theorem. Suppose that $G = P^{-1}P$. Then the above question has an afirmative answer if and only if $\alpha_g \mathcal{L}_g$ commutes with $\alpha_h \mathcal{L}_h$, for every $g, h \in P$. In this case the extension V is unique and if $g \in G$ is written as

$$g = p^{-1}q,$$

with $p, q \in P$, one has that

$$V_g = \mathcal{L}_p \alpha_q.$$

The reader acquainted with Larsen's paper is perhaps curious as to what is the precise relationship between the crossed-products.

However, while there may be some relationship between $\mathcal{T}(A, G, V)$ and the Toeplitz algebra defined by Larsen it seems that our notion of redundancy is significantly different from hers so it is unlikely that the crossed-products will coincide.

AN EXAMPLE

Let $\mathbb{Q}_+^*$ be the multiplicative group of positive non-zero rationals and let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$ be the (multiplicative) subsemigroup of integers.

Let $\mathbb{T}$ be the complex unit circle and for every integer $n \in \mathbb{N}^*$ let

$$\theta_n : z \in \mathbb{T} \mapsto z^n \in \mathbb{T}.$$

Since $\theta_n \theta_m = \theta_{nm}$ we get an action of $\mathbb{N}^*$ on $\mathbb{T}$, and hence also an action α by endomorphisms of $C(X)$:

$$\alpha_n : f \mapsto f \circ \theta_n.$$

Given $n \in \mathbb{N}^*$, let $\mathcal{L}_n$ be the operator on $C(X)$ given by

$$\mathcal{L}_n(f)\big|_z = \frac{1}{n} \sum_{w^n = z} f(w), \quad \forall\, f \in C(\mathbb{T}), \quad \forall\, z \in \mathbb{T}.$$

It is easy to show that $\mathcal{L}_n$ is a transfer operator for α_n and moreover that $\alpha_n \mathcal{L}_n$ and $\alpha_m \mathcal{L}_m$ commute for every n and m. Thus, by the Theorem above, if

$$q = \frac{n}{m}$$

is any rational number the map

$$V_q = \mathcal{L}_m \alpha_n$$

does not depend on the representation of q as a fraction and moreover one has that $\{V_q\}_{q \in \mathbb{Q}_+^*}$ is an interaction group on $C(\mathbb{T})$.

Here is a formula for V_q:

$$V_q(f)\big|_z = \frac{1}{m} \sum_{w^m = z} f(w^n)$$

Notice that f is computed at the n^{th} power of all m^{th} roots of z, that is on all branching values of $z^{n/m}$.

Leaving this aside for a while let

$$\mathcal{G} = \big\{(x,q,y) \in \mathbb{T} \times \mathbb{Q}_+^* \times \mathbb{T} : \exists n,m \in \mathbb{N}^*,\ q = \frac{n}{m},\ x^n = y^m\big\}.$$

(this is to suggest that $y = x^{\frac{n}{m}}$). This becomes a groupoid with the operations

$$(x,q,y)(y,p,z) := (x,pq,z)$$

and

$$(x,q,y)^{-1} := (y,q^{-1},x).$$

Theorem. The groupoid C*-algebra $C^*(\mathcal{G})$ is isomorphic to the crossed-product $C(\mathbb{T}) \rtimes_V \mathbb{Q}_+^*$.

More topics:

(i) Embedding A into $A \rtimes_V G$

(ii) Counter-example for extension problem (expectations do not commute)

(iii) Examples out of action and conditional expectation.

(iv) Examples out of lattice ordered groups.