NONCOMMUTATIVE CARTAN SUB-ALGEBRAS OF C*-ALGEBRAS

Ruy Exel
Florianópolis

1 · Definition. A *-subalgebra B of a von Neumann algebra A is called a Cartan subalgebra if

(i) B is maximal abelian,

(ii) the normalizer of B in A, namely $N(B) := \{u \in U(A) : uBu^* = B\}$ generates A,

(iii) there exists a faithful normal conditional expectation of A onto B. //

2 · Theorem. [Feldman and Moore, 1977] The most general example of a Cartan subalgebra of a von Neumann algebra with separable predual is

$$L^\infty(X, \mu) \subseteq W^*(R, \sigma),$$

where R is a countable standard measured equivalence relation on (X, μ), and σ is a two-cocycle. //

In the paper:

A. Kumjian,
On C*-diagonals,
Canad. J. Math. **38** (1986), no. 4, 969–1008.

a different definition of normalizer, more suitable to study C*-algebras, was introduced:

3 · Definition. [Kumjian] If B is a closed *-subalgebra of a C*-algebra A, the normalizer of B in A is the set

$$N(B) := \{a \in A : aBa^* \subseteq B,\ a^*Ba \subseteq B\}. \ /\!/$$

This was in turn based on a previous definition by Renault in his thesis, where partial isometries were considered.

In the paper mentioned above Kumjian gave a generalization of Feldman and Moore's Theorem to the context of C*-algebras. Kumjian's hypotheses are rather strong and they imply, in particular, that pure states extend uniquely from the subalgebra.

Recently Renault found an extension of Kumjian's ideas:

4 · Definition. [Renault] A *-subalgebra B of a C*-algebra A is called a C*-Cartan subalgebra if

(i) B contains an approximate unit of A,

(ii) B is maximal abelian,

(iii) $N(B)$ (Kumjian's definition) generates A,

(iv) there exists a faithful conditional expectation of A onto B. //

5 · Theorem. [Renault] The most general example of a C*-Cartan subalgebra of a separable C*-algebra is

$$C_0\big(G^{(0)}\big) \subseteq C^*_{\text{red}}(G, \sigma),$$

where G is a Hausdorff, second countable, essentially principal, étale groupoid and σ is a two-cocycle. //

6 · **Corollary.** [Renault] If B is a C*-Cartan subalgebra of a separable C*-algebra A then the conditional expectation from A to B is unique. //

We wish to find generalizations of the last two results to situations in which B is not abelian.

Observe that

$$\left.\begin{array}{rl} (\textsc{Max}) & B' \cap A \subseteq B \\ \\ (\textsc{Ab}) & B \text{ is abelian} \end{array}\right\} \iff B \text{ is maximal abelian}$$

One could attempt to define a generalized Cartan (?) subalgebra by replacing maximal abeliannes in the definition of a C*-Cartan subalgebra with ($\textsc{Max}$).

Recall:

7 · **Theorem.** [Takesaki's Book IX.4.3] Let B be a weakly closed *-subalgebra of a von Neumann algebra A such that $B' \cap A \subseteq B$. Then there exists at most one normal conditional expectation from A to B. //

This indicates that perhaps the Corollary above may be generalized by eliminating condition ($\textsc{Ab}$).

8 · **Example.** With $\mathcal{K}$ denoting the algebra of compact operators on an infinite dimensional Hilbert space, let $A = C([0,1]) \otimes \mathcal{K}$, and $B = 1 \otimes \mathcal{K}$.

Notice that $f \in B' \cap A$ if and only of

$$f(x)k = kf(x), \quad \forall\, x \in [0,1], \quad \forall\, k \in \mathcal{K},$$

which implies that $f = 0$, and hence $f \in B$. Thus $B' \cap A \subseteq B$.

It is easy to prove that in fact the pair (A, B) is a generalized Cartan pair according to the proposed (?) definition above.

However, there are lots of conditional expectations from A to B. Just take any measure μ on $[0,1]$ and put

$$E = \left(\int_0^1 d\mu(x)\right) \otimes I \ : \ C([0,1]) \otimes \mathcal{K} \longrightarrow 1 \otimes \mathcal{K}.$$

This is bad news! So we must reformulate everything if we are to obtain a positive result.

Virtual commutants.

9 · **<u>Definition</u>.** Let B be a closed *-subalgebra of a C*-algebra A. A virtual commutant of B in A is a pair (J, ϕ), where J is an ideal in B, and

$$\phi : J \to A$$

is a B-bimodule map. //

10 · **<u>Example</u>.** Suppose $B \subseteq A \subseteq C$, and let $x \in B' \cap C$. Put

$$\phi : b \in J \mapsto bx \in A,$$

where

$$J = \{b \in B : bx \in A\}.$$

Then (J, ϕ) is a virtual commutant. Moreover any virtual commutant is of this form! //

11 · Definition. A subalgebra B of a C*-algebra A is said to satisfy condition $(\textsc{Max}')$ if for every virtual commutant (J, ϕ) of B in A, the range of ϕ is contained in B. //

One has that

$$(\textsc{Max}') \ \Rightarrow \ (\textsc{Max}),$$

and although $(\textsc{Max})$ and $(\textsc{Max}')$ are not equivalent, it is easy to prove that

$$(\textsc{Max}'{+}\textsc{Ab}) \ \Leftrightarrow \ (\textsc{Max}{+}\textsc{Ab}),$$

so $(\textsc{Max}')$ is a natural condition to consider.

We thus propose:

12 · Definition. $B \subseteq A$ is a generalized Cartan subalgebra if it satisfies all of the above conditions of a C*-Cartan subalgebra, except that in place of maximal abeliannes we require only $(\textsc{Max}')$. //

13 · First Main Theorem. If B is a generalized Cartan subalgebra of a separable C*-algebra A then the conditional expectation from A to B is unique. //

We now wish to describe our generalization of Feldman–Moore–Kumjian–Renault to the above context. As it stands it is obviously impossible since the unit space of a groupoid leads to an abelian algebra!

So we need to use a generalization of the notion of groupoids in which the unit space is noncommutative!

Fortunately this exists. It is Sieben's notion of Fell bundles over inverse semigroups:

14 · Definition. [Sieben, talk at Groupoid Fest, 1998, unpublished] Let S be an inverse semigroup. A Fell bundle over S is a quadruple

$$\mathcal{A} = \Big(\{A_s\}_{s\in S},\ \{\mu_{s,t}\}_{s,t\in S},\ \{\mathrm{star}_s\}_{s\in S},\ \{j_{t,s}\}_{s,t\in S,\, s\leq t}\Big)$$

where, for each $s, t \in S$,

(a) A_s is a complex Banach space,

(b) $\mu_{s,t} : A_s \times A_t \to A_{st}$ is a bilinear map,

(c) $\mathrm{star}_s : A_s \to A_{s^*}$ is a conjugate-linear isometric map, and

(d) $j_{t,s} : A_s \hookrightarrow A_t$ is a linear isometric map for every $s \leq t$.

It is moreover required that for every $r, s, t \in S$, and every $a \in A_r$, $b \in A_s$, and $c \in A_t$,

(i) $(ab)c = a(bc)$,

(ii) $(ab)^* = b^*a^*$,

(iii) $a^{**} = a$,

(iv) $\|ab\| \leq \|a\|\|b\|$,

(v) $\|aa^*\| = \|a\|^2$,

(vi) $aa^* \geq 0$, in B_{rr^*}.

(vii) if $r \leq s \leq t$, then $j_{t,r} = j_{t,s} \circ j_{s,r}$,

(viii) if $r \leq r'$, and $s \leq s'$, then the diagrams

$$\begin{array}{ccc} A_r \times A_s & \xrightarrow{\mu_{r,s}} & A_{rs} \\ {\scriptstyle j_{r',r}\times j_{s',s}} \downarrow & & \downarrow {\scriptstyle j_{r's',rs}} \\ A_{r'} \times A_{s'} & \xrightarrow{\mu_{r',s'}} & A_{r's'} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} A_s & \xrightarrow{\mathrm{star}_s} & A_{s^*} \\ {\scriptstyle j_{s',s}} \downarrow & & \downarrow {\scriptstyle j_{s',s}} \\ A_{s'} & \xrightarrow{\mathrm{star}_{s'}} & A_{s'^*} \end{array}$$

commute. //

Given such a Fell bundle one may define both a full cross-sectional C*-algebra $C^*(\mathcal{A})$ and a reduced cross-sectional C*-algebra $C^*_{\mathrm{red}}(\mathcal{A})$.

In either case the algebra is generated by a representation of the A_s's and, if $s \leq t$, one has that "$A_s \subseteq A_t$".

Why does this generalize groupoids?

If G is an étale groupoid then the collection of all open slices (also called bisections or G-sets) is an inverse semigroup.

For every slice $U \subseteq G$, let A_U be the set of elements in $C^*(G)$ supported in U. Then $\{A_U\}_U$ is a Fell bundle and its cross-sectional C*-algebra is isomorphic to $C^*(G)$. If G is second countable then one may restrict to a countable subsemigroup of all slices.

Reference:

R. Exel,
Inverse semigroups and combinatorial C*-algebras,
Bull. Braz. Math. Soc. (N.S.), **39** (2008), 191 - 313.

Most likely this also holds for twisted groupoids as well.

15 · **Second Main Theorem.** If "$B \subseteq A$" is a generalized Cartan pair, with A separable, then there exists a Fell bundle $\mathcal{A}$ over a countable inverse semigroup S and an isomorphism from A to $C^*_{\text{red}}(\mathcal{A})$ which carries B onto $C^*(\mathcal{E})$, where $\mathcal{E}$ is the restriction of $\mathcal{A}$ to the idempotent semilattice of S. //

Highlights of the proof.

Fix a generalized Cartan pair "$B \subseteq A$". Recall that

$$N(B) := \{a \in A : aBa^* \subseteq B,\ a^*Ba \subseteq B\}.$$

Observe that $N(B)$ is not a linear subspace.

16 · **Definition.** A slice is any closed linear subspace $M \subseteq N(B)$, such that $BM, MB \subseteq M$. //

If $a \in N(B)$ then $\overline{BaB}$ is a slice containing a. Therefore every element of $N(B)$ is contained in some slice.

Since $N(B)$ is supposed to generate A, one has that

$$A = \overline{\sum_{M \in \mathfrak{S}_{A,B}} M}.$$

If M is a slice then $M^*M \subseteq B$, so M may be viewed as a right Hilbert B-module with inner product

$$\langle m, n \rangle = m^* n, \quad \forall\, m, n \in M.$$

By Kasparov's stabilization Theorem there exists a basis, that is, a sequence $\{u_i\}_{i \in \mathbb{N}}$ in M such that

$$m = \sum_{i \in \mathbb{N}} u_i \langle u_i, m \rangle = \sum_{i \in \mathbb{N}} u_i u_i^* m, \quad \forall\, m \in M.$$

Let $P : A \to B$ be the conditional expectation (required by definition of Cartan subalgebra).

Here is the most important computation of this whole program: for $m, n \in M$

$$\sum_{i=1}^{\infty} P(u_i) P(u_i^*) m n^* =$$

$$= \sum_{i=1}^{\infty} P(u_i) P(u_i^* m n^*) =$$

$$= \sum_{i=1}^{\infty} P(u_i) u_i^* m P(n^*) =$$

$$= \sum_{i=1}^{\infty} P(u_i u_i^* m) P(n^*) =$$

$$= P\Big(\sum_{i=1}^{\infty} u_i u_i^* m \Big) P(n^*) =$$

$$= P(m) P(n^*).$$

This proves:

17 · **<u>Lemma</u>.** The series

$$\sum_{i \in \mathbb{N}} P(u_i) P(u_i^*)$$

converges in the strict topology of the multiplier algebra of $R(M) = MM^*$ (closed linear span) and if τ is the sum then

$$\tau m n^* = P(m) P(n^*) = m n^* \tau, \quad \forall\, m, n \in M.$$

Therefore τ lies in the center of said multiplier algebra. //

With $m = n$ we have

$$\tau^{1/2} m m^* \tau^{1/2} = P(m) P(m^*),$$

from where we deduce that:

18 · **<u>Corollary</u>.** If M is a slice then for every $m \in M$ one has that $\|\tau^{1/2} m\| = \|P(m)\|$, and hence the map

$$\phi : P(m) \mapsto \tau^{1/2} m$$

is well defined and extends to a map defined on $\overline{P(M)}$ which is a virtual commutant. //

Since we are assuming (Max') we conclude that $\tau^{1/2} m \in B$, for every $m \in M$, and hence also that $\tau M \subseteq B$.

We then need a technical fact:

19 · Proposition. $P(M) \subseteq \tau M M^*$ (closed linear span). //

This implies that

$$P(M) \subseteq \tau M M^* = M M^* \tau = M(\tau M)^* \subseteq M B^* \subseteq M.$$

Therefore M is invariant under P! Hence

$$\begin{aligned} M &= \operatorname{Im}(P|_M) \oplus \operatorname{Ker}(P|_M) \\ &= \big(B \cap M\big) \oplus \big(\operatorname{Ker}(P) \cap M\big) \\ &= \big(B \cap M\big) \oplus \big(B \cap M\big)^\perp \qquad \text{(as Hilbert Modules).} \end{aligned}$$

So P is detemined on M, and uniqueness follows because $A = \overline{\sum_{M \in \mathfrak{S}_{A,B}} M}$.

Fell bundles from generalized Cartan pairs.

As we have already seen, a slice is any closed linear subspace $M \subseteq N(B)$ that is a B-bimodule.

If M and N are slices then MN (closed linear span) is also a slice and hence the set $\mathfrak{S}_{A,B}$ formed by all slices becomes a semigroup.

It is elementary to check that M^* is a slice for each slice M.

In addition, given a slice M we have that

$$M M^* M \subseteq M B \subseteq M.$$

Thus M is a right Hilbert module over M^*M so, by Cohen-Hewitt, $MM^*M = M$. Therefore $\mathfrak{S}_{A,B}$ is actually an inverse semigroup.

The Fell bundle is then obvious: For each slice M put $A_M = M$, and hence

$$\{A_M\}_{M \in \mathfrak{S}_{A,B}}$$

is a Fell bundle.

When dealing with inverse semigroups there is often information in excess, so one may hope to reduce the size of $\mathfrak{S}_{A,B}$.

20 · **Theorem.** Let $\mathfrak{S}$ be a *-subsemigroup of $\mathfrak{S}_{A,B}$ such that

(i) $A = \overline{\sum_{M\in\mathfrak{S}} M}$,

(ii) for every $M, N \in \mathfrak{S}$, and every $a \in M \cap N$, there exists $K \in \mathfrak{S}$ such that $a \in K \subseteq M \cap N$.

Consider the Fell bundle

$$\mathcal{A} = \{A_M\}_{M\in\mathfrak{S}}$$

and the restriction of $\mathcal{A}$ to the idempotent semilattice of S

$$\mathcal{E} = \{A_M\}_{M\in E(\mathfrak{S})}.$$

Then $C^*_{\text{red}}(\mathcal{A})$ is isomorphic to A via an isomorphism which carries $C^*(\mathcal{E})$ onto B. //

Since A is separable, one may easily find a separable $\mathfrak{S}$ satisfying the conditions above!

Construction of the algebras.

Let us suppose we are given a Fell bundle $\mathcal{A} = \{A_s\}_{s\in S}$ over the inverse semigroup S.

21 · **Definition.** The full cross-sectional C*-algebra of A, denoted $C^*(\mathcal{A})$, is defined to be the universal C*-algebra generated by the disjoint union

$$\bigcup_{s\in S} A_s,$$

subject to the relations given by the multiplication and adjoint operations above, and also such that whenever $s \leq t$, and $a_s \in A_s$, one has that

$$a_s = j_{t,s}(a_s).$$

Easy to show this exists. As always one should provide representations to show it is nontrivial.

• **Special Case: Every element in S is idempotent.**

This is to say that S is a semilattice. Given $e, f \in S$, pick $a \in A_e$. Then $ef \leq f$, so may speak of the composition

$$b \in A_f \mapsto ab \in A_{ef} \overset{j_{f,ef}}{\longmapsto} A_f.$$

This defines a map $A_e \to \mathcal{M}(A_f)$ (multiplier algebra), hence a map

$$\lambda_e : A_e \to \prod_f \mathcal{M}(A_f).$$

22 · **Proposition.** $C^*(\mathcal{A})$ is isomorphic to the closed *-subalgebra generated by $\bigcup_{e \in S} \lambda_e(A_e)$.

• **General Case.**

Given $\mathcal{A}$, consider the restriction $\mathcal{E} = \{A_e\}_{e \in E(S)}$, where $E(S)$ is the idempotent semilattice of S.

By case above we have a nice picture of $C^*(\mathcal{E})$, so may use it as a starting point for analyzing $C^*(\mathcal{A})$.

Fell bundles over groups suggest: construct a Hilbert module over $C^*(\mathcal{E})$, and a regular (?) representation of $C^*(\mathcal{A})$ as operators on this Hilbert module.

Big trouble: regular representations require conditional expectations, but there is none in sight (this is related to similar headaches in non-Hausdorff groupoids).

Somehow it is still possible to induce representations, which is basically the same as extending states. Here is how to do it: pick a state ϕ on $C^*(\mathcal{E})$.

Suppose that $s \in S$ is such that $s \geq e$, for some idempotent e. Then $se = e$, so

$$A_s A_e \subseteq A_e \subseteq C^*(\mathcal{E}).$$

Using an approximate unit $\{u_i\}$ for A_e define

$$\phi^s : A_s \to \mathbb{C}$$

by

$$\phi^s(a) = \lim_i \phi(a u_i).$$

Thus one gets functionals on each A_s, which put together gives

$$\Phi : \bigoplus_s A_s \to \mathbb{C}.$$

Assuming that the initial state ϕ is pure, and with a lot more work, it is possible to prove that Φ is positive and vanishes on

$$a_s \delta_s - j_{t,s}(\alpha_s) \delta_t,$$

so Φ factors through a state on $C^*(\mathcal{A})$.

23 · <u>Definition</u>. The reduced C*-algebra of $\mathcal{A}$, denoted $C^*_{\mathrm{red}}(\mathcal{A})$, is the image of $C^*(\mathcal{A})$ under the direct sum of all GNS representations associated to the Φ's obtained above.

24 · <u>Corollary</u>. The "inclusion" $A_s \to C^*(\mathcal{A})$ is one-to-one.

25 · Corollary. If $s_1, s_2, \ldots s_n$ are pairwise disjoint elements in S, then

$$\bigoplus_{i=1}^{n} A_{s_i} \to C^*(\mathcal{A})$$

is one-to-one.

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THE END