INTERACTIONS

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The starting point for this talk is the theory of crossed products by endomorphisms introduced in:

Exel, R., "A new look at the crossed-product of a C*-algebra by an endomorphism", *Ergodic Theory Dynam. Systems*, **23** (2003), 1733–1750, [arXiv:math.OA/0012084].

- Includes earlier definition of crossed product by an endomorphism when range is a hereditary subalgebra (essentialy the only situation in which that earlier theory is useful).

- Gives sensible algebras even if the endomorphism does not have hereditary range.

- (E-. 2000) First crossed product construction to give Cuntz–Krieger algebras from Markov subshifts!

- (E-. Vershik, 2002) Recovers Arzumanian–Vershik algebras (1978).

- (E-. 2003) KMS states on it are closely related to Gibbs states of Statistical Mechanics.

- (Kajiwara, Watatani, 2003) Gives very interesting results when applied to rational maps on the Julia set. For example the map

$$f(z) = z^2 - 2,$$

gives the Cuntz algebra $\mathcal{O}_\infty$ (needs classification Theorem).

Let us briefly review it: Start with a C*-algebra A and a *-endomorphism

$$\alpha : A \to A$$

Also need a **transfer operator**, namely a positive linear map

$$\mathcal{L} : A \to A$$

such that

$$\mathcal{L}\big(\alpha(a)b\big) = a\mathcal{L}(b).$$

Basic example when α is injective:

$$\mathcal{L} = \alpha^{-1} \circ E,$$

where E is a conditional expectation onto the range of α.

Make a Hilbert A–A–bimodule (correspondence) $\mathcal{X}$ as follows:

(i) $\mathcal{X} = A$, as a linear space,

(ii) Inner product: $\langle x, y\rangle = \mathcal{L}(x^* y)$,

(iii) Left module structure: $a \cdot x = ax$,

(iv) Right module structure: $x \cdot a = x\alpha(a)$.

Definition. The crossed-product of A by α, relative to $\mathcal{L}$, is the C*-algebra denoted by

$$A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

obtained as the Cuntz–Pimsner algebra of $\mathcal{X}$.

Remark. If α is not injective it is still possible to define the crossed-product, but Pimsner's original paper cannot be used. I used universal C*-algebras and **redundancies**, but you may use relative Cuntz–Pimsner algebras of Muhly/Solel. Katsura also have nice ways to do this. See Brownlowe-Raeburn as well.

Alternate definition (universal C*-algebra approach): Define the "Toeplitz" algebra $\mathcal{T}(A, \alpha, \mathcal{L})$ to be the universal C*-algebra generated by a copy of A and an isometry S subject to the relations

$$Sa = \alpha(a)S,$$

$$S^* a S = \mathcal{L}(a),$$

for all $a \in A$.

Let $\mathcal{X} \subseteq \mathcal{T}(A, \alpha, \mathcal{L})$ be given by

$$\mathcal{X} = AS$$

(closed linear span) and notice that for all $a, b, c \in A$

$$(aS)(bS)^*(cS) = aSS^*b^*cS = aS\ \mathcal{L}(b^*c) = a\alpha\big(\mathcal{L}(b^*c)\big)S \in \mathcal{X},$$

so

$$\mathcal{X}\mathcal{X}^*\mathcal{X} \subseteq \mathcal{X},$$

so that $\mathcal{X}$ is a **ternary ring of operators**.

A ternary ring of operators, or TRO, may be defined as any closed linear subspace $\mathcal{X}$ of a C*-algebra such that $\mathcal{X}\mathcal{X}^*\mathcal{X} \subseteq \mathcal{X}$. There is also an axiomatic definition in which the ternary operation

$$(x, y, z) \in \mathcal{X} \times \mathcal{X} \times \mathcal{X} \mapsto xy^*z \in \mathcal{X}$$

plays a central role. TRO's as well as ternary C*-rings were introduced in Zettl's thesis (1983).

Note that $\mathcal{X}\mathcal{X}^*\mathcal{X} \subseteq \mathcal{X}$ implies:

- $\mathcal{X}\mathcal{X}^*$ is a *-subalgebra of $\mathcal{T}(A, \alpha, \mathcal{L})$,

- $\mathcal{X}$ is a left–$\mathcal{X}\mathcal{X}^*$–Hilbert module,

Observe that $\mathcal{X}$ is also a left–A–module.

Definition. A pair (a, k), where $a \in A$, and $k \in \mathcal{X}\mathcal{X}^*$, is called a redundancy if

$$ax = kx, \quad \forall\, x \in \mathcal{X}.$$

Since $\mathcal{X} = AS$, this is equivalent to

$$abS = kbS, \quad \forall\, b \in A.$$

Definition. The crossed-product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is the quotient of $\mathcal{T}(A, \alpha, \mathcal{L})$ by the closed two sided ideal generated by all elements $a - k$, where (a, k) is a redundancy.

Recall relations that hold in the crossed product:

$$Sa = \alpha(a)S,$$

$$S^*aS = \mathcal{L}(a),$$

Notice how asymmetric these formulas are! We have no formula like "$SaS^* = \ldots$" or "$aS = \ldots$"

The cause for this asymmetry is the fact that time evolution is often irreversible.

If we are speaking of a classical irreversible system, say a continuous surjective (possibly non-injective) map

$$T : X \to X,$$

where X is a compact space, think of T as time evolution: if x represents the state of a physical system then $T(x)$ represents the state of the same system on unit of time into the **future**.

What about the **past**? The trouble is that a point x in X may have more than one pre-image under T.

Given $x \in X$ consider the set $T^{-1}(\{x\})$ of all possible past configurations of our system.

Suppose we are given a probability distribution μ_x on $T^{-1}(\{x\})$ to tell us the likelihood of each of these possible past configuration.

Given a continuous scalar valued function f on X (a.k.a. an observable), one may define

$$\mathcal{L}(f)\big|_x = \int_{T^{-1}(\{x\})} f(y)\, d\mu_x(y), \quad \forall\, x \in X.$$

$\mathcal{L}(f)$ represents the expected value of the observable f one unit of time into the past. Supposing that $\mathcal{L}(f)$ is continuous for every f, one checks without difficulty that $\mathcal{L}$ is a transfer operator for the endomorphism α of $C(X)$ defined by

$$\alpha : f \in C(X) \mapsto f \circ \alpha \in C(X).$$

Thus one may apply the above definition to form $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$. Arguably, this turns out to be the "right" C*-algebra to be studied in connection to our dynamical system!

In this talk we wish to take a step toward the study of systems whose future behavior presents the **same degree of uncertainty** as its past.

Inspired by the previous crossed-product construction we postulate that our given algebra of observables A should be embedded in a larger algebra B containing a partial isometry S which **governs time evolution**.

Time evolution itself will be thought of as the interaction between A and S, meaning the **commutation relations** between S and the elements of A. These commutation relations will be required to have the form

$$SaS^* = \mathcal{V}(a)SS^*, \quad \text{and} \quad S^*aS = \mathcal{H}(a)S^*S,$$

for all $a \in A$, where $\mathcal{V}$ and $\mathcal{H}$ are positive linear operators on A. One should think of these maps as corresponding to the past and future evolution. Which is which is not an issue since the situation will be **absolutely symmetric**.

In particular, since we are assuming that $\mathcal{V}$ and $\mathcal{H}$ are positive, this will imply that

$$\mathcal{V}(a) \ \diamond \ SS^*, \quad \text{and} \quad \mathcal{H}(a) \ \diamond \ S^*S. \qquad (\diamond = \text{commutes})$$

For the time being fix $A \subseteq B \ni S$ as above. Given a in A observe that

$$\mathcal{V}(a)S = \mathcal{V}(a)SS^*S = SS^*\mathcal{V}(a)S = S\mathcal{H}(\mathcal{V}(a))S^*S =$$

$$= \mathcal{V}(\mathcal{H}(\mathcal{V}(a)))SS^*S = \mathcal{V}(\mathcal{H}(\mathcal{V}(a)))S,$$

thus it is sensible to assume that

$$\mathcal{V} = \mathcal{V}\mathcal{H}\mathcal{V}$$

and by symmetry

$$\mathcal{H} = \mathcal{H}\mathcal{V}\mathcal{H}.$$

Given a and b in A, let us compute SaS^*bS in the following two ways:

$$SaS^*bS = Sa\mathcal{H}(b)S^*S = \mathcal{V}\big(a\mathcal{H}(b)\big)SS^*S = \mathcal{V}\big(a\mathcal{H}(b)\big)S,$$

while

$$SaS^*bS = \mathcal{V}(a)SS^*bS = \mathcal{V}(a)S\mathcal{H}(b)S^*S = \mathcal{V}(a)\mathcal{V}\big(\mathcal{H}(b)\big)S.$$

Thus we also assume

$$\mathcal{V}(a\mathcal{H}(b)) = \mathcal{V}(a)\mathcal{V}(\mathcal{H}(b)).$$

Since both $\mathcal{V}$ and $\mathcal{H}$ are positive, and hence preserve the involution, this implies that

$$\mathcal{V}(\mathcal{H}(a)b) = \mathcal{V}(\mathcal{H}(a))\mathcal{V}(b).$$

This amounts to

$$\mathcal{V}(xy) = \mathcal{V}(x)\mathcal{V}(y),$$

if **either** x or y belong to $\mathcal{H}(A)$.

By symmetry we also assume that

$$\mathcal{H}(xy) = \mathcal{H}(x)\mathcal{H}(y),$$

if either x or y belong to $\mathcal{V}(A)$.

Under these assumptions we have:

Proposition.

(i) $\mathcal{V}(A)$ and $\mathcal{H}(A)$ are closed *-subalgebras of A,

(ii) $E_{\mathcal{V}} := \mathcal{V} \circ \mathcal{H}$ is a conditional expectation onto $\mathcal{V}(A)$,

(iii) $E_{\mathcal{H}} := \mathcal{H} \circ \mathcal{V}$ is a conditional expectation onto $\mathcal{H}(A)$.

(iv) $\mathcal{V}$ restricts to an isomorphism from $\mathcal{H}(A)$ onto $\mathcal{V}(A)$.

(v) $\mathcal{H}$ restricts to an isomorphism from $\mathcal{V}(A)$ onto $\mathcal{H}(A)$.

(vi) These are inverses of each other.

Definition. If A is a C*-algebra then a pair $(\mathcal{V}, \mathcal{H})$ of positive linear maps

$$\mathcal{V}, \mathcal{H} : A \to A$$

will be called an **interaction** if

(i) $\mathcal{V}\mathcal{H}\mathcal{V} = \mathcal{V}$,

(ii) $\mathcal{H}\mathcal{V}\mathcal{H} = \mathcal{H}$,

(iii) $\mathcal{V}(xy) = \mathcal{V}(x)\mathcal{V}(y)$, if either x or y belong to $\mathcal{H}(A)$,

(iv) $\mathcal{H}(xy) = \mathcal{H}(x)\mathcal{H}(y)$, if either x or y belong to $\mathcal{V}(A)$.

Main goal: to start with an interaction and to **reconstruct** B and S as above.

Still assuming $A \subseteq B \ni S$, let $\mathcal{Y}$ be the linear subspace of B given by

$$\mathcal{Y} = ASA = \Big\{ \sum_{i=1}^{n} a_i S b_i : n \in \mathbb{N},\ a_i, b_i \in A \Big\}.$$

Proposition. Given $a_1, a_2, b_1, b_2, c_1, c_2 \in A$ we have that

$$(a_1 S a_2)(b_1 S b_2)^*(c_1 S c_2) = a_1 \mathcal{V}(a_2 b_2^*)\ S\ \mathcal{H}(b_1^* c_1) c_2.$$

Proof. Apply formulas. □

This shows that $\mathcal{Y}$ is closed under the ternary operation

$$[x, y, z] := x y^* z.$$

In other words $\mathcal{Y}$ is a TRO.

Also $\mathcal{Y}$ is an A–A–bimodule and we have

$$[\xi, a\eta, \zeta] = [\xi, \eta, a^*\zeta], \quad \text{and} \quad [\xi, \eta a, \zeta] = [\xi a^*, \eta, \zeta], \qquad (\dagger)$$

for all $\xi, \eta, \zeta \in \mathcal{Y}$ and all $a \in A$.

Definition. A **generalized correspondence** over A is a TRO which is also an A–A–bimodule satisfying ($\dagger$).

In particular, both

$$\mathcal{K}_\ell := \mathcal{Y}\mathcal{Y}^*, \quad \text{and} \quad \mathcal{K}_r := \mathcal{Y}^*\mathcal{Y}$$

are closed *-subalgebras of B, and $\mathcal{Y}$ is a $\mathcal{K}_\ell$–$\mathcal{K}_r$–Hilbert-bimodule with the left and right inner-products given by

$$\langle m, n\rangle_\ell = mn^*$$

and

$$\langle m, n\rangle_r = m^*n,$$

for any $m, n \in \mathcal{Y}$.

Even though $\mathcal{Y}$ is an A–A–bimodule **it is not necessarily a right or left Hilbert module as there is no reason for $\langle m, n\rangle_r$ or $\langle m, n\rangle_\ell$ to lie in A !!**

We now wish to obtain a formula for the norm of an element of $\mathcal{Y}$ in terms of the maps $\mathcal{V}$ and $\mathcal{H}$.

Proposition. If $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are in A then

$$\Big\| \sum_{i=1}^{n} a_i^* S b_i \Big\| = \big\| \mathcal{V}_n\big(\mathcal{H}_n(aa^*)\big)^{1/2} \mathcal{V}_n(bb^*)^{1/2} \big\|,$$

where

$$a = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

so that aa^* and bb^* are $n \times n$ matrices over A, while $\mathcal{V}_n$ and $\mathcal{H}_n$ are the corresponding operators on $M_n(A)$.

RECONSTRUCTION

Now let us abandon B and S definitively and stay just with A and an abstract interaction $(\mathcal{V}, \mathcal{H})$.

The first step will be to reconstruct the generalized correspondence $\mathcal{Y}$. Let

$$\mathcal{X} = A \otimes_{\mathbb{C}} A,$$

and equip $\mathcal{X}$ with the ternary operation

$$[\,\cdot\,,\,\cdot\,,\,\cdot\,] : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to \mathcal{X}$$

defined by

$$[a_1 \otimes a_2, b_1 \otimes b_2, c_1 \otimes c_2] = a_1 \mathcal{V}(a_2 b_2^*) \ \otimes \ \mathcal{H}(b_1^* c_1) c_2.$$

$$\left(\begin{array}{c} \text{Recall that a while ago we had} \\ (a_1 S a_2)(b_1 S b_2)^* (c_1 S c_2) = a_1 \mathcal{V}(a_2 b_2^*) \ S \ \mathcal{H}(b_1^* c_1) c_2. \end{array} \right)$$

Given $x \in \mathcal{X}$ of the form

$$x = \sum_{i=1}^{n} a_i^* \otimes b_i,$$

define

$$\|x\| = \left\| \mathcal{V}_n\big(\mathcal{H}_n(aa^*)\big)^{1/2} \mathcal{V}_n(bb^*)^{1/2} \right\|.$$

Theorem. $\|\cdot\|$ is a well defined seminorm on $\mathcal{X}$ and its completion is a generalized correspondence over A.

Proof. A mighty headache! □

Once this is granted it is easy to imitate the linking algebra construction to find a C*-algebra B and a partial isometry S satisfying our relations: embed $\mathcal{X}$ in $B(H)$ i.e, such that

$$[x, y, z] = xy^* z.$$

One may prove that this will automatically give representations λ and ρ on A on H such that

$$\lambda(a)x = ax$$

$$x\rho(a) = xa,$$

for all x in $\mathcal{X}$ and a in A. Set

$$\pi(a) = \begin{bmatrix} \lambda(a) & 0 \\ 0 & \rho(a) \end{bmatrix} \in B(H \oplus H)$$

and let

$$S = \begin{bmatrix} 0 & 1 \otimes 1 \\ 0 & 0 \end{bmatrix}.$$

Then π is a representation of A, S is a partial isometry, and

$$S^*\pi(a)S = \pi(\mathcal{H}(a))S^*S, \quad \text{and} \quad S\pi(a)S^* = \pi(\mathcal{V}(a))SS^*,$$

for all a in A.

Thus we have reconstructed the concrete situation from an abstract interaction !!!

Should we live with this happily ever after? Only if we had never seen Pimsner's paper!

The question is: what is the "covariance algebra" for $(\mathcal{V}, \mathcal{H})$?

Definition. Given a generalized correspondence $\mathcal{X}$ over A, let $\mathcal{T} = \mathcal{T}(A, \mathcal{X})$ be the universal C*-algebra generated by $A \cup X$ subject to all algebraic relations in either $\mathcal{X}$ and A, including the ternary operation in $\mathcal{X}$, plus the bimodule relations.

Let us imitate the passage from the <u>Toeplitz–Cuntz–Pimsner</u> algebra to the <u>Cuntz–Pimsner</u> algebra using redundancies:

Definition. Let $\mathcal{X}$ be a generalized correspondence over the C*-algebra A. By a **right redundancy** we shall mean a pair $(a, k) \in \mathcal{T} \times \mathcal{T}$, such that $a \in A$, $k \in \mathcal{X}^*\mathcal{X}$, and

$$xa = xk, \quad \forall\, x \in \mathcal{X}.$$

Likewise, by a **left redundancy** we shall mean a pair $(a, k) \in \mathcal{T} \times \mathcal{T}$, such that $a \in A$, $k \in \mathcal{X}\mathcal{X}^*$, and

$$ax = kx, \quad \forall\, x \in \mathcal{X}.$$

Definition. The **redundancy ideal** is the closed two-sided ideal of $\mathcal{T}$ generated by the elements $a - k$, for all left and right redundancies (a, k).

In fact, unless we assume some nondegeneracy properties, we should not take all redundancies (a, k) but only those such that a lie in certain ideals. This is an idea of Katsura and it has the purpose of avoiding an otherwise nonzero intersection between A and the redundancy ideal which is undesirable.

Definition. Let $\mathcal{X}$ be a generalized correspondence over a C*-algebra A. The covariance algebra for the pair $(A, \mathcal{X})$, denoted $C^*(A, \mathcal{X})$, is the quotient of $\mathcal{T}$ by the redundancy ideal.

This is a very tentative definition which we nevertheless believe to be of interest given its similarities with the enormously popular Cuntz-Pimsner construction. It is the right construction to deal with interactions.

The main questions brought about by the above definition are:

(1) Are the canonical embeddings of A and $\mathcal{X}$ into $C^*(A, \mathcal{X})$ injective? Equivalently, is the intersection between the redundancy ideal and A, or $\mathcal{X}$, the zero ideal?

(2) Can one say anything useful about $C^*(A, \mathcal{X})$, compute its K-theory, or find a concrete faithful representation of it?

(3) Is there a **Fock space representation** of $\mathcal{T}(A, \mathcal{X})$ similar to the one given by Pimsner?

At least we can show this generalizes Cuntz-Pimsner algebras: given a correspondence $\mathcal{X}$ over A let us forget the A–valued inner-product but keep the ternary operation

$$[\xi, \eta, \zeta] = \xi \langle \eta, \zeta \rangle .$$

Theorem. $\mathcal{X}$ is a generalized correspondence and $C^*(A, \mathcal{X})$ is isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{\mathcal{X}}$.

This means that the A–valued inner-product is **unnecessary** for the Cuntz-Pimsner construction!

But, if you need it, you can recover it as a redundancy: if $a = \langle \eta, \zeta \rangle$, then the pair $(a, \eta^* \zeta)$ in $\mathcal{T} \times \mathcal{T}$ is a right-redundancy since

$$\xi a = \xi \langle \eta, \zeta \rangle = [\xi, \eta, \zeta] = \xi \eta^* \zeta$$

so

$$\eta^* \zeta = \langle \eta, \zeta \rangle$$

in the covariance algebra!