KMS STATES FOR GENERALIZED GAUGE ACTIONS ON CUNTZ-KRIEGER ALGEBRAS

(An application of the Ruelle-Perron-Frobenius Theorem)

by Ruy Exel

BASED ON:

R. Exel, “A New Look at The Crossed-Product of a C*-algebra by an Endomorphism”, preprint, 2000, math.OA/0012084.

R. Exel, “Crossed-Products by Finite Index Endomorphisms and KMS states”, preprint, 2001, math.OA/0105195.

R. Exel, “KMS states for generalized gauge actions on Cuntz-Krieger algebras (An application of the Ruelle-Perron-Frobenius Theorem)”, preprint, 2001, math.OA/0110183.

Let A be a unital C*-algebra and

$$\sigma : A \to A$$

be a monomorphism (injective endomorphism) with $\sigma(1) = 1$. We want to define a new notion of *crossed-product* of A by σ.

Starting points:

1) $A \rtimes_\sigma \mathbb{N}$ should be generated by a copy of A and an isometry S.

2) For $a \in A$ we want $S^* a S \in A$.

3) For $a \in A$ we want $Sa = \sigma(a)S$.

In view of (3) it should be noticed that the following relations have been considered:

$$\begin{aligned} S\,a\,S^* &= \sigma(a) \\ S\,a &= \sigma(a)\,S \\ a &= S^*\,\sigma(a)\,S \end{aligned}$$

each being weaker than the previous one. We will therefore **not** require the top one.

In view of (2) observe that we must choose beforehand the map

$$\mathcal{L} : a \in A \mapsto S^* a S \in A.$$

Definition. A *transfer operator* for the pair (A, σ) is a positive linear map

$$\mathcal{L} : A \to A$$

such that $\mathcal{L}(1) = 1$ and $\mathcal{L}(a)b = \mathcal{L}(a\sigma(b))$.

Definition. Given A, σ, and $\mathcal{L}$ as above we define $\mathcal{T}(A, \sigma, \mathcal{L})$ to be the universal C*-algebra generated by a copy of A and an isometry S subject to the relations

i) $Sa = \sigma(a)S$, and

ii) $S^*aS = \mathcal{L}(a)$.

For example, when σ is an automorphism and $\mathcal{L} = \sigma^{-1}$ then $\mathcal{T}(A, \sigma, \mathcal{L})$ turns out to be the Toeplitz extension of the crossed-product considered by Pimsner and Voiculescu for the computation of $K_*(A \rtimes_\sigma \mathbb{Z})$.

It is therefore not reasonable to define the crossed-product to be $\mathcal{T}(A, \sigma, \mathcal{L})$. It will be a quotient of $\mathcal{T}(A, \sigma, \mathcal{L})$.

THE CROSSED-PRODUCT AS A QUOTIENT OF $\mathcal{T}(A,\sigma,\mathcal{L})$

Working within $\mathcal{T}(A,\sigma,\mathcal{L})$, observe that $M := \overline{AS}$ is a right Hilbert-A-module under the inner-product $\langle x,y\rangle = x^*y$. If $x = aS$ and $y = bS$ then

$$\langle x,y\rangle = (aS)^*bS = S^*a^*bS = \mathcal{L}(a^*b).$$

The algebra of generalized compact operators in the sense of Rieffel is given by

$$K(M) = \overline{MM^*} = \overline{ASS^*A}.$$

Obviously M is a left $K(M)$-module. It is also a left A-module. So it could be that there exists $k \in K(M)$ and $a \in A$ which act in the same way as left multiplication operators on M.

Definition. A *redundancy* is a pair (a,k) such that $a \in A$, $k \in \overline{ASS^*A}$, and

$$abS = kbS, \quad \forall\, b \in A.$$

Definition. The crossed-product $A \rtimes_{\sigma,\mathcal{L}} \mathbb{N}$ is defined to be the quotient of $\mathcal{T}(A,\sigma,\mathcal{L})$ by the closed two-sided ideal generated by the differences $a - k$, for all redundancies (a,k).

Let A be an $n \times n$ matrix with $A_{ij} \in \{0,1\}$, having no zero rows or columns.

Consider the "Markov space"

$$\Sigma_A = \Big\{ x \in \prod_{i \in \mathbb{N}} \{1, 2, \dots, n\} : A_{x_i, x_{i+1}} = 1 \text{ for all } i \geq 0 \Big\},$$

and the "Markov sub-shift"

$$\sigma : (x_0, x_1, x_2, \dots) \in \Sigma_A \longmapsto (x_1, x_2, x_3, \dots) \in \Sigma_A.$$

This induces the unital monomorphism

$$\sigma : f \in C(\Sigma_A) \longmapsto f \circ \sigma \in C(\Sigma_A).$$

Now we need a transfer operator: for $f \in C(\Sigma_A)$ put

$$\mathcal{L}(f)\big|_x = \frac{1}{\#\sigma^{-1}(x)} \sum_{y \in \sigma^{-1}(x)} f(y).$$

Theorem. $C(\Sigma_A) \rtimes_{\sigma, \mathcal{L}} \mathbb{N}$ is isomorphic to $\mathcal{O}_A$.

The isomorphism we found maps $C(\Sigma_A)$ to the subalgebra generated by the elements of the form $S_{i_1} \dots S_{i_k} S_{i_k}^* \dots S_{i_1}^*$, where the S_i are the standard generating partial isometries.

The isometry S in $C(\Sigma_A) \rtimes_{\sigma, \mathcal{L}} \mathbb{N}$ is mapped to the isometry part in the polar decomposition of $T := \sum_{i=1}^n S_i$, that is

$$S \longmapsto T(T^*T)^{-1/2}.$$

Back to the general case let $(A, \sigma, \mathcal{L})$ be as above. Given a unitary element u in the center of A it is easy to show that the correspondence

$$\begin{array}{ccc} a & \mapsto & a \\ S & \mapsto & uS \end{array}$$

extends to give an automorphism of $A \rtimes_{\sigma,\mathcal{L}} \mathbb{N}$.

Given a positive element H in the center of A we have that H^{it} is unitary and hence we get an automorphism γ_t of $A \rtimes_{\sigma,\mathcal{L}} \mathbb{N}$ determined by

$$\gamma_t : \quad \begin{array}{ccc} a & \mapsto & a \\ S & \mapsto & H^{it}S \end{array}$$

It is easy to see that $\{\gamma_t\}_{t\in\mathbb{R}}$ is a one-parameter automorphism group of $A \rtimes_{\sigma,\mathcal{L}} \mathbb{N}$.

In the case of $\mathcal{O}_A$ one takes a positive continuous function H on Σ_A and γ_t is determined by

$$\gamma_t(S_j) = H^{it} S_j.$$

When $H = e$ (Neper's number) we get the usual gauge action on $\mathcal{O}_A$.

We want to compute the KMS states for the gauge action in the general case. For this it will be necessary to introduce conditional expectations

$$A \rtimes_{\sigma,\mathcal{L}} \mathbb{N} \to A.$$

This turns out to be a major difficulty!

Let (A, σ) be as above.

How do we obtain a transfer operator? One canonical way is to choose a conditional expectation

$$E : A \to \sigma(A)$$

and set $\mathcal{L}$ to be the composition

$$A \xrightarrow{E} \sigma(A) \xrightarrow{\sigma^{-1}} A$$

In fact the transfer operator leading up to the Cuntz-Krieger algebra above is of this form.

Definition. (Jones, Kosaki, Watatani) A conditional expectation E is said to be of *index-finite type* if one can find elements $u_1, \ldots, u_m \in A$ such that

$$a = \sum_{i=1}^{m} u_i E(u_i^* a), \quad \forall\, a \in A.$$

In this case one defines the *index* of E by

$$\mathrm{ind}(E) = \sum_{i=1}^{m} u_i u_i^*.$$

Theorem. Assume that E is of index-finite type. Then there exists a conditional expectation $G : A \rtimes_{\sigma,\mathcal{L}} \mathbb{N} \to A$ such that

$$G(aS^n S^{*m} b) = \begin{cases} aI_n^{-1}b & , \text{ if } n = m, \\ \\ 0 & , \text{ if } n \neq m. \end{cases}$$

where

$$I_n = \mathrm{ind}(E)\sigma\big(\mathrm{ind}(E)\big) \ldots \sigma^{n-1}\big(\mathrm{ind}(E)\big).$$

If A is commutative this is the unique conditional expectation which is invariant under the scalar gauge action (i.e. the gauge action with $H = e$).

Main Theorem. Suppose that E is of index-finite type and $E(ab) = E(ba)$ for all $a, b \in A$ (e.g. when A is commutative). Let H be a central self-adjoint element with $H > 1$ and consider the associated gauge action. Then for all $\beta > 0$ the correspondence

$$\psi \mapsto \nu = \psi|_A$$

is a bijection from the set of KMS$_\beta$ states ψ on $A \rtimes_{\sigma,\mathcal{L}} \mathbb{N}$ and the set of states ν on A such that

$$\nu(f) = \nu\Big(\mathcal{L}\big(\mathrm{ind}(E) H^{-\beta} f\big)\Big), \quad \forall\, f \in A. \tag{$\dagger$}$$

The inverse of the above correspondence is given by

$$\nu \mapsto \psi = \nu \circ G,$$

where G is the conditional expectation given above.

In the case of $\mathcal{O}_A$ the the right hand side of ($\dagger$) becomes

$$\sum_{y \in \sigma^{-1}(x)} H(t)^{-\beta} f(t),$$

which coincides with the Ruelle operator $\mathcal{L}_{H^{-\beta}}(f)$, or $\mathcal{L}_\beta(f)$ for short, so that equation ($\dagger$) becomes

$$\nu = \mathcal{L}^*_\beta(\nu).$$

Theorem. Suppose that the zero-one $n \times n$ matrix A is irreducible and aperiodic (i.e. there exists a positive integer m such that all entries of A^m are positive). Suppose that H is a Hölder continuous function on Σ_A with $H > 1$ and let γ be the associated gauge action on $\mathcal{O}_A$. Then there exists a unique KMS state for γ.

Let us describe how to get this state. By the Ruelle-Perron-Frobenius Theorem, for each $\beta \geq 0$ there exists a unique pair (λ, ν) where $\lambda > 0$, ν is a probability measure on Σ_A, and

$$\mathcal{L}_\beta^*(\nu) = \lambda\nu.$$

One may then show that the map

$$\beta \mapsto \lambda = \lambda(\beta)$$

is continuous, strictly decreasing, and satisfies

$$\lambda(0) > 1 \quad \text{and} \quad \lim_{\beta \to \infty} \lambda(\beta) = 0.$$

Thus there exists a unique β_c such that $\lambda(\beta_c) = 1$. Let ν be such that

$$\mathcal{L}_{\beta_c}^*(\nu) = \lambda(\beta_c)\nu = \nu.$$

Then by the *Main Theorem* above $\nu \circ G$ is a KMS_{β_c} state on $\mathcal{O}_A$ and it is easy to show this is unique.