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The Laguerre functions in the inversion of the Laplace transform

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Abstract. We present a numerical inversion method for the Laplace transform based on the Fourier series of Laguerre functions. We assume that the values of the Laplace transform are given in a finite interval and that they contain noise. The domain of the restored function is $(0, \infty)$. The convergence of the algorithm is examined and we present a rule for the choice of the Laguerre function parameter. The algorithm is applied to test problems and the results confirm that our algorithm is competitive with others recently presented for this ill-posed problem.

1. Introduction

Spectral methods have become increasingly popular in recent years, with applications to the numerical solution of differential equations and to automatic computations for a wide class of physical problems. These methods appear to be competitive with finite difference and finite element methods. The spectral methods are essentially of Ritz–Galerkin type: they involve representing the solution of a problem as a truncated series of known functions of independent variables. The convergence properties of spectral methods are due to the rapid convergence of expansions of smooth functions in series of some orthogonal functions.

Methods based on expanding the solution in a Fourier series of orthogonal functions of polynomials have also been used in integral equations. For example Selezov *et al* use the Fourier–Legendre series in [10], and the Chebyshev polynomials are used in [7].

It seems to us that the use of orthogonal polynomials with domain $(0, \infty)$ in the numerical inversion of Laplace transforms is a more natural choice. In [3] we employed the Laguerre polynomials in an iterative method for this problem. It was mentioned in that paper that the strong oscillations of this orthogonal set of functions, that appear when the degree is large, are responsible for the poor results when the independent variable increases. As is known, the Laguerre expansion requires many more terms to resolve functions of a given complexity than either Chebyshev or Legendre expansions (see [5] for example); the reason is that significant weight is given to large values of x in Laguerre series. However, as we will see in this paper, in some situations it is possible to find a compromise solution: we obtain approximations $(0, \infty)$ with relatively little effort.

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2. An implicit scheme for the Laplace inversion problem

Let us write the Laplace transform problem in the following form

$$(Ax)(s) = \int_0^\infty e^{-st} x(t)\, dt \tag{2.1}$$

where $A : X \longrightarrow Y$, $X = L^2(R_+)$ and $Y = L^2([c,d]), c > 0$; in the inversion problem the data are given in $[c,d]$. By direct calculations we can see that the adjoint of A in this case is $A^* : Y \longrightarrow X$

$$(A^* v)(t) = \int_c^d e^{-ts} v(s)\, ds. \tag{2.2}$$

Let

$$L_m(t) = \sum_{k=0}^{m} \binom{m}{m-k} \frac{(-t)^k}{k!}$$

be the classical Laguerre polynomials and

$$\phi_k^p(t) = \sqrt{2p} e^{-pt} L_k(2pt) \tag{2.3}$$

the Laguerre functions. Here the constant $p > 0$ is a shift parameter. The functions $\phi_k^p(t)$ constitute an orthonormal basis for $L^2(R_+)$ [12].

We assume that only an approximation to $y(s)$, $y_\varepsilon \in Y$, is available and this approximation is such that

$$\|y - y_\varepsilon\| \leqslant \varepsilon \tag{2.4}$$

where $\varepsilon > 0$ is a known error bound.

The inversion of the Laplace transform is an example of the so-called 'ill-posed problem'; for this reason there is no universal algorithm to solve it. In solving this problem numerically one is faced with the question of numerical instabilities: small perturbations of the data produce large fluctuations in the output data.

To obtain an approximation for the solution of (2.1) we must discretize the problem. Let V_N be the finite dimensional subspace of X spanned by $\phi_i^p(t)$, $i \leqslant N$; we set $A_N = AP_N$, where P_N is the orthogonal projection of X onto V_N.

A very interesting approach to this kind of problem is the use of regularization methods that are generated by certain suitable functions [9]. Implicit schemes are methods in this class: if we take

$$g_k(t) = \frac{1}{t}\left[1 - \left(\frac{\mu}{\mu + t}\right)^k\right] \qquad t \neq 0 \qquad \mu > 0$$

an approximation, in V_N, of the solution x_* for (2.1), which is nearest to x_0, is given by $R_{N,k} y_\varepsilon$ with

$$R_{N,k} = (I - g_k(B)B)P_N x_0 + g_k(B)A_N^*$$

where $B = A_N^* A$. It can be shown that $x_{N,k} = R_{N,k} y_\varepsilon$ is given by the iterations

$$\begin{aligned} &x_{N,0} = x_0 = 0 \\ &(A_N^* A_N + \mu I) x_{N,k} = \mu x_{n,k-1} + A_N^* y_\varepsilon \qquad k = 1, 2, \ldots . \end{aligned} \tag{2.5}$$

We define $\psi_i^p(s) = \int_0^\infty \mathrm{e}^{-st} \phi_i^p(t)\, \mathrm{d}t$, i.e. the Laplace transform of the Laguerre functions $\phi_i^p(t)$. As we know

$$L_i(t) = \sum_{k=0}^{i} \binom{i}{i-k} \frac{(-t)^k}{k!}$$

and by direct calculations we obtain

$$\psi_i^p(s) = \sqrt{2p} \int_0^\infty \mathrm{e}^{-(s+p)t} L_i(2pt)\, \mathrm{d}t = \sqrt{2p} \frac{1}{s+p} \left[1 - \frac{2p}{s+p}\right]^i . \tag{2.6}$$

In the finite dimensional space V_N the solution of (2.5) can be written in the form

$$x_{N,k}(t) = \sum_{i=0}^{N} a_i \phi_i^p(t).$$

If we construct a matrix M with entries

$$m_{ij} = \int_c^d \psi_i^p(s) \psi_j^p(s)\, \mathrm{d}s$$

and a vector f with components

$$f_i = \int_c^d \tilde{y}_\varepsilon \psi_i^p(s)\, \mathrm{d}s$$

where $\tilde{y}_\varepsilon$ is a suitable smooth version of the data y_ε in the discrete case, then (2.5) is equivalent to the following sequence of systems of equations for determining the coeficient vector $a^k = (a_j^k)$, $j = 1, \ldots, N+1$ and $k = 1, \ldots$:

$$(M + \mu I) a^k = \mu a^{k-1} + f. \tag{2.7}$$

Observe that the matrix of this system does not change as we perform successive iterations, therefore the Cholesky decomposition is carried out only once. The next step is to obtain the convergence rate for this procedure. The following theorem gives us an estimate for $\|A - A_N\|$.

Theorem 2.1. Let A be the Laplace transform operator and P_N the orthogonal projection onto the subspace spanned by the Laguerre functions (2.3), $i \leqslant N$. Then there exists a θ, with $|\theta| \leqslant C < 1$ such that

$$\|A(I-P_N)\| \leqslant \left[\frac{2p(d-c)}{(d+p)(c+p)}\right]^{1/2} \frac{\theta^{N+1}}{\sqrt{1-\theta^2}}. \tag{2.8}$$

Proof. We know that

$$\|A(I-P_N)\| = \|(I-P_N)A^*\| = \sup_{\|v\|=1} \{\|(I-P_N)A^*v\|\}.$$

Let $z(t) = (A^*v)(t),\ v \in Y$ such that $\|v\| = 1$, then

$$\|(I-P_N)A^*v\| = \|(I-P_N)z(t)\| = \left\| z(t) - \sum_{i=0}^{N} b_i \phi_i(t) \right\|$$

where $b_i = \int_0^\infty z(t)\phi_i^p(t)\,\mathrm{d}t$ are the Laguerre–Fourier coeficients. In other words, we need the asymptotic rates of convergence, in L_2-sense, of the Laguerre function expansion for functions $z(t) = (A^*v)(t)$, for some $v \in Y$.

Changing the variables in the Laguerre polynomials we can see that

$$L_i(2pt) = \frac{\mathrm{e}^{2pt}}{i!}\frac{\mathrm{d}^i}{\mathrm{d}t^i}(t^i\mathrm{e}^{-2pt})$$

so

$$\phi_i^p(t) = \frac{\sqrt{2p}\mathrm{e}^{pt}}{i!}\frac{\mathrm{d}^i}{\mathrm{d}t^i}(t^i\mathrm{e}^{-2pt}).$$

Using (2.2), the Laguerre–Fourier coefficients are

$$b_i = \int_0^\infty \phi_i^p(t) \int_c^d \mathrm{e}^{-ts} v(s)\,\mathrm{d}s\,\mathrm{d}t.$$

By successive integration by parts we get

$$b_i = \frac{\sqrt{2p}}{i!}(-1)^i \int_0^\infty t^i \int_c^d (s-p)^i \mathrm{e}^{-(s+p)t} v(s)\,\mathrm{d}s\,\mathrm{d}t.$$

Now we can change the order of integration, thus obtaining

$$b_i = (-1)^i \frac{\sqrt{2p}}{i!} \int_c^d (s-p)^i v(s) \int_0^\infty \mathrm{e}^{-st}\mathrm{e}^{-pt} t^i\,\mathrm{d}t\,\mathrm{d}s.$$

We recognize the inner integral as a shifted Laplace transform of t^i. Finally

$$b_i = (-1)^i \frac{\sqrt{2p}}{i!} \int_c^d (s-p)^i v(s) \frac{i!}{(s+p)^{i+1}}\,\mathrm{d}s$$
$$= (-1)^i \sqrt{2p} \int_c^d \frac{1}{s+p}\left(\frac{s-p}{s+p}\right)^i v(s)\,\mathrm{d}s.$$

If we use the Schwarz inequality, noting that $\|v\| \doteq 1$, we obtain

$$|b_i| \leqslant \sqrt{2p}\left\{\int_c^d \frac{1}{(s+p)^2}\left(\frac{s-p}{s+p}\right)^{2i} \mathrm{d}s\right\}^{1/2}\left\{\int_c^d |v(s)|^2 \,\mathrm{d}s\right\}^{1/2}$$

$$\leqslant \left\{\frac{1}{2i+1}\left(\frac{s-p}{s+p}\right)^{2i+1}\Bigg|_{s=c}^{s=d}\right\}^{1/2}$$

$$\leqslant \left\{\frac{1}{2i+1}\left[\left(\frac{d-p}{d+p}\right)^{2i+1} - \left(\frac{c-p}{c+p}\right)^{2i+1}\right]\right\}^{1/2}.$$

Then, there exists a $\theta(i)$, with

$$\frac{c-p}{c+p} \leqslant \theta(i) \leqslant \frac{d-p}{d+p}$$

such that

$$|b_i| \leqslant \left\{\frac{2p(d-c)}{(d+p)(c+p)}\right\}^{1/2} \theta^i(i).$$

Now let $\theta = \sup_{i \geqslant N}|\theta(i)|$; as $\left|\dfrac{c-p}{c+p}\right| < 1$, $\left|\dfrac{d-p}{d+p}\right| < 1$ and $|\theta| < 1$ we have

$$\sum_{i=N+1}^{\infty} |b_i|^2 \leqslant \left\{\frac{2p(d-c)}{(d+p)(c+p)}\right\}\frac{\theta^{2N+2}}{1-\theta^2} \qquad \square$$

The next corollary tells us that taking into account (2.8) it is possible to determine the parameter p.

Corollary 2.1: If the data are in the interval $[c, d]$ then if we choose $p^* = (cd)^{1/2}$ the estimates in (2.8) will be minimal.

Proof. The function $f(x) = (x-p)/(x+p)$ is increasing, so $f(c) \leqslant \theta \leqslant f(d)$; also $g(p) = (c-p)/(c+p)$ and $h(p) = (d-p)/(d+p)$ are decreasing functions, therefore for each $p \geqslant 0$

$$|\theta(p)| \leqslant \max\{|h(p)|, |g(p)|\}.$$

Let p^* be such that $|h(p)| = |g(p)|$. There are two possibilities:

(i) If $p < p^*$ then $|\theta(p)| \leqslant |h(p)|$ and $h(p) > h(p^*) > 0$.
(ii) If $p > p^*$ then $|\theta(p)| \leqslant |g(p)|$ and $g(p^*) < g(p) < 0$.

This shows that $p = p^*$ is the best choice. But $p^* = (cd)^{1/2}$ is the unique root of $(c-p)/(c+p) = (d-p)/(d+p)$. $\square$

Now, if we take $p = p^* = (cd)^{1/2}$ in (2.7) we will have

$$\|A - A_N\| \leqslant \beta_N = \left\{\frac{(d-c)(d+p^*)}{d(c+p^*)}\right\}^{1/2}\left(\frac{d-p^*}{d+p^*}\right)^{N+1}.$$

The first part of the following corollary provides a convergence result; in the second part a convergence rate is given for solutions of (2.1), which are smooth in some sense. The proof follows from theorem 3.1 in [9].

Corollary 2.2. Let $x_{N,k}$ be given by (2.5) (which is equivalent to (2.7)) and let x^* be a solution of (2.1).

(i) If $k = k(N, \varepsilon)$ is such that $\varepsilon\sqrt{k} \to 0$, $\beta_N\sqrt{k} \leqslant c$ and $k \to \infty$ as $N \to \infty$ and $\varepsilon \to 0$ then $x_{N,k} \to x^*$.

(ii) If $x^* - x_0 = (A^*A)^{q/2}z$, $\|z\| \leqslant \rho$, $x^* = (A^*A)^{q/2}v$, $\|v\| \leqslant \rho$ and

$$c_1\left(\left(\frac{\varepsilon}{\rho}\right)^{1/(q+1)} + \beta_N\right) \leqslant k^{-1/2} \leqslant c_2\left(\left(\frac{\varepsilon}{\rho}\right)^{1/(q+1)} + \beta_N^{\min\{1/q,1\}}\right)$$

with some positive constants C_1 and C_2, then

$$\|x_{N,k} - x^*\| \leqslant e_q\{(\rho\varepsilon^q)^{1/(q+1)} + \rho\beta_N^{\min(q,1)}\} \tag{2.9}$$

e_q is independent of ε, N and ρ and $q \longrightarrow e_q$ is bounded in $(0, q_0]$ for any $q_0 \geqslant q/2 > 0$.

The convergence speed of $x_{N,k}$ towards x^* depends on the iteration number k, whose choice depends on the noise level ε and the finite dimensional operator approximation A_N. The central question for regularization methods is the problem of how to choose the regularization parameter; it depends on the quantities that appear during the calculations, like the residue. The earliest methods of this type are based on the discrepancy principle, i.e. matching the error of the approximate solution to the accuracy of the initial data of the problem; this was first done by Morozov [8] and Arcangeli [1]. The principle was later extended to iterative methods. In [9, 11] convergence-rate estimates were obtained, according to which the method has optimal order if the iterations are stopped in accordance with the discrepancy principle. We used rule 2 proposed in [9] in our final algorithm. The convergence rate for this parameter selection is given in theorem 3.3 of that paper.

3. Numerical examples

The examples presented in this section will give some idea of the performance of the algorithm proposed in this paper. We will take some examples used in [2, 4] but we will present only the results obtained from data with noise. The level of noise in the data was simulated by

$$y_\varepsilon(s) = y(s) + \delta\sin(100s) \qquad 0.1 \leqslant s \leqslant 4$$

with the values $\delta = 10^{-4}$ and $\delta = 10^{-2}$, which correspond to $\varepsilon = 1.39 \times 10^{-4}$ and $\varepsilon = 1.39 \times 10^{-2}$ respectively. As the data are taken on the interval $0.1 \leqslant s \leqslant 4$, corollary 2.1 in this paper leads us to the best choice for the parameter p of $p^* = 0.6324$. The approximations were calculated using finite dimensional spaces, V_N, with $N \leqslant 15$.

The three first examples are presented in tables and figures in which we plotted the results. In these tables and in the fourth example we use the following notation.

Data parameters:

N = maximal degree of the polynomials in V_N.

Tol = $T\varepsilon$ (T and ε were given in the algorithm).

Measured parameters:

Res = residuals when the iteration process stopped.

Err = approximation error (Err $= \|x(t) - x_{N,k}(t)\|_{L^2(R^+)}$).

Iter = number of iterations required to get the residual tolerance.

$$L = \left[\sum_{i=1}^{30}\left\{x\left(\frac{i}{2}\right) - x_{N,k}\left(\frac{i}{2}\right)\right\}^2 \Big/ 30\right]^{1/2}$$

$$L_e = \left[\sum_{i=1}^{30}\left\{x\left(\frac{i}{2}\right) - x_{N,k}\left(\frac{i}{2}\right)\right\}^2 \mathrm{e}^{-i/2} \Big/ \left\{\sum_{i=1}^{30} \mathrm{e}^{-i/2}\right\}\right]^{1/2}.$$

We have considered the last two errors for sake of comparison with the results of [2] and [4].

Finally we remark that the parameter μ used in the algorithm was chosen in such way that the first iteration yields a reasonable approximation to the solution $x(t)$; here we took advantage of the fact that $x_{N,1}(t)$ is the Thikonov regularization solution [6].

Example 1. For this example we took

$$y(s) = \frac{1}{s + 0.5} \qquad x(t) = \mathrm{e}^{-t/2} \qquad N = 10.$$

In table 1 we present the results concerned with the two levels of noise. We do not plot the restored function in the case $\delta = 10^{-4}$ because it practically coincides with the original one; for $\delta = 10^{-2}$ we obtain a good approximation as we can see in figure 1.

Table 1. Results for example 1 where $y(s) = 1/(s + 0.5)$, $x(t) = \mathrm{e}^{-t/2}$ and $N = 10$.

	$\delta = 10^{-2}$	$\delta = 10^{-4}$
Tol	1.5×10^{-2}	1.5×10^{-4}
Res	1.39×10^{-2}	1.39×10^{-4}
Err	6.0×10^{-2}	1.6×10^{-3}
L	7.47×10^{-4}	3.49×10^{-5}
L_e	2.98×10^{-3}	1.44×10^{-4}
Iter	1	1

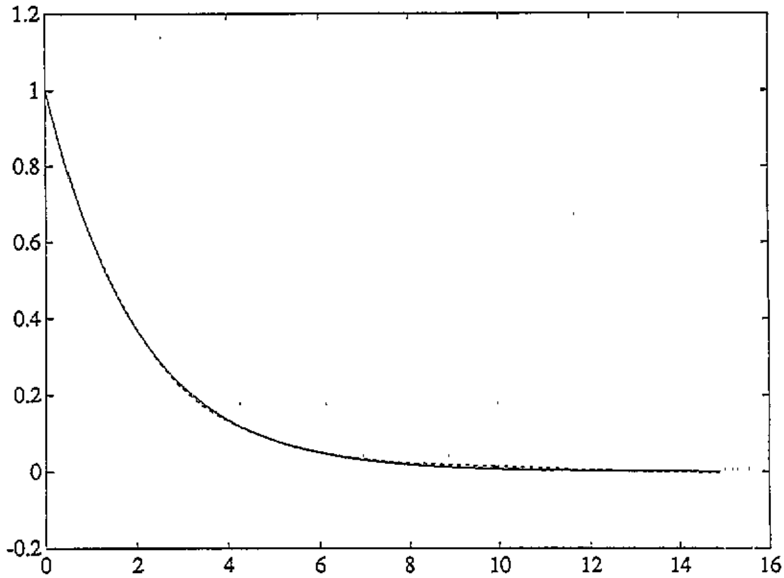

Figure 1. Comparison of $x(t) = e^{-t/2}$ (full curve) with the recovery function using $N = 10$ and $\delta = 10^{-2}$ (broken curve).

Example 2. For this example we took

$$y(s) = \frac{1}{(s+1)^2} \qquad x(t) = te^{-t} \qquad N = 10.$$

The results are shown in table 2 and plotted in figure 2.

Table 2. Results for example 2 where $y(s) = 1/(s+1)^2$, $x(t) = te^{-t}$ and $N = 10$.

	$\delta = 10^{-2}$	$\delta = 10^{-4}$
Tol	1.5×10^{-2}	1.5×10^{-4}
Res	1.39×10^{-2}	1.42×10^{-4}
Err	2.0×10^{-2}	8.1×10^{-3}
L	6.02×10^{-4}	1.54×10^{-4}
L_e	2.78×10^{-3}	4.79×10^{-4}
Iter	2	2

Example 3. For this example we took

$$y(s) = \frac{1}{(s+1)^2+1} \qquad x(t) = e^{-t}\sin(t) \qquad N = 10.$$

Table 3 and figure 3 give the results obtained.

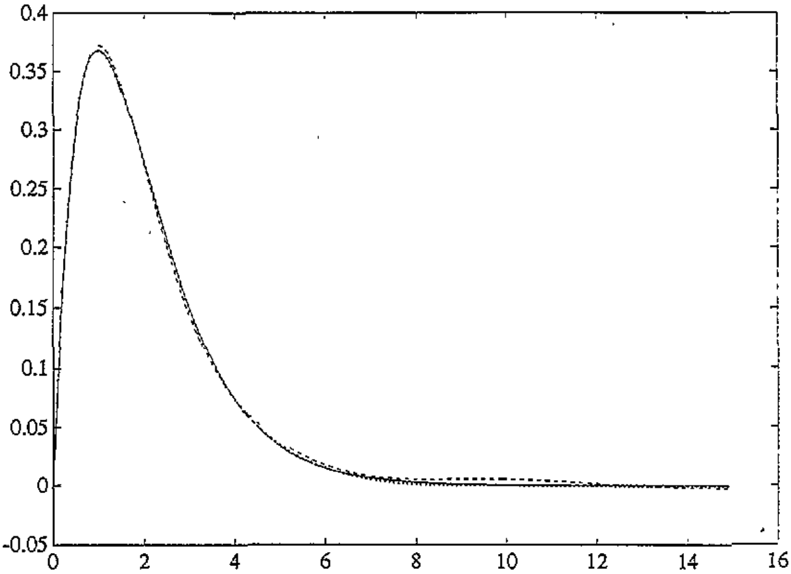

Figure 2. Comparison of $x(t) = \mathrm{e}^{-t}$ (full curve) with recoveries using $N = 10$, $\delta = 10^{-2}$ (broken curve) and $\delta = 10^{-4}$ (dotted curve).

Table 3. Results for example 3 where $y(s) = 1/((s+1)^2+1)$, $x(t) = \mathrm{e}^{-t}\sin(t)$ and $N = 10$.

	$\delta = 10^{-2}$	$\delta = 10^{-4}$
Tol	1.5×10^{-2}	1.5×10^{-4}
Res	1.40×10^{-2}	1.42×10^{-4}
Err	2.4×10^{-2}	9.0×10^{-3}
L	6.59×10^{-4}	1.07×10^{-3}
L_e	5.0×10^{-3}	2.32×10^{-3}
Iter	2	2

Example 4. In this example we used

$$y(s) = \tan^{-1}(1/s) \qquad x(t) = \sin(t)/t$$

and tested $N = 10$ and $N = 15$; we used noisy data with $\delta = 10^{-4}$. We can see in figure 4 that there was no significant improvement in going from $N = 10$ to $N = 15$; this fact still persists when $N > 15$. All this lead us suspect that the smoothness of $x(t)$ is not enough to recover the function in a large interval. In fact $x'(t) \notin L^2(R^+)$ so $P_N x(t)$ is a poor appoximation for $x(t)$ [5]; as an illustration we plotted $x(t)$ and $P_{12}x(t)$ in figure 5. However a small number of iterations was needed to get Res $= 1.5 \times 10^{-4}$. The error $e(t) = x(t) - x_{15,k}(t)$ is plotted in figure 6.

4. Conclusion

As is well known every Laplace transform numerical inversion method breaks down for some functions and therefore the verification by different methods can greatly

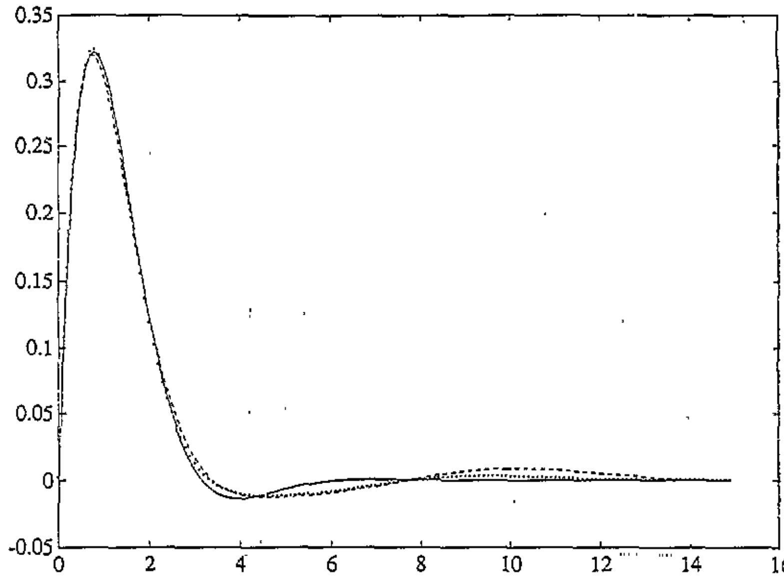

Figure 3. Comparison of $x(t) = e^{-t}\sin(t)$ (full curve) with recoveries using $N = 10$, $\delta = 10^{-2}$ (broken curve) and $\delta = 10^{-4}$ (dotted curve).

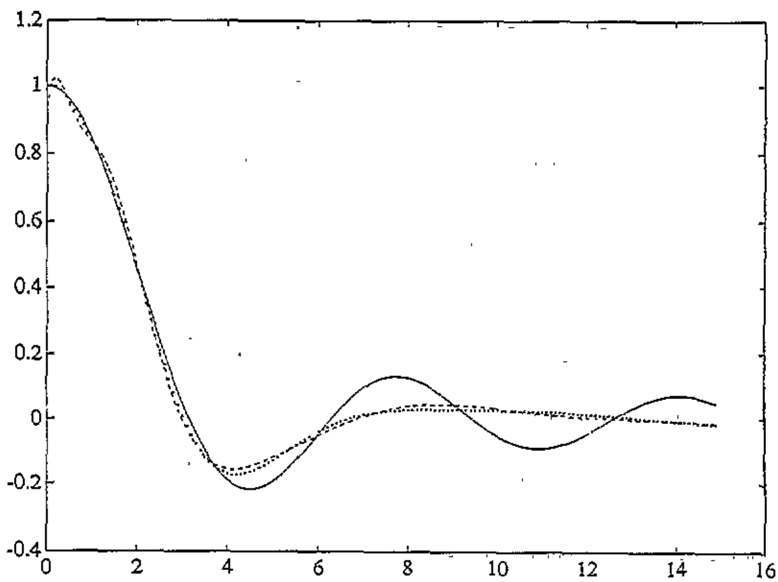

Figure 4. Comparison of $x(t) = \sin(t)/t$ (full curve) with recoveries using $\delta = 10^{-4}$, $N = 10$ (broken curve) and $N = 15$ (dotted curve).

increase confidence in the results achived. The method proposed in [3] was substantially improved with the aid of the Laguerre functions. The algorithm proposed in this paper was tested on a class of examples considered in the recent literature and they confirm that it is competitive with other algorithms recently presented for problems with noisy numerical data (for example, [2]).

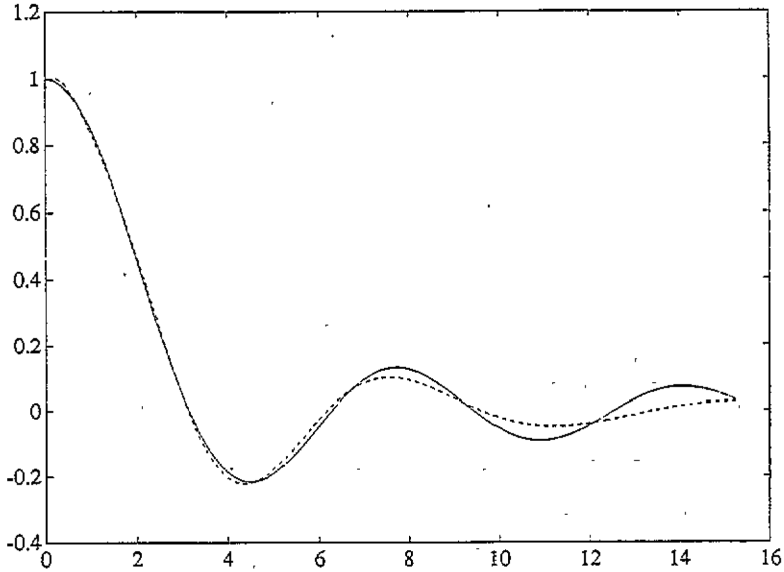

Figure 5. Comparison of $x(t) = \sin(t)/t$ (full curve) and the orthogonal projection of $x(t)^t$ (broken curve) on the Laguerre functions subspace ($N = 12$).

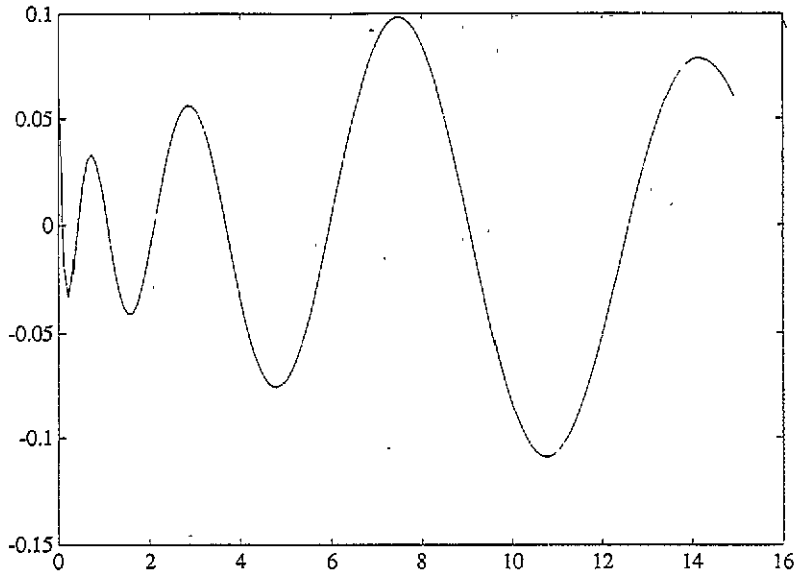

Figure 6. Error of the restored function ($N = 15$) $x_{15}(t)$.

Some words about the hypothesis of corollary 2.2 are needed. In fact we did not present examples in which $x^* \in R(A^*A)^{q/2}$: it is very difficult to find such functions when A is the Laplace transform operator. Although we have not yet found any of these examples, the theoretical results presented in section 2 are very important in substantiating the convergence results of our algorithm.

References

[1] Arcangeli R 1966 Pseudo-solution de l'equation $Ax = y$ *C. R. Acad. Sci.* A **263** 282–5

[2] Brianzi P and Frontini M 1991 On the regularized inversion of the Laplace transform *Inverse Problems* **7** 355–68

[3] Cunha C and Viloche F 1991 An iterative method for the numerical inversion of Laplace transform *Technical Report* 66/91 IMECC/UNICAMP

[4] Davis B and Martin B 1979 Numerical inversion of the Laplace transform: a survey and comparison of methods *J. Comput. Phys.* **33** 1–32

[5] Gottlied D and Orszag S 1977 *Numerical Analysis of Spectral Methods* (Philadelphia, PA: SIAM)

[6] Groetsch C W, King J T and Murio D 1982 Asymptotic analysis of a finite element for Fredholm equations of the first kind *Treatment of Integral Equations by Numerical Methods* ed C T Baker and G F Muller (New York: Academic)

[7] Gur'yanova K N, Medevedea L A and Orlava Yu 1988 Numerical inversion of Laplace transform by means of Fourier series in orthogonal polynomials *Investigations in the Theory of Approximation* **120** 13–7 (in Russian)

[8] Morozov A 1966 On the solution of functional equations by the method of regularization *Sov. Math. Dokl.* **7** 414–7

[9] Plato R and Vainikko G 1990 On the regularization of projection methods for solving ill-posed problems *Numer. Math.* **57** 63–79

[10] Selezov T and Korsunski S V 1988 Numerical inversion of the Laplace transform on the basis of Fourier–Bessel expansion *Dokl. Akad. Nauk Ukrainian SSR* A **11** 25–8

[11] Vainikko G 1982 The discrepancy principles for a class of regularization methods *USSR Comput. Maths. Math. Phys.* **22** (3) 1–19

[12] Zygmund A 1959 *Trigonometric Series* vol I (Cambridge: Cambridge University Press)