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A note on existence and uniqueness of solutions for a 2D bioheat problem

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ABSTRACT
We consider a 2D bioheat model on a domain with curvilinear polygonal boundary and boundary conditions involving heat transfer between blood vessels and tissue. Based on elliptic regularity on polygons and semigroup theory, we obtain a Fourier series representation for the solution in H^2 settings. The results become useful in that they provide theoretical support for numerical approaches recently published in the literature.

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1. Introduction

Modeling of thermal energy transport in living tissues has become crucial in applications such as cancer treatment, burn therapy, cryosurgery, laser irradiation, and others.[1–5] As a result, several analytical and numerical methods have appeared dealing with initial boundary-value problems involving bioheat transfer equations for different geometries and boundary conditions.[3,5–8] In particular, after the seminal paper by Pennes [9], there have been several studies to characterize the thermal tissues properties, which gave rise to various bioheat transfer equations with applications in distinct scenarios. However, despite the efforts of researchers to analyze and describe solutions for certain problems in medical applications, e.g. [2,4,10–12], rigorous analyses concerning existence and uniqueness are still lacking. The goal of this note is to partially fill in this gap by providing a Fourier-based analysis approach for a 2D bioheat transfer equation with convective boundary conditions on a rectangle, whose numerical treatment and application in perfusion coefficient inverse estimation problems have been given in [7,13]; related work concerning inverse estimation problems involving the bioheat model can be found in [1,3,5,14–17]. Our existence and uniqueness analysis is based on elliptic regularity theory on polygons in $H^2(\mathcal{O})$ settings for a domain $\mathcal{O}$ with curvilinear polygonal boundary [18] and internal angles equal to $\pi/2$, along with semigroup theory. By assuming so, our approach covers the traditional case where the tissue occupies a rectangular region.[3]

The rest of the note is organized as follows. In Section 2, the boundary conditions are taken into account to transform the original problem into a standard Cauchy problem involving an elliptic operator. In Section 3, the spectral problem for the elliptic operator associated with Pennes' equation is solved and an analytical semigroup of contractions on $L^2(\mathcal{O})$ is generated. Proceeding this way, it is established not only a computable Fourier series representation for the bioheat solution problem, but also a rigorous theoretical treatment is provided in order to justify eigenfunction expansions methods very often seen in the literature.[5,7] In particular, a highly accurate method for computing

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eigenvalues and eigenfunctions of the associated elliptic operator is also introduced and illustrated by way of several numerical examples. The note ends with some conclusions in Section 4.

Throughout the paper, $H^s(\mathcal{O})$ stands for the Sobolev space of functions with derivatives of order less than or equal to s in $L^2(\mathcal{O})$, with respective norm denoted by $\|.\|_{s,2,\mathcal{O}}$.

2. Bioheat equation

Let $\mathcal{O} \subset \mathbb{R}^2$ denote an open, bounded, and connected domain with boundary $\Gamma = \cup_{i=1}^4 \overline{\Gamma}_i$, where $\overline{\Gamma}_i$ (the closure of the open arc Γ_i) is a C^∞ curve.[18] Let $\mathbf{s}_i := \overline{\Gamma}_i \cap \overline{\Gamma}_{i+1}$, $1 \leq i \leq 3$, $\mathbf{s}_4 =: \overline{\Gamma}_4 \cap \overline{\Gamma}_1$ and assume that $\overline{\Gamma}_i$ follows $\overline{\Gamma}_{i+1}$ in an anticlockwise direction. Also, assume that for each $1 \leq i \leq 4$, $\omega_i = \pi/2$, where ω_i is the internal angle of the polygon with the vertex in s_i. Let $\mathbf{x} = (x, y)$ and let $\nu_i = (\nu_x, \nu_y)$, $\tau_i = (-\nu_y, \nu_x)$ be the unit outward normal vector field to Γ_i and the tangent vector field to Γ_i, respectively. The assumption on ω_i is purely technical and introduced to avoid the presence of singular solutions.[19] Let the temperature of a perfused tissue occupying the region $\mathcal{O}$ be denoted by $U = U(\mathbf{x}, t)$ where t stands for the time variable. The boundary Γ_1 represents the upper skin surface, while the boundary Γ_3 corresponds to a wall between the tissue and an adjoint large blood vessel.[6,7] The Pennes' bioheat model with convective boundary conditions that we are interested is given by

$$\rho c\, U_t - \kappa \Delta U + w_b c_b (U - U_a) = q_m + q_e \quad \text{in} \quad \mathcal{O} \times]0, +\infty[, \tag{1}$$

$$\frac{\partial U}{\partial \nu_i} = 0 \quad \text{on} \quad \Gamma_i \times]0, +\infty[,\ i = 2, 4 \tag{2}$$

$$\kappa \frac{\partial U}{\partial \nu_3} + hU = hU_\infty \quad \text{on} \quad \Gamma_3 \times]0, +\infty[, \tag{3}$$

$$U = U_b \quad \text{on} \quad \Gamma_1 \times]0, +\infty[, \tag{4}$$

$$U = U_0 \quad \text{in} \quad \mathcal{O} \times \{0\}, \tag{5}$$

where ρ, c, h and κ are positive constants that stand for the density, the specific heat, the heat transfer coefficient, and the thermal conductivity of the tissue, respectively; c_b is a positive constant denoting the blood specific heat, w_b is the mass flow rate of blood per unit volume of tissue such that $w_b \geq 0$ a.e. in $\mathcal{O}$, $w_b \in L^\infty(\mathcal{O})$, and $U_a \in L^\infty(\mathcal{O})$ is the temperature of arterial blood. Additionally, $U_0 \in L^2(\mathcal{O})$, $U_b \in H^{3/2}(\Gamma_1)$ and $U_\infty \in H^{1/2}(\Gamma_3)$, are the initial temperature of the tissue, the skin surface temperature and the environmental temperature (in the adjacent blood vessel), respectively. Finally, for each $T > 0$, $q_m, q_e \in C^\sigma\left(]0, T]; L^2(\mathcal{O})\right)$, $0 < \sigma < 1$, stand for the metabolic heat generation per unit volume and the volumetric rate of external heat, respectively. The boundary condition (3) attempts to simulate the heat transfer between the tissue and the adjoint blood vessel in Γ_3, while (2) is an adiabatic condition. In the upper skin surface, the temperature is prescribed and gives rise to the boundary condition (4).

The key idea behind our existence and uniqueness analysis is to transform the original initial and boundary value problem (6)–(10) into a Cauchy problem through a suitable change of variables involving a function $U_{\text{ext}} \in H^2(\mathcal{O})$ satisfying (2)–(4). The existence proof of

such a function depends on mild conditions on function U_b (the skin surface temperature) and is postponed to Section 3 (see Lemma 3.4). In fact, setting $V = U - U_{\text{ext}}$, the original problem (6)–(10) can be expressed as

$$V_t + \mathcal{L}V = f \quad \text{in} \quad \mathcal{O} \times]0, +\infty], \tag{6}$$

$$\frac{\partial V}{\partial \nu_i} = 0 \quad \text{on} \quad \Gamma_i \times]0, +\infty[,\ i = 2, 4 \tag{7}$$

$$\kappa \frac{\partial V}{\partial \nu_3} + hV = 0 \quad \text{on} \quad \Gamma_3 \times]0, +\infty[, \tag{8}$$

$$V = 0 \quad \text{on} \quad \Gamma_1 \times]0, +\infty[, \tag{9}$$
$$V = U_0 - U_{\text{ext}} \quad \text{in} \quad \mathcal{O} \times \{0\}, \tag{10}$$

where $\mathcal{L} : \mathcal{A} \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is the elliptic operator given by $\mathcal{L} = (\rho c)^{-1}(-\kappa\Delta + c_b w_b I)$, with

$$\mathcal{A} := \{v \in H^2(\mathcal{O}), v \text{ satisfying (7)–(9)}\} \tag{11}$$

and $f = (\rho c)^{-1}(q_m + q_e + c_b w_b U_a) - \mathcal{L}U_{\text{ext}}$. Proceeding this way, problem (6)–(10) can be reformulated as the following Cauchy problem:

P: Find $\mathcal{V} \in C([0, +\infty[; L^2(\mathcal{O})) \cap C^1(]0, +\infty[; H^2(\mathcal{O})) \cap \mathcal{A}$ such that

$$\frac{d\mathcal{V}}{dt} + \mathcal{L}\mathcal{V} = \mathfrak{f}, \; t > 0; \quad \mathcal{V}(0) = U_0 - U_{\text{ext}}, \tag{12}$$

where $\mathfrak{f} : [0, +\infty[\to L^2(\mathcal{O})$ is given by $\mathfrak{f}(t) = f(., t)$, for all $t \geq 0$.

3. Existence and uniqueness

For our existence and uniqueness analysis, we will show that $-\mathcal{L}$ is the infinitesimal generator of an analytical semigroup of contractions on $L^2(\mathcal{O})$. We start by deriving a set of technical results.

Lemma 3.1: *Given any $\gamma_0 \in \mathbb{C}$ such that $Re(\gamma_0) \geq 0$ and any complex-valued $g \in L^2(\mathcal{O})$, there exists a unique complex-valued $\widetilde{U} \in \mathcal{A}$ such that*

$$(-\mathcal{L} - \gamma_0 I)\widetilde{U} = g \quad a.\,e.\ in\ \mathcal{O}. \tag{13}$$

Proof: In the context of complex-valued functions, let $H^1_{\Gamma_1}(\mathcal{O}) := \{\varphi \in H^1(\mathcal{O}),\ \varphi|_{\Gamma_1} = 0\}$. As a preliminary Step, we first prove that the weak formulation of (13) admits a unique solution. For this, we recall that such a formulation reads: find $\widetilde{U} \in H^1_{\Gamma_1}(\mathcal{O})$ such that

$$a_{\gamma_0}(\widetilde{U}, \varphi) := \kappa(\rho c)^{-1}\left[(\nabla\widetilde{U}, \nabla\varphi)_{\mathcal{O}} + h(\widetilde{U}, \varphi)_{\Gamma_3}\right] + (((\rho c)^{-1}c_b w_b + \gamma_0)\widetilde{U}, \varphi)_{\mathcal{O}} = -(g, \varphi)_{\mathcal{O}} \tag{14}$$

for all $\varphi \in H^1_{\Gamma_1}(\mathcal{O})$, where $(\cdot, \cdot)_\Omega$ denotes the standard (complex) inner product in $L^2(\Omega)$. Having introduced the bilinear form a_{γ_0}, Poincaré's inequality in $H^1_{\Gamma_1}(\mathcal{O})$, Hölder's inequality and the trace theorem yield

$$Re[a_{\gamma_0}(\widetilde{U}, \widetilde{U})] \geq C_1\|\widetilde{U}\|^2_{1,2,\mathcal{O}}, \; |a_{\gamma_0}(\widetilde{U}, \varphi)| \leq C_2\|\widetilde{U}\|_{1,2,\mathcal{O}}\|\varphi\|_{1,2,\mathcal{O}}, \tag{15}$$

where C_1, C_2 depend on $(\mathcal{O}, h, c_b, w_b, \gamma_0, \kappa, \rho, c)$. As (15) ensures that the bilinear form a_{γ_0} is bounded and coercive, Lax-Milgram theorem shows that problem (14) has a unique solution and our preliminary result is proved. Proceeding analogously, given arbitrary real-valued $\widetilde{f} \in L^2(\mathcal{O})$ and $\psi \in H^{1/2}(\Gamma_3)$, there exists a unique (real-valued) $\widetilde{u} \in H^1_{\Gamma_1}(\mathcal{O})$ that is the weak solution of the boundary-value problem

$$-\mathcal{L}\widetilde{u} = \widetilde{f} \quad \text{in} \quad \mathcal{O}, \tag{16}$$
$$\frac{\partial\widetilde{u}}{\partial\nu_i} = 0 \quad \text{on} \quad \Gamma_i,\ i = 2, 4 \tag{17}$$
$$\frac{\partial\widetilde{u}}{\partial\nu_3} = \psi \quad \text{on} \quad \Gamma_3, \tag{18}$$
$$\widetilde{u} = 0 \quad \text{on} \quad \Gamma_1. \tag{19}$$

Also notice that for all $\varphi \in H^1_{\Gamma_1}(\mathcal{O})$ such $\widetilde{u}$ satisfies

$$a_0(\widetilde{u}, \varphi) - h(\rho c)^{-1}\kappa(\widetilde{u}, \varphi)_{\Gamma_3} = -(\widetilde{f}, \varphi)_{\mathcal{O}} + ((\rho c)^{-1}\kappa\psi, \varphi)_{\Gamma_3}. \tag{20}$$

Based on these results, the regularity of $\widetilde{U}$ can be improved. In fact, for the case of dealing with real-valued functions, consider the spaces D^1, D^2 defined by

$$D^1 = \{\varphi \in H^1(\mathcal{O}),\ \mathcal{L}\varphi \in L^2(\mathcal{O}),\ \varphi \text{ satisfying (17)–(19) for } \psi = 0\},$$

$$D^2 = \{\varphi \in H^2(\mathcal{O}),\ \varphi \text{ satisfying (17)–(19) for } \psi = 0\}.$$

We now observe that elliptic PDE theory on polygons ensures that there exists $u_r \in H^2(\mathcal{O})$ and $\{\sigma_i\}_{i=1}^{\ell}$, $\sigma_i \in D^1 \backslash D^2$, ℓ being the codimension of D^2 as a subspace of D^1, such that $\widetilde{u} = u_r + \sum_{i=1}^{\ell} c_i\sigma_i$ [19, Theorem 3.2.4]. Let $\vartheta_j = 0$ if $j \in \{2, 3, 4\}$, $\vartheta_1 = \pi/2$ and $\lambda_{j,m} = (\vartheta_j - \vartheta_{j+1} + m\pi)/\omega_j$, for $j \in \{1, 2, 3, 4\}$ and $m \in \mathbb{Z}$. As $w_j = \pi/2$, it is easy to check that for each $j \in \{1, 2, 3, 4\}$ and $m \in \mathbb{Z}$, $\lambda_{j,m} \notin]-1, 0[$. As a result, from [19, Proposition 3.2.1], it follows that D^2 is a closed subspace of D^1 with $\ell = 0$, thus $\widetilde{u} \in H^2(\mathcal{O})$. Let $\widetilde{u}_1 \in H^2(\mathcal{O})$ be the solution of (16)–(19) corresponding to $\widetilde{f} = Re(g + \gamma_0\widetilde{U})$ and $\psi = Re(-h\widetilde{U})$; analogously let $\widetilde{u}_2 \in H^2(\mathcal{O})$ be the solution of (16)–(19) corresponding to $\widetilde{f} = Im(g + \gamma_0\widetilde{U})$ and $\psi = Im(-h\widetilde{U})$. Letting $\widetilde{u} = \widetilde{u}_1 + i\widetilde{u}_2$, $\varphi = \widetilde{U} - \widetilde{u}$, subtracting (14) from (20), Poincaré's inequality yields

$$\|\widetilde{U} - \widetilde{u}\|^2_{0,2,\mathcal{O}} \le (\kappa(\rho c)^{-1}/C_1)\|\nabla(\widetilde{U} - \widetilde{u})\|^2_{0,2,\mathcal{O}} = 0.$$

This implies that $\widetilde{U} = \widetilde{u}$ a. e. in $\mathcal{O}$ and hence $\widetilde{U} \in \mathcal{A}$. □

Lemma 3.2: *Under the assumption that $v \in \mathcal{A}$, there exists $C = C(\mathcal{O})$ such that*

$$\|v\|_{2,2,\mathcal{O}} \le C(\|\mathcal{L}v\|_{0,2,\mathcal{O}} + \|v\|_{1,2,\mathcal{O}}). \tag{21}$$

Proof: We observe that the trace operators $T_1 : H^2(\mathcal{O}) \to H^{3/2}(\Gamma_1)$, $T_1(\varphi) = \varphi|_{\Gamma_1}$, $T_s : H^1(\mathcal{O}) \to H^{1/2}(\Gamma_3)$, $T_s(\varphi) = \varphi|_{\Gamma_3}$, and $T_i : H^2(\mathcal{O}) \to H^{1/2}(\Gamma_i)$, $T_i(\varphi) = \dfrac{\partial\varphi}{\partial\nu_i}|_{\Gamma_i}$, $2 \le i \le 4$, are all linear, continuous, and surjective [18, Theorem 1.5.2.1]. Hence, $\mathcal{B} := \{\varphi \in H^2(\mathcal{O}), T_i(\varphi) = 0,\ i = 1, 2, 4\}$ is a closed subspace of $H^2(\mathcal{O})$ and thus a Hilbert space with the induced inner product. In addition, for all $\psi \in H^{1/2}(\Gamma_3)$, we can find a function $\varphi \in \mathcal{B}$ satisfying (18). It is not difficult to see that the operator $\widetilde{T} : \mathcal{B} \to H^{1/2}(\Gamma_3)$ given by $\widetilde{T}(\varphi) = T_3(\varphi)$ is linear, continuous, surjective and has a continuous right inverse $\widetilde{T}_R^{-1}$ which is a bijection between $H^{1/2}(\Gamma_3)$ and $Ker(\widetilde{T})^{\perp} \subset \mathcal{B}$ [20, Proposition 4.6.1]. As a consequence, for each $v \in \mathcal{A}$ and $w := \widetilde{T}_R^{-1}(T_s(-\kappa^{-1}h\ v))$, for some constant $C = C(\mathcal{O}, h, \kappa)$, we have

$$\|w\|_{2,2,\mathcal{O}} \le C\|T_s(v)\|_{1/2,2,\Gamma_3} \le C\|v\|_{1,2,\mathcal{O}}. \tag{22}$$

Now notice that $u := w - v$ satisfies (17)–(19) with $\psi = 0$. Notice also that Lemma 3.1 and the closed range mapping theorem imply that the embedding $I : D^2 \to D^1$ is continuous. Therefore, from the open mapping theorem, $I^{-1} : D^1 \to D^2$ is also continuous and we get (21) for u. Inequality (22) and the definition of u yield (21) for v. □

In view of the Lemmas 3.1, 3.2 and well-known results from spectral theory for self-adjoint operators with compact inverse [21, Theorems 7, p.39], there exists a non-decreasing sequence of real positive eigenvalues $\{\lambda_k\}_{k=1}^{+\infty}$ of $\mathcal{L}$ such that $\lim_{k\to+\infty}\lambda_k = +\infty$ and an orthonormal basis $\{\psi_k\}_{k=1}^{+\infty}$ of $L^2(\mathcal{O})$ consisting of real-valued eigenfunctions of $\mathcal{L}$. Accordingly, for each $\varphi \in L^2(\mathcal{O})$, we can write $\varphi = \sum_{k=1}^{+\infty} c_k\psi_k$, where $c_k := (\varphi, \psi_k)_{L^2(\mathcal{O})}$, and we are able to characterize the elements of the set $\mathcal{A}$.

Lemma 3.3: *Let $\mathcal{A}$ be defined in (11) and $\varphi \in L^2(\mathcal{O})$. Then, $\varphi \in \mathcal{A}$ if and only if*

$$\sum_{k=1}^{+\infty} c_k^2 \lambda_k^2 < +\infty.$$

Proof: Let $\mathcal{A}$ be endowed with the inner product $B(u,v) := (-\mathcal{L}u, -\mathcal{L}v)_{\mathcal{O}} + a_0(u,v)$. Notice that due to Lemma 3.2, the corresponding induced norm is equivalent to the standard norm on $\mathcal{A}$. Now, since for each $k \in \mathbb{N}$ and $u \in \mathcal{A}$, $B(u,\psi_k) = (\lambda_k + \lambda_k^2)(u,\psi_k)_{\mathcal{O}}$, it is clear that $\{\psi_k\}_{k=1}^{+\infty}$ is an orthogonal basis for $\mathcal{A}$. In particular, since $c_k = (\varphi,\psi_k)_{L^2(\mathcal{O})} = B(\varphi,\psi_k)/B(\psi_k,\psi_k)$, if $\varphi \in \mathcal{A}$, we have

$$\sum_{k=1}^{+\infty} c_k \psi_k = \varphi,$$

where convergence is in the sense of $H^2(\mathcal{O})$, and hence

$$\sum_{k=1}^{+\infty} \|c_k \mathcal{L}\psi_k\|_{0,2,\mathcal{O}}^2 = \sum_{k=1}^{+\infty} c_k^2 \lambda_k^2 < +\infty.$$

Conversely, suppose that $\varphi \in L^2(\mathcal{O})$ and

$$\sum_{k=1}^{+\infty} c_k^2 \lambda_k^2 < +\infty. \tag{23}$$

From (14), taking $\gamma_0 = 0$ and $g = -\lambda_k \psi_k$, it is easy to check that

$$\|\nabla \psi_k\|_{0,2,\mathcal{O}}^2 \leq \kappa^{-1} \rho c \lambda_k. \tag{24}$$

Then, combining (23) and (24), we have

$$\sum_{k=1}^{+\infty} \|c_k \nabla \psi_k\|_{0,2,\mathcal{O}}^2 \leq \kappa^{-1} \rho c \sum_{k=1}^{+\infty} c_k^2 \lambda_k < +\infty$$

and

$$\sum_{k=1}^{+\infty} \|c_k \mathcal{L}\psi_k\|_{0,2,\mathcal{O}}^2 < +\infty.$$

Hence, in view of Lemma 3.2, $\varphi \in \mathcal{A}$. □

Before establishing the main result of the section regarding existence and uniqueness of solutions for (6)–(10), we address the existence of $U_{\text{ext}} \in H^2(\mathcal{O})$ satisfying (2)–(4).

Lemma 3.4: *For $U_b \in H^{3/2}(\Gamma_1)$, let*

$$\widetilde{U}_b(\mathbf{x}) := \begin{cases} 0 \quad \text{if} \quad \mathbf{x} \in \Gamma_i, \ i = 2,3,4 \\ \frac{\partial U_b}{\partial \tau_1}(\mathbf{x}) \quad \text{if} \quad \mathbf{x} \in \Gamma_1 \end{cases}. \tag{25}$$

Then for $\widetilde{U}_b \in H^{1/2}(\Gamma)$, there exist $U_{\text{ext}} \in H^2(\mathcal{O})$ satisfying (2)–(4).

Proof: We first observe that the geometric assumptions on Γ, the definition (25) and [19, Theorem 2.1] imply that there exists $\overline{U}_{\text{ext}} \in H^2(\mathcal{O})$ such that

$$\frac{\partial \overline{U}_{\text{ext}}}{\partial \nu_i} = 0 \quad \text{on} \quad \Gamma_i, \ i = 2,4$$

$$\begin{aligned}\frac{\partial \overline{U}_{\text{ext}}}{\partial \nu_3} &= h\kappa^{-1}U_\infty \quad \text{on} \quad \Gamma_3,\\ \overline{U}_{\text{ext}} &= U_b \quad \text{on} \quad \Gamma_1.\end{aligned}$$

Now it is easy to check that the weak formulation of the problem

$$\begin{aligned} -\mathcal{L}\widehat{U} &= 0 \quad \text{in} \quad \mathcal{O},\\ \frac{\partial \widehat{U}}{\partial \nu_i} &= 0 \quad \text{on} \quad \Gamma_i,\ i=2,4\\ \kappa\frac{\partial \widehat{U}}{\partial \nu_3} + h\widehat{U} &= -h\overline{U}_{\text{ext}} \quad \text{on} \quad \Gamma_3,\\ \widehat{U} &= 0 \quad \text{on} \quad \Gamma_1 \end{aligned} \tag{26}$$

is analogous to problem defined by (14). As a consequence, arguing as in Lemma 3.1, there exists a unique $\widehat{U} \in H^1_{\Gamma_1}(\mathcal{O})$ which is the weak solution of (26). Notice also that the problem

$$\begin{aligned} -\mathcal{L}U^* &= 0 \quad \text{in} \quad \mathcal{O},\\ \frac{\partial U^*}{\partial \nu_i} &= 0 \quad \text{on} \quad \Gamma_i,\ i=2,4\\ \kappa\frac{\partial U^*}{\partial \nu_3} &= -h(\overline{U}_{\text{ext}} + \widehat{U}) \quad \text{on} \quad \Gamma_3,\\ U^* &= 0 \quad \text{on} \quad \Gamma_1. \end{aligned} \tag{27}$$

is analogous to problem defined by (16)–(19). Then, proceeding again as in the Lemma 3.1, there exists a unique $U^* \in H^2(\mathcal{O})$ satisfying (27). From the weak formulation of (26) and (27), we obtain

$$\|\widehat{U} - U^*\|^2_{0,2,\mathcal{O}} \le (1/C_1)\|\nabla(\widehat{U} - U^*)\|^2_{0,2,\mathcal{O}} = 0.$$

This implies that $\widehat{U} = U^*$ a. e. in $\mathcal{O}$ and $\widehat{U} \in H^2(\mathcal{O})$. The lemma holds if we consider the function $U_{\text{ext}} = \widehat{U} + \overline{U}_{\text{ext}}$. □

The assumption $\widetilde{U}_b \in H^{1/2}(\Gamma)$ includes the relevant cases where U_b is constant everywhere or U_b is constant in a neighborhood of each vertex s_i. In view of the assumptions on $\mathcal{O}$, we are able to generalize Lemma 3.4 to the case where the heat fluxes vary along the boundaries Γ_2 and Γ_4. However, this requires suitable compatibility conditions involving the data on the boundaries Γ_1, Γ_2, and Γ_4, as seen in [19, Theorem 2.1]. It is worth emphasizing also that the assertion of Lemma 3.4 is not a straightforward consequence of the trace theorem in [22] as (2)–(3) can be regarded as a Robin condition with discontinuous coefficients. Similar results for rectilinear polygonal regions obtained throughout a different procedure can be found in [23].

Proceeding with our analysis, now for each $t \ge 0$, consider the map $E(t) : L^2(\mathcal{O}) \to L^2(\mathcal{O})$ such that $E(t)\varphi = \sum_{k=0}^{+\infty} c_k e^{-\lambda_k t}\psi_k$. It is not difficult to check that the family $\{E(t)\}_{t=0}^{+\infty}$ is a C_0 semigroup of contractions on $L^2(\mathcal{O})$ (see [24, Section 2.1]). In addition, for each $t > 0$ and $\varphi \in L^2(\mathcal{O})$, we have

$$\|\nabla(E(t)\varphi)\|^2_{0,2,\mathcal{O}} \le \kappa^{-1}\rho c\sum_{k=1}^{+\infty} c_k^2 e^{-2\lambda_k t}\lambda_k \le \kappa^{-1}\rho c t^{-2}\sum_{k=1}^{+\infty} c_k^2\lambda_k^{-1} < +\infty \tag{28}$$

and

$$\|\mathcal{L}(E(t)\varphi)\|^2_{0,2,\mathcal{O}} \le \sum_{k=1}^{+\infty} c_k^2 e^{-2\lambda_k t}\lambda_k^2 \le t^{-2}\sum_{k=1}^{+\infty} c_k^2 < +\infty. \tag{29}$$

Thus $E(t)\varphi \in \mathcal{A}$ because of Lemma 3.2. In particular, this implies that for $t > 0$ and $\varphi \in L^2(\mathcal{O})$,

$$\frac{d}{dt}(E(t)\varphi) = -\sum_{k=1}^{+\infty} \lambda_k e^{-\lambda_k t} c_k \psi_k = -\mathcal{L}(E(t)\varphi).$$

Therefore, from Lemma 3.3, $\frac{\mathrm{d}}{\mathrm{d}t}(E(t)\varphi)|_{t=0} = -\mathcal{L}\varphi$ iff $\varphi \in \mathcal{A}$ and $-\mathcal{L} : \mathcal{A} \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ is the infinitesimal generator of $\{E(t)\}_{t=0}^{+\infty}$. On the other hand, since Lemma 3.1 ensures that each $\gamma_0 \le 0$ lies in the resolvent set of $\mathcal{L}$ and since (15) guarantees that for all $u \in \mathcal{A}$,

$$\int_{\mathcal{O}} \mathcal{L}u\, \overline{u}\, d\mathbf{x} = a_0(u, u) \ge C_1 \|u\|_{1,2,\mathcal{O}}^2,$$

it follows that the numerical range of $\mathcal{L}$ is contained in $[C_1, +\infty)$. Based on these results, following the proof of Theorem 7.2.7 in [25], it follows that $E(t)$ can be extended to an analytical semigroup in any sector $\{\lambda \in \mathbb{C}, |arg(\lambda)| < \delta\}$, $0 < \delta < \pi/2$. We observe that this result can also be obtained as a consequence of the well-known Lummer-Philips theorem. However, we emphasize that our approach yields $\{E(t)\}_{t=0}^{+\infty}$ explicitly. From the theoretical and practical point of view, this results in an existence and uniqueness theorem for the solution $\mathcal{V}$ of problem (6)–(10) expressed in series form.

Theorem 3.1: *There exists a unique solution $U \in C([0, +\infty[; L^2(\mathcal{O})) \cap C^1(]0, +\infty[; H^2(\mathcal{O}))$ for problem (1)–(5). In addition, for each $U_{\mathrm{ext}} \in H^2(\mathcal{O})$ satisfying the assumptions of Lemma 3.4, such solution can be expressed as $U = \mathcal{V} + U_{\mathrm{ext}}$, where*

$$\mathcal{V}(t) = \sum_{k=1}^{+\infty} (U_0 - U_{\mathrm{ext}}, \psi_k)_{\mathcal{O}}\, e^{-\lambda_k t} \psi_k + \sum_{k=1}^{+\infty} \int_0^t (\mathfrak{f}(s), \psi_k)_{\mathcal{O}}\, e^{-\lambda_k (t-s)} \psi_k \mathrm{d}s. \tag{30}$$

Proof: Since for construction $\{E(t)\}_{t=0}^{+\infty}$ is an analytical semigroup, existence and uniqueness of solution for problem **P** stated in (12) as well as the representation (30) are straightforward consequences of Corollary 4.3.3 in [25]. This proves existence and uniqueness of solution for (1)–(5). □

We note in passing that, in order to cover more general physical problems involving the bioheat model, the thermal properties of ρ c, h and k should be allowed to be spatially dependent as seen in some works dealing with one-dimensional problems, see, e.g. [26,27]. If this were the case, an analysis similar to the one we have just performed is also possible, but special assumptions on these functions are required. Also, as mentioned before, we can establish the existence of the function U_{ext} in the case where the heat fluxes vary on the boundaries Γ_2 and Γ_4. As a result, we can deal with the homogeneous boundary conditions (7)–(9) and deal with the problem as in the case of null heat fluxes. These cases are beyond the scope of this note and therefore are omitted here. Other than that, notice that the representation (30) is precisely the Fourier series solution for problem **P**. It has proved useful when determining the temperature field in the inverse problem of estimating the blood perfusion coefficient [5,15] for particular choices of the coefficient $c_b w_b$. The perfusion estimation problem plays a key role in areas as hyperthermia and optical tomography and has attracted the attention of several researchers.[2,3,28–31] Obviously, the Fourier series solution cannot be determined in closed form in general as the problem of determining eigenvalues and eigenfunction for $\mathcal{L}$ is difficult. However, we notice that for the particular case where $\mathcal{O}$ is the rectangle $]0, 1[\times]0, M[$, and $\rho c = 1$, if $c_b w_b(x, y) = p(x) + q(y)$, with $p \in L^\infty(]0, 1[)$, $q \in L^\infty(]0, M[)$ being non-negative functions, then the spectral problem can be handled straightforwardly through the method of separation of variables. Indeed, proceeding this way, the following regular Sturm–Liouville problems are derived

$$\begin{aligned} &X''(x) + \kappa^{-1}(\mu^2 - p(x))X(x) = 0,\ \ 0 < x < 1 \\ &X'(0) = X'(1) = 0, \end{aligned} \tag{31}$$

and

$$\begin{aligned} &Y''(y) + \kappa^{-1}(\gamma^2 - q(y))Y(y) = 0,\ 0 < y < M \\ &Y(M) = 0,\ \kappa Y'(0) - hY(0) = 0 \end{aligned}. \tag{32}$$

Then, it can be proved that the eigenvalues $\{\mu_k^2\}_{k=1}^{\infty}$ of (31) and the eigenvalues $\{\gamma_k^2\}_{k=1}^{\infty}$ of (32) satisfy $\lim_{k\to+\infty} \gamma_k^2 = +\infty$, $\lim_{k\to+\infty} \mu_k^2 = +\infty$, and that the corresponding eigenfunctions $X_k \in H^2(]0,1[)$, $Y_k \in H^2(]0,M[)$ form orthogonal bases of $L^2(]0,1[)$ and $L^2(]0,M[)$, respectively. Moreover, it is not difficult to prove that the infinity family $\{X_iY_j\}_{i,j=1}^{\infty}$ is an orthogonal basis for $L^2(]0,1[\times]0,M[)$. With these results at hand, eigenvalues $\{\lambda_k\}_{k=1}^{+\infty}$ and eigenfunctions $\{\psi_k\}_{k=1}^{+\infty}$ of $\mathcal{L}$ can be determined in several ways. In particular, the eigenpairs $\{\psi_k,\lambda_k\}_{k=1}^{+\infty}$ can be obtained through the following procedure: for given $m \in \mathbb{N}$ and $k = 1 + m(m-1)/2$, define

$$\lambda_k = \mu_1^2 + \gamma_m^2, \quad \lambda_{k+1} = \mu_2^2 + \gamma_{m-1}^2, \quad \ldots, \quad \lambda_{k+m-1} = \mu_m^2 + \gamma_1^2. \tag{33}$$

and

$$\psi_k = \widehat{X}_1\widehat{Y}_m, \quad \psi_{k+1} = \widehat{X}_2\widehat{Y}_{m-1}, \quad \ldots, \quad \psi_{k+m-1} = \widehat{X}_m\widehat{Y}_1. \tag{34}$$

The purpose of the above enumeration of the eigenfunctions ψ_k is to capture low frequencies first. In such a case, it is often seen that a few terms are usually enough for the truncated series to produce good approximation to the solution of the bioheat transfer problem. To illustrate this, we can consider the case where $p = 0$ and $q > 0$ with q constant. In this event, elementary calculations show that

$$\mu_i^2 = \kappa(i-1)^2\pi^2, \qquad \gamma_j^2 = \kappa\beta_j^2 + q, \tag{35}$$

where β_j is a root of the nonlinear equation

$$\beta \cot(\beta M) = -\kappa^{-1}h.$$

In addition, the orthonormal eigenfunctions are $\widehat{X}_i = X_i/N_i$, where

$$X_i(x) = \cos(\kappa^{-1/2}\mu_i x), \quad N_i = \begin{cases} \sqrt{2}/2, & i \neq 1, \\ 1, & i = 1 \end{cases} \tag{36}$$

and $\widehat{Y}_j = Y_j/M_j$, where

$$Y_j(y) = \sin(\beta_j(M-y)), \quad M_j = \left(\frac{M}{2} - \frac{1}{4\beta_j}\sin(2\beta_j M)\right)^{1/2}. \tag{37}$$

Furthermore, when both U_∞ and U_b are constant, we can take

$$U_{\text{ext}}(y) = U_b + h\kappa^{-1}M^{-1}(U_\infty - U_b)y(y-M).$$

In this case, from (30), we can see that the Fourier series solution to (1)–(5) becomes

$$\begin{aligned} U(x,y,t) = U_b + \frac{h(U_\infty - U_b)}{\kappa M}y(y-M) + \sum_{k\in\mathcal{J}_m}^{+\infty} a_k e^{-\lambda_k t}\psi_k(x,y) \\ + \sum_{k\in\mathcal{J}_m}^{+\infty} \int_0^t (\mathfrak{f}(s),\psi_k)_{\mathcal{O}}\, e^{-\lambda_k(t-s)}\psi_k(x,y)\mathrm{d}s, \end{aligned} \tag{38}$$

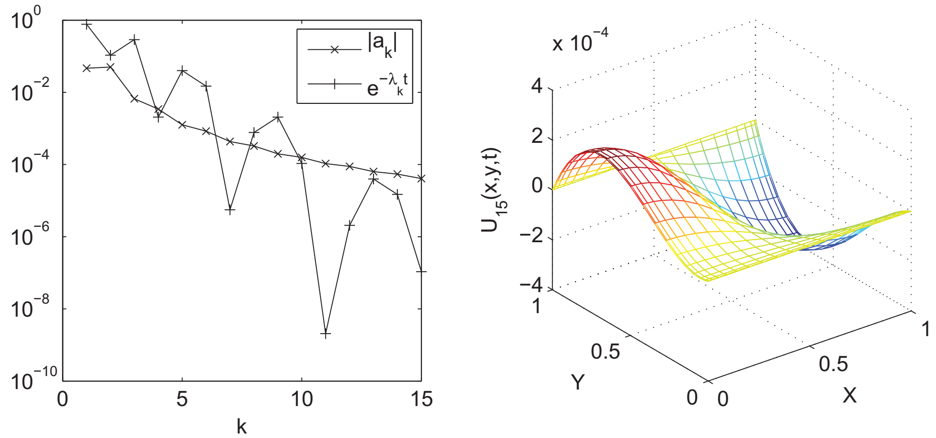

Figure 1. Left: Usual behavior of coefficients a_k and exponential terms $e^{-\lambda_k t}$ for the series solution of the bioheat problem. Right: Solution of bioheat problem at $t = 0.1$. In this case, the maximum error in a regular grid of 20×20 points between the exact solution $U(x_i, y_j, t)$ and the approximate solution obtained by truncating the series to 15 terms, which we denote here by $U_{15}(x_i, y_j, t)$, is $\max_{i,j} |U(x_i, y_j, t) - U_{15}(x_i, y_j, t)| = 4.35 \times 10^{-7}$.

where $a_k = (U_0 - U_{\text{ext}}, \psi_k)_{\mathcal{O}}$ and where for each $m \geq 1$, $\mathcal{J}_m = \{\ell \in \mathbb{N}/\ \ell = i + m(m-1)/2,\ i = 1, \dots, m\}$.

For illustration purposes and completeness, the first 15 exponential terms $e^{-\lambda_k t}$ and corresponding coefficients a_k of the Fourier series (38) are displayed in Figure 1. In this illustration, we consider a dimensionless counterpart of the bioheat model (6)–(10) where $\mathcal{O}$ is the rectangle $]0,1[\times]0,M[$, $q = 0.15$, and where the solution to the bioheat model is given by

$$U(x, y, t) = e^{c_1 t} \cos(c_2 t) y^2 (y - M) \cos(\pi x) + hU_\infty (y - M) y/M,$$

for $h = 0.025$, $c_1 = -50$, $c_2 = 6\pi$, $U_\infty = 0.01$, and $M = 1$. The average decay of the coefficients a_k in absolute value and the exponential terms $e^{-\lambda_k t}$ at $t = 0.1$ confirm that the main features of the solution are indeed captured with a few terms of the series. For additional numerical results of the Fourier approach as well as comparisons with a pseudospectral based approach for the problem, the reader is referred to [7].

Why only a few eigenvalues λ_k are sufficient to capture the most important features of the Fourier-based solution to the bioheat model is now justified by the theorem below.

Theorem 3.2: *Assume that* $\mathcal{O} =]0,1[\times]0,M[$, $c_b w_b(x,y) = p(x) + q(y)$, *with* $p \in L^\infty(]0,1[)$, $q \in L^\infty(]0,M[)$ *being nonnegative functions, and that the eigenvalues* λ_k *of the elliptic operator* $\mathcal{L} : \mathcal{A} \subset L^2(\mathcal{O}) \to L^2(\mathcal{O})$ *are ordered as in* (33). *Then for* $j = 0, 1, \dots, m-1$ *we have*

$$\lambda_{k+j} = \mu_{j+1}^2 + \gamma_{m-j}^2 \geq \kappa j^2 \pi^2 + \gamma_{m-j}^2 + m_p,$$

where μ_i^2 *and* γ_j^2 *are eigenvalues of the Sturm–Liouville problems* (31) *and* (32), *respectively, and* $m_p = \operatorname{ess\,inf}_{0 \leq x \leq 1} |p(x)|$.

Proof: Let λ_i and $\check{\lambda}_i$ be the eigenvalues of $\mathcal{L}$ associated to the cases $p(x) \geq 0$ a. e. in $[0,1]$ and $p(x) = 0$ a. e. in $[0,1]$, respectively, and let $a_0(u,v)$ and $a_1(u,v)$ be the corresponding bilinear forms, i.e.

$$a_0(u,v) = \kappa(\nabla u, \nabla v)_{\mathcal{O}} + h(u,v)_{\Gamma_3} + ((p(x) + q(y))u, v)_{\mathcal{O}}$$

and

$$a_1(u,v) = \kappa(\nabla u, \nabla v)_{\mathcal{O}} + h(u,v)_{\Gamma_3} + (q(y)u, v)_{\mathcal{O}}.$$

It is immediate to see that $a_0(u,v)$ and $a_1(u,v)$ are coercive and continuous on $H^1_{\Gamma_1}(\mathcal{O}) \times H^1_{\Gamma_1}(\mathcal{O})$, and that for each $u \in H^1_{\Gamma_1}(\mathcal{O})$ with $\|u\|_{0,2,\mathcal{O}} = 1$, we have

$$a_0(u,u) \geq a_1(u,u) + m_p.$$

Assume temporarily that λ_i and $\check{\lambda}_i$ are ordered in nondecreasing form. Then, the well-known Min–Max Theorem [21, Theorem 10, p.102], implies

$$\begin{aligned} \lambda_i &= \max_{V_{i-1} \subset H^1_{\Gamma_1}(\mathcal{O})} [\min\{a_0(u,u),\ u \in V_{i-1}^{\perp}, \|u\|_{1,2,\mathcal{O}} = 1\}] \\ &\geq \max_{V_{i-1} \subset H^1_{\Gamma_1}(\mathcal{O})} [\min\{a_1(u,u),\ u \in V_{i-1}^{\perp}, \|u\|_{1,2,\mathcal{O}} = 1\}] + m_p = \widetilde{\lambda}_i + m_p, \end{aligned} \tag{39}$$

where the maximum is over all subspaces $V_{i-1} \subset H^1_{\Gamma_1}(\mathcal{O})$ of dimension $i-1$. Now notice that for the case $p(x) = 0$, the ith eigenvalue of the Sturm–Liouville problem (31) is given by $\kappa(i-1)^2\pi^2$, see (35). This shows that if we enumerate the eigenvalues $\check{\lambda}_k$ according to (33), we have $\check{\lambda}_{k+j} = \kappa j^2\pi^2 + \gamma^2_{m-j}$, for $j = 0, \dots, m-1$. This and (39) imply

$$\lambda_{k+j} = \mu^2_{j+1} + \gamma^2_{m-j} \geq \widetilde{\lambda}_{k+j} + m_p = \kappa j^2\pi^2 + \gamma^2_{m-j} + m_p,$$

and the proof follows. □

Since the Fourier series (38) involves exponential factors $e^{-\lambda_k t}$, it is now clear that only a few eigenvalues will play some role in the solution.

There is a physical motivation to assume the particular case where q_m, q_e, and U_a are all constants.[4,5] If this is the case, the Fourier series solution (30), as well as the explicit representation of X_i and Y_j given in (36)–(37), allow us to obtain the following regularity result.

Theorem 3.3: *Assume that $\mathcal{O} =]0,1[\times]0,M[$, $c_b w_b = q > 0$, $\rho c = 1$, $U_0 \in L^2(\mathcal{O})$ and that q_m, q_e, U_a, U_b, U_∞ are all constants. Then, problem (1)–(5) has a unique solution $U \in C^\infty(]0,+\infty[\times \overline{\mathcal{O}})$ given by*

$$U(x,y,t) = \eta(y) + \sum_{k=1}^{\infty} a_k e^{-\lambda_k t} \psi_k(x,y), \tag{40}$$

where $a_k = (U_0 - \eta, \psi_k)_{\mathcal{O}}$ and

$$\begin{aligned} \eta(y) = \frac{G}{q} &+ \left(\frac{qU_b - G}{q}\right) \cosh\left(\sqrt{q\kappa^{-1}}(y-M)\right) \\ &+ \frac{hG - hqU_\infty + (U_b q - G)(\sqrt{\kappa q}\,\mathsf{s} + h\,\mathsf{c})}{q(\sqrt{\kappa q}\,\mathsf{c} + h\,\mathsf{s})} \sinh\left(\sqrt{q\kappa^{-1}}(y-M)\right), \end{aligned} \tag{41}$$

with $\mathsf{c} = \cosh(\sqrt{q\kappa^{-1}}M)$, $\mathsf{s} = \sinh(\sqrt{q\kappa^{-1}}M)$ and $G = qU_a + q_m + q_e$.

Proof: It is straightforward to see that the function $\eta(y)$ defined in (41) satisfies both the differential equation $-\kappa\,\eta''(y) + q\,\eta(y) = G$ and the boundary conditions (2)–(4). In view of Theorem 3.1, the unique solution U of the bioheat problem (1)–(5) can be expressed as $U = V + \eta$, where V solves the Cauchy problem (6)–(10), with $U_{\text{ext}} = \eta$ and $f = 0$. Moreover, from (36)–(37), for each integer $m \geq 1$, $k = 1 + m(m-1)/2$ and $i \in \{1, \dots, m\}$, we get $\psi_k \in C^\infty(\overline{\mathcal{O}})$ and

$$\sup_{\mathbf{x} \in \overline{\mathcal{O}}} \left| \frac{\partial^{|\alpha|} \psi_{k+i-1}}{\partial x^{\alpha_1} \partial y^{\alpha_2}}(\mathbf{x}) \right| = O\left(\frac{\mu_i^{\alpha_1} \beta_{m-i+1}^{\alpha_2}}{M_j} \right) \tag{42}$$

for any $\alpha = (\alpha_1, \alpha_2) \in \mathbb{N} \times \mathbb{N}$. It is easy to check that, for each $j \in \mathbb{N}$,

$$s_j(t) = \sup_{k \in \mathbb{N}} e^{-2\lambda_k t} \lambda_k^j$$

is a continuous function in any interval $[\epsilon, T]$, $\epsilon > 0$. Now, based on (42), we can differentiate

$$V(x, y, t) = \sum_{k=1}^{+\infty} a_k e^{-\lambda_k t} \psi_k(x, y)$$

term by term with respect to the variables x, y, and t as many times as we desire, thus obtaining a uniformly convergent series in $\overline{\mathcal{O}} \times [\epsilon, T]$, for any $\epsilon > 0$. This concludes the proof. □

3.1. *Highly accurate method for computing eigenpairs*

As already mentioned, except for the case where $c_b w_b$ is constant, the problem of determining the spectral information for $\mathcal{L}$ is difficult. The objective of this section is to describe an algorithm that intends to alleviate this difficulty for the case where $c_b w_b$ is split into two functions as described above, in which case we are able to construct approximations to the eigenpairs of $\mathcal{L}$ based on eigenpairs of the Sturm–Liouville problems (31)–(32). The underlying idea is to approximate eigenvalues and eigenfunctions of the continuous problems using eigenvalues and eigenvectors of standard matrix eigenvalue problems obtained after discretization of (31) and (32), respectively. To this end, due to its high accuracy and low computational cost compared with finite difference methods, we choose to use the Chebyshev pseudospectral method (CPS) based on the well-known Chebyshev differentiation matrix.[32,33] For simplicity, we will consider a mesh consisting of $N + 1$ grid points on $[0, 1]$ based on the $N + 1$ Chebyshev-Gauss Lobatto points in each directions (which means we set $M = 1$):

$$x_i = y_i = \frac{1}{2}\left[1 - \cos(\pi i/N)\right], \quad i = 0, \ldots, N,$$

and denote the $(N + 1) \times (N + 1)$ Chebyshev differentiation matrix by D. If $\mathbf{v} = [v_0, \ldots, v_N]^T$ is a vector consisting of values of function $v(x)$ at the grid points x_i, we recall that highly accurate approximations to $v'(x_i)$, $v''(x_i)$, etc. can be produced by performing products of the form $D\mathbf{v}$, $D^2\mathbf{v}$, etc. i.e. by taking $v'(x_i) \approx (D\mathbf{v})_i$, $v''(x_i) \approx (D^2\mathbf{v})_i$, etc. For the discretization of the Sturm–Liouville problems, it is convenient to express the Chebyshev differentiation matrix as

$$D = [c_0, \ldots, c_N] = \begin{bmatrix} r_0^T \\ \vdots \\ r_N^T \end{bmatrix}, \quad c_i, r_i \in \mathbb{R}^{N+1}.$$

With these representations for D, the second order differentiation matrix can be expressed as

$$D^2 = c_0 r_0^T + \cdots + c_N r_N^T.$$

To discretize the Sturm–Liouville problem (31), let $\mathbf{X} = [X(x_0), \ldots, X(x_N)]^T$. Then,

$$[X''(x_0), \ldots, X''(x_N)]^T \approx D^2\mathbf{X} = c_0 r_0^T \mathbf{X} + c_1 r_1^T \mathbf{X} + \cdots + c_{N-1} r_{N-1}^T \mathbf{X} + c_N r_N^T \mathbf{X}.$$

Taking into account that

$$r_0^T \mathbf{X} \approx X'(x_0) = 0 = X'(x_N) \approx r_N^T \mathbf{X},$$

neglecting approximation errors and denoting by $\widetilde{\mathbf{X}}$ the vector of approximations to $\mathbf{X}$, the collocation Chebyshev pseudospectral method produces the standard matrix eigenvalue problem

$$A_{\mathbf{X}}\widetilde{\mathbf{X}} = \mu^2\widetilde{\mathbf{X}}, \quad A_{\mathbf{X}} = P - \kappa D_1 D_2 \in \mathbb{R}^{(N+1)\times(N+1)}, \tag{43}$$

being

$$P = \operatorname{diag}\left(p(x_0), \ldots, p(x_N)\right), \quad D_1 = [c_1, \ldots, c_{N-1}], \; D_2 = \begin{bmatrix} r_1^T \\ \vdots \\ r_{N-1}^T \end{bmatrix}. \tag{44}$$

Thus, our method takes as approximations to the eigenvalues μ_i^2 the eigenvalues of matrix $A_{\mathbf{X}}$ and as approximations to pointwise values of eigenfunctions X_i the components of corresponding eigenvectors of $A_{\mathbf{X}}$.

Similarly, to discretize the Sturm–Liouville problem (32), let $\mathbf{Y} = [Y(x_0), \ldots, Y(x_N)]^T$. As before, the vector of second-order derivatives can be approximated as

$$[Y''(x_0), \ldots, Y''(x_N)]^T \approx D^2\mathbf{Y} = DD\mathbf{Y} = D\begin{bmatrix} r_0^T\mathbf{Y} \\ \widehat{D}_2^T\mathbf{Y} \end{bmatrix} = D\begin{bmatrix} r_0^T\mathbf{Y} \\ \mathbf{0} \end{bmatrix} + D\begin{bmatrix} \mathbf{0} \\ \widehat{D}_2^T\mathbf{Y} \end{bmatrix},$$

where $\mathbf{0} \in \mathbb{R}^N$ is a vector of all zeros. Now taking into account the boundary conditions

$$Y(0) = \frac{\kappa}{h}Y'(0) \approx \frac{\kappa}{h}r_0^T\mathbf{Y}, \quad Y(x_N) = 0,$$

neglecting approximation errors and denoting by $\check{Y}$, the vector that approximates the N first components of $\mathbf{Y}$, we obtain a matrix eigenvalue problem

$$A_{\mathbf{Y}}\check{Y} = \gamma^2\check{Y}, \quad A_{\mathbf{Y}} = Q - \kappa\left(\left[\frac{h}{\kappa}\widehat{c}_0, \mathbf{0}, \ldots, \mathbf{0}\right] + \widehat{D}_1\widehat{D}_2\right) \in \mathbb{R}^{N\times N}, \tag{45}$$

where $Q = \operatorname{diag}(q(y_0), \ldots, q(y_{N-1}), \widehat{c}_0$ comprises the N first components of c_0, $\mathbf{0} \in \mathbb{R}^N$ is as above, $[\widehat{D}_1]_{i,j} = D_{i,j}$ for $0 \le i \le N-1,\ 1 \le j \le N$, and $[\widehat{D}_2]_{i,j} = D_{i,j}$ for $1 \le i \le N,\ \ 0 \le j \le N-1$.

Thus, approximate eigenpairs for the elliptic operator $\mathcal{L}$ can be constructed following (33) and (34) based on eigenpairs $\{\mu_i^2, \widetilde{X}_i\}$ of $A_{\mathbf{X}}$ and eigenpairs $\{\gamma_j^2, \check{Y}_j\}$ of $A_{\mathbf{Y}}$, taking as pointwise approximation to the jth eigenfunction Y_j, the vector $\widetilde{Y}_j = \begin{bmatrix} \check{Y}_j \\ 0 \end{bmatrix}$. Based on this, approximations to the eigenfunctions ψ_k on the mesh can be constructed using rank one matrices calculated as,

$$\psi_1(x_i, y_j) \approx [\widetilde{X}_1\widetilde{Y}_1^T]_{i,j}, \quad \psi_2(x_i, y_j) \approx [\widetilde{X}_1\widetilde{Y}_2^T]_{i,j}, \quad \psi_3(x_i, y_j) \approx [\widetilde{X}_2\widetilde{Y}_1^T]_{i,j}, \ \ldots \tag{46}$$

Remark 3.1: It is worth observing that if the method of separation of variables works and the eigenpairs of the Sturm–Liouville problems (31)–(32) are enumerated according to (33)–(34), then m computed accurate approximations to eigenpairs of the Sturm–Liouville problems will result in $m(m+1)/2$ accurate approximations to eigenpairs of the elliptic operator.

Thus, if the goal is to compute eigenmodes of the elliptic operator through some numerical method, a few eigenpairs of the Sturm–Liouville problems are required and these must be computed to high accuracy. As we will see, that is precisely what the Chebyshev pseudospectral method does. In fact, to illustrate this, we consider a bioheat equation for which $\kappa = 1$, $p(x) = 0$, and $q(y) = 0.15$. In this case, both Sturm–Liouville problems have closed form solutions given by (35)–(37). Figure 2 displays some continuous eigenmodes given in (36) and (37), as well as the corresponding approximations produced by the Chebyshev pseudospectral method with $N = 20$. As we can observe, except for

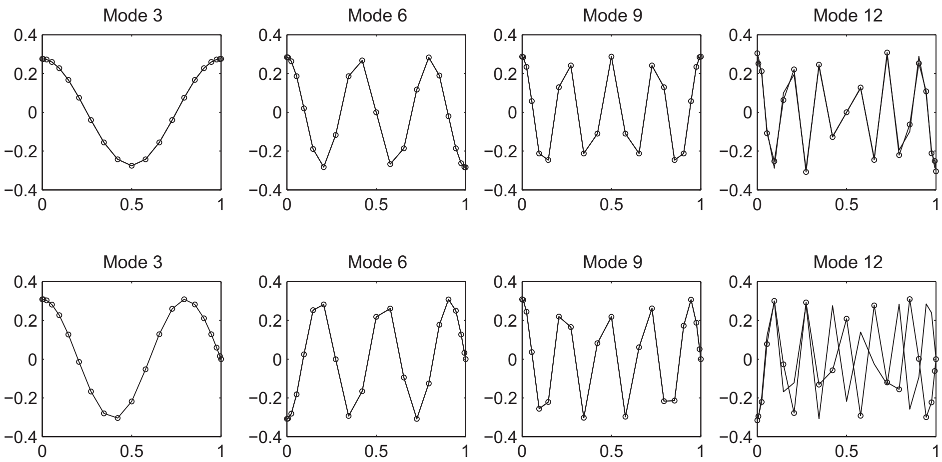

Figure 2. Top: Continuous and discrete eigenmodes of Sturm Liouville problem (31). Bottom: Continuous and discrete eigenmodes of Sturm–Liouville problem (32). In both cases, lines linking small circles correspond to discrete eigenmodes.

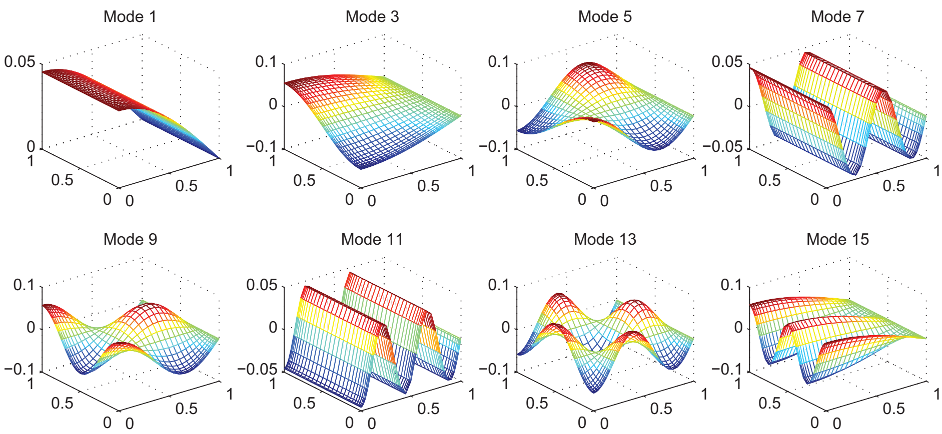

Figure 3. Approximate eigenmodes produced by Chebyshev pseudospectral method.

mode 12 corresponding to the Sturm–Liouville problem (32), the computed modes agree well with the continuous ones. To be more precise, the numerical results revealed that modes corresponding to low frequencies are calculated more accurately, in fact, with regard to the Sturm–Liouville problem (32), eigenvalue 5 is accurate to eight digits, eigenvalue 9 is accurate to 6 digits, but eigenvalue 12 is not at all accurate and with relative error 19%. From this observation and Remark 3.1, we conclude that the first 5 modes of the Sturm–Liouville problems, computed with high precision, will result in 15 highly accurate modes for the elliptic operator, as seen in Figure 4, which have been shown sufficient for the solution of the bioheat problem in series form to be meaningful, see Figure 1. We notice that accuracy can be improved by simply increasing the number of grid points. As an example, by taking $N = 30$, the difference between the first 5 continuous eigenvalues and the five computed ones is just at the level of rounding errors. Approximations to some of the first 15 modes of $\mathcal{L}$ according to (46) produced by the Chebyshev method with $N = 30$ are displayed in Figure 3.

Other methods can be employed to solve the Sturm–Liouville problems (31)–(32). These include finite differences methods, finite element methods, spectral methods, and multidomain spectral-type methods. While the former are well suited for very simple domains, the latter are preferable

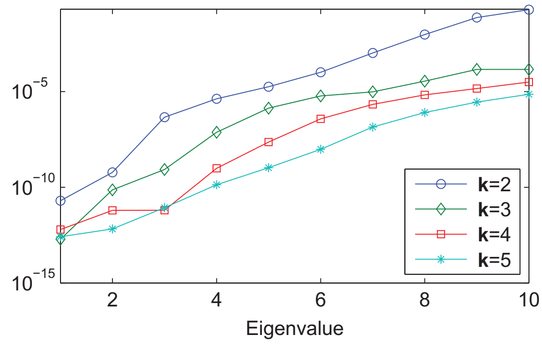

Figure 4. Relative error for first 10 eigenvalues of Sturm–Liouville problem (31) computed by multidomain collocation method for several subdomains with $N = 10$ for each subdomain.

for boundary value problems in domains with a complicated geometry and solutions with different regularity in different parts of the domain. In multidomain spectral-type methods, we divide the domain of interest into smaller subdomains which are individually mapped onto a reference domain, e.g. the interval $[-1, 1]$ for 1D problems.[34,35] The approximation to the global solution arises by combining the solutions computed via a spectral/pseudospectral method relative to each subdomain. We close this note by briefly describing a multidomain method for the Sturm–Liouville problem (31) based on the CPS method. Let $\mathbf{k} \geq 1$ be an integer and $\mathbf{s}_j \in \mathbb{R}$, $0 \leq j \leq \mathbf{k}$, such that $0 = \mathbf{s}_0 < \mathbf{s}_1 < \cdots < \mathbf{s}_{\mathbf{k}-1} < \mathbf{s}_{\mathbf{k}} = 1$. To derive the pseudospectral approximation, we subdivide the interval $I = [0, 1]$ into $\mathbf{k}$ subintervals $I_j = [\mathbf{s}_{j-1}, \mathbf{s}_j]$, $1 \leq j \leq \mathbf{k}$. Then, solving the Sturm–Liouville problem (31) via a multidomain collocation method is equivalent to finding $\mathbf{k}$ N_j-degree polynomials X_j, $1 \leq j \leq \mathbf{k}$, such that

$$X_j''(x) + \kappa^{-1}(\mu^2 - p(x))X_j(x) = 0, \quad x \in I_j,\ 1 \leq j \leq \mathbf{k}, \tag{47}$$

$$X_j(\mathbf{s}_j) = X_{j+1}(\mathbf{s}_j), \quad 1 \leq j \leq \mathbf{k} - 1, \tag{48}$$

$$X_j'(\mathbf{s}_j) = X_{j+1}'(\mathbf{s}_j), \quad 1 \leq j \leq \mathbf{k} - 1, \tag{49}$$

$$X_1'(\mathbf{s}_0) = X_{\mathbf{k}}'(\mathbf{s}_{\mathbf{k}}) = 0. \tag{50}$$

To keep the notation simple, we take the polynomial order N_j to be equal to N across all subdomains. Thus, to determine the polynomials X_j, we define a new set of nodes $\xi_\ell^{N,j} \in I_j, 0 \leq \ell \leq N$, $1 \leq j \leq \mathbf{k}$, and then collocate the differential Equation (47) at the nodes in I_j in such a way that the constraints (48)–(50) are satisfied. This gives rise to a block-diagonal $(\mathbf{k}N + 1) \times (\mathbf{k}N + 1)$ matrix eigenvalue problem which can be handled via numerical linear algebra tools. Of course, polynomials of different degree can be used in each domain depending on the regularity of the solution. The number of blocks and the number of collocation points in each subdomain is arbitrary but the use of many subdomains and polynomials of low degree is often found in literature.[35] Convergence analyses of multidomain collocation methods for 1D problems can be found in [35, Chapter 11]. To illustrate the effectiveness of the multidomain collocation method, we solve (31) using polynomials of the same degree as the solutions are regular in the whole domain. Errors associated to computed eigenvalues are plotted in Figure 4. Good accuracy with relatively few nodes and small number of subdomains is apparent. A similar observation applies to the accuracy of corresponding eigenmodes (not shown here).

We finally observe that for the general case where the coefficient $c_b w_b$ is a function that cannot be separated as above, the problem of constructing approximations to the eigenpairs $\{\lambda_k, \psi_k\}$ can be handled by discretizing the elliptic operator. In such a case, approximate eigenpairs can be obtained by solving suitable matrix eigenvalue problems. This is beyond the scope of the present paper.

4. Conclusion

An analysis on existence and uniqueness of solutions for a 2D bioheat equation with convective boundary conditions has been carried out. As main result, a Fourier series-based solution was obtained based on a suitable transformation of the original problem into a standard Cauchy problem, as well as on elliptic regularity on polygons and semigroup theory. The analysis shows that the series constructed this way can converge quickly in cases that appear in applications. Although second-order parabolic equations have been widely discussed in the literature, very little attention has been devoted to the convergence analysis of Fourier series-based solutions for the bioheat model. In this event, we believe our results become useful in that they provide theoretical support for numerical methods, successfully used so far but with no convergence analysis, see, e.g. [5,7].

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