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Sensitivity eigenanalysis for single shift-invariant subspace-based methods

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Abstract

A signal eigenvalue sensitivity analysis for subspace-based methods that exploit the shift-invariance property present in the signal subspace is considered. It is proved that signal eigenvalues are rather insensitive to small perturbations in the data provided the dimension of the problem is large enough and the eigenvalues themselves are not extremely close to each other. In addition, bounds on the signal eigenvalue error that depend on both the largest canonical angle between the exact and approximate signal subspace and the dimension of the data matrix are provided. The theory is illustrated by a numerical example where a signal taken from the literature is analysed. © 2000 Elsevier Science B.V. All rights reserved.

Zusammenfassung

Es wird eine Signaleigenwert-Sensitivitätsanalyse für Unterraummethoden, welche die im Signalunterraum vorhandene Verschiebungsinvarianz ausnützen, betrachtet. Es wird gezeigt, daß Signaleigenwerte eher unempfindlich gegenüber kleinen Störungen der Daten sind, falls die Dimension des Problems groß genug ist und die Eigenwerte selbst nicht zu nahe beieinander liegen. Weiters werden Schranken für den Fehler der Signaleigenwerte angegeben, welche vom größten kanonischen Winkel zwischen dem exakten und dem angenäherten Signaluterraum sowie von der Dimension der Datenmatrix abhängen. Die vorgestellte Theorie wird durch ein numerisches Beispiel veranschaulicht, in dem ein der Literatur entnommenes Signal analysiert wird. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Nous considérons dans cet article l'analyse de sensitivité des valeurs propres d'un signal pour des méthodes de sous-espaces qui exploitent la propriété d'invariance en décalage dans le sous-espace du signal. Il est prouvé que les valeurs propres d'un signal sont assez insensibles à de petites perturbations sur les données pourvu que la dimension du problème soit assez grande et que les valeurs propres elles-mêmes ne soient pas extrêmement proches des unes des autres. De plus, nous fournissons des bornes pour l'erreur des valeurs propres du signal, qui dépendent à la fois du plus grand angle canonique entre les sous-espaces exacts et approchés du signal, et de la dimension de la matrice de données. La théorie est illustrée par un exemple numérique où un signal pris dans la littérature est analysé. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Eigenvalue perturbation analysis; Subspace-based approaches; Exponential modelling

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1. Introduction

The problem of estimating model parameters such as decay constants, signal frequencies, amplitudes, plane waves, etc., from data corresponding to noisy damped/undamped exponential signals, has become an active area of research in recent years. This problem has applications in a number of areas which include modal analysis, radar and sonar signal processing, nuclear magnetic resonance (NMR), acoustic, and speech processing, among others.

One popular class of algorithms for the problem that has encountered prominence because of its higher resolution and lower computational cost compared with classical methods is the subspace-based approach. Algorithms in this class start by computing an estimate of the so-called *signal subspace* which is nothing but the column or row subspace of a certain matrix containing the data (for noise-free data). This subspace contains underlying signal information which is then used for determining the parameters of interest which can be done in a great variety of schemes. Subspace-based methods are commonly divided into two classes [27]: (a) subspace fitting techniques [18,21,24,25]; and (b) single shift-invariant (SSI) methods. The last class covers Kung's method [12] also called HSVD in NMR [29], the matrix pencil method of Hua and Sarkar [10], OPIA of Bazán and Bavastri [3], subspace rotation methods such as ESPRIT and PRO-ESPRIT [16,31] and some total least-squares-based methods such as HTLS [29] and TLS-ESPRIT [15], among others. A survey on subspace SVD-based methods can be found in [27].

We shall consider a class of methods that exploit the so-called *shift-invariance* structure present in the signal space. A particularly nice feature of these methods is that the desired model parameters emerge after solving a typically small eigenvalue problem. These eigenvalues are often referred to as *signal eigenvalues* or *poles*. Since in practical applications only an approximate signal subspace is available, as it is estimated from data corrupted by noise, and since poles emerge as eigenvalues of nonsymmetric matrices (which may be very sensitive to eigenvalue computation), an issue of interest is to analyse the sensitivity of the poles to perturbations on the data.

Several authors have developed analyses on pole sensitivity and error estimation in connection with classical techniques, namely those where poles emerge from the roots of large polynomials, concerning which there exists a vast literature (see, for instance, [8,9,23,19], just to quote a few references). However, relatively little work has been done with respect to pole sensitivity for SSI subspace methods: a few contributions can be found in [6,10,11,22,30]. It must be emphasized however, that, with the exception of the work by De Groen and De Moor [6], the remaining contributions present results that heavily depend on the particular nature of the noise (e.g., its elements are independent Gaussian zero-mean random numbers). Furthermore, as pointed out in [27], these results are considered to hold only asymptotically. So an eigenvalue sensitivity analysis using finite measurements without using any hypothesis on the nature of the noise seems to be suitable.

We present a signal eigenvalue error analysis that applies for a wide class of subspace-based methods, holding for finite measurements but free of statistical hypotheses. We deduce informative signal eigenvalue error estimates, obtaining, in particular, a stronger bound for the eigenvalue error than that by De Groen and De Moor. Our main result is that the signal eigenvalues become insensitive to small perturbations on the data whenever the dimension of the data matrix is large enough and the signal eigenvalues are not extremely close to each other. In particular, we show that slightly damped signals benefit of favourable mathematical conditions that facilitate the task of retrieving the model parameters even if the data are relatively noisy. Also, we obtain eigenvalue error estimates as a function that depends on both the subspace angle between the exact and approximate signal subspace and the dimension of the data matrix. These results provide insight into the problem when the signal subspace is estimated via the singular value decomposition.

An outline of the paper is as follows. In Section 2 we describe the model assumptions needed to perform our analysis and review the underlying

idea behind SSI subspace methods. Sensitivity analysis is presented in Section 3 where we show that the sensitiviy of the signal eigenvalues is governed by the conditioning of a related rectangular Vandermonde matrix whose columns span the signal subspace. This result is then used in Section 4 where we present our pole error estimates. Numerical results that illustrate our theoretical analysis are reported in Section 5. We finally present some conclusions in Section 6.

2. Model assumptions and generalities

We first introduce the notation used throughout the paper. For $A \in \mathbb{C}^{M\times N}$, $\|A\|$ and $\|A\|_F$ denote the spectral and Frobenius norm of A, respectively. A^* denotes the conjugate transpose of A and $A^\dagger$ denotes its Moore–Penrose pseudo-inverse. The singular values of A are denoted by $\sigma_i(A)$ and arranged in decreasing order, i.e., $\sigma_1(A) \geqslant \sigma_2(A) \geqslant \cdots \geqslant \sigma_p(A)$, where $p = \min(M, N)$. The 2-norm condition number of A, $\kappa_2(A)$, is defined by $\kappa_2(A) = \|A\|\,\|A^\dagger\|$.

We assume that the available data arise as $\tilde{h}_k = h_k + \varepsilon_k$, $k = 0,1,\ldots,L-1$, where h_k is an unknown sampled signal of the form

$$h_k = \sum_{l=1}^{n} r_l\, \mathrm{e}^{(d_l + \imath\omega_l)k\Delta t} = \sum_{l=1}^{n} r_l\, z_l^k, \tag{2.1}$$

with $r_l \in \mathbb{C}$, $\imath = \sqrt{-1}$, $d_l \leqslant 0$, $\omega_l \in \mathbb{R}$, $z_j \neq z_k$ for $j \neq k$, Δt being the sampling interval, n the number of spectral components contained in the signal and ε_k the noise. For the moment we assume that the data are free of noise. Most of shift-invariant methods start by arranging the available data in a Hankel or Toeplitz matrix $Y_{M\times N}$ of order $M \times N$, $M, N \geqslant n$, whose column space, denoted here by $\mathscr{S}_M$ and from here on referred to as the *signal space*, is spanned by the columns of an $M \times n$ Vandermonde matrix of the form

$$W_M = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ \vdots & \vdots & \vdots & \vdots \\ z_1^{M-1} & z_2^{M-1} & \cdots & z_n^{M-1} \end{bmatrix} \tag{2.2}$$

(note that $\operatorname{rank}[W_M] = n$). The underlying idea used for SSI techniques, we are concerned with in this work, relies on the fact that if $\mathscr{A}$ denotes the matrix consisting of all rows of W_M excluding the last and $\mathscr{B}$ the matrix consisting of all rows excluding the first, then

$$\mathscr{A}Z = \mathscr{B} \quad \text{where } Z = \operatorname{diag}(z_1, \ldots, z_n).$$

But if $\hat{V}$ is any matrix of order $M \times n$, whose columns span $\mathscr{S}_M$, and if A and B are obtained from $\hat{V}$ as $\mathscr{A}$ and $\mathscr{B}$ were from W_M, then there exists an $n \times n$ matrix T, say, known as a *transition matrix*, such that

$$AT = B. \tag{2.3}$$

Obviously T is similar to Z. SSI techniques recover such a transition matrix from the available data and extract d_l and ω_l from its eigenvalues. Once the signal eigenvalues z_l are available, the weights r_l are readily computed by solving a linear least-squares (LS) problem. The same work can be performed using a matrix $\hat{U} \in \mathbb{C}^{N\times n}$ with columns spanning the row subspace of $Y_{M\times N}$, though in this case Eq. (2.3) involves a transition matrix similar to Z^*.

Different choices of $\hat{V}$ ($\hat{U}$) and different forms of solving (2.3) have resulted in a number of subspace-based methods. For example, if $\hat{V}$ is chosen to be the matrix formed by the left singular vectors associated with the nonzero singular values of $Y_{M\times N}$ and (2.3) is solved in the standard least-squares (LS) sense, then one obtains Kung's method [12]:

$$T = A^\dagger B. \tag{2.4}$$

Note that, since A is almost an isometry, $A^\dagger$ in (2.4) need not be computed by matrix inversion.

$$I = \hat{V}^*\hat{V} = A^*A + xx^* \Rightarrow A^\dagger = (A^*A)^{-1}A^*$$

$$= (I - xx^*)^{-1}A^* = \left(I + \frac{xx^*}{1 - x^*x}\right)A^*,$$

where x^* denotes the last row of $\hat{V}$ and I the $n \times n$ identity matrix. If Eq. (2.3) is solved in the total least-squares (TLS) sense, then one obtains an algorithm called HTLS. This method was recently introduced by Van Huffel and co-workers, and it is

currently used in time-domain analysis of NMR data [29]. Another popular method, known as the state-space balanced realization approach (SSR), chooses $\hat{V}$ by weighting the columns of the same matrix used by Kung's method [13]. Related methods are the standard ESPRIT [16] and the matrix pencil method of Hua and Sarkar [10].

When the data are noise corrupted, the signal space is estimated from a perturbed data matrix $\tilde{Y}_{M\times N} = Y_{M\times N} + E$, where E contains the noise, but the procedure is essentially the same. In this case however, the shift-invariance equation (2.3) is no longer compatible. The question that arises is then how the signal eigenvalues change when the transition matrix suffers perturbations as a consequence of the errors on the data. In the remainder of this work, we analyse the problem in the case where A and B in (2.3) arise from any orthonormal basis of the approximate signal subspace, and always assume that the number n is known in advance. In particular, we shall focus on the case where the signal subspace basis is estimated from the SVD of the data matrix $\tilde{Y}_{M\times N}$. This covers, of course, the eigenvalue sensitivity analysis for all above mentioned shift-invariant methods, including those methods that exploit the shift-invariance equation (2.3) for solving another class of problems such as that of system pole extraction from input–output measurements [12,30].

3. Signal eigenvalue sensitivity

We start by assuming that an orthonormal basis for the exact signal subspace $\mathscr{S}_M$ is available. Let V_s be an $M\times n$ matrix whose columns form such a basis. The following theorem gives information on the conditioning of the eigenvalue problem related to all SSI methods that use the shift-invariance equation (2.3).

Theorem 1. *Suppose* $T\in\mathbb{C}^{n\times n}$ *satisfies the shift-invariance equation* (2.3). *Then, for* $M \geqslant n+1$

$$T = (V_s^* W_M) Z (V_s^* W_M)^{-1}. \tag{3.1}$$

Proof. Since the columns of W_M also span the signal subspace $\mathscr{S}_M$, there exists a nonsingular matrix $F\in\mathbb{C}^{n\times n}$ such that

$$V_s = W_M F. \tag{3.2}$$

Decompose V_s and W_M as

$$V_s = \begin{bmatrix} A \\ x^* \end{bmatrix} = \begin{bmatrix} y^* \\ B \end{bmatrix},$$

$$W_M = \begin{bmatrix} W_{M-1} \\ e^* Z^{M-1} \end{bmatrix} = \begin{bmatrix} e^* \\ W_{M-1} Z \end{bmatrix},$$

where e denotes the vector in $\mathbb{R}^n$ of all ones. From this and (3.2), it follows that $B = W_{M-1} Z F$, $A = W_{M-1} F$ and $A^\dagger = F^{-1} W_{M-1}^\dagger$. Substituting $A^\dagger$ and B into (2.4), we get

$$T = F^{-1} W_{M-1}^\dagger W_{M-1} Z F = F^{-1} Z F,$$

since $W_{M-1}^\dagger W_{M-1} = I$. The statement of the theorem results then by observing from (3.2) that $F^{-1} = V_s^* W_M$ and $F = W_M^\dagger V_s = (V_s^* W_M)^{-1}$. □

Since it is straightforward to see that $\kappa_2(V_s^* W_M) = \kappa_2(W_M)$, where $\kappa_2(W_M)$ denotes the 2-norm condition number of W_M, Theorem 1 shows that the sensitivity of the signal eigenvalues to perturbations on T is essentially governed by $\kappa_2(W_M)$. We now present an upper bound for $\kappa_2(W_M)$ which shows that under conditions that hold in many applications, this condition number is not as severe as one could usually suppose.

Theorem 2. *Define* $\alpha = \max|z_l|$ *and* $\beta = \min|z_l|$. *Also define*

$$\delta = \min_{\substack{1\leqslant j,k\leqslant n \\ j\neq k}} |z_j - z_k|, \tag{3.3}$$

$$\phi_M = \sqrt{\frac{1 + \alpha^2 + \cdots + \alpha^{2(M-1)}}{1 + \beta^2 + \cdots + \beta^{2(M-1)}}} \tag{3.4}$$

and

$$D_M^2 = \|T\|_F^2 - (|z_1|^2 + \cdots + |z_n|^2). \tag{3.5}$$

Then, for all $M \geqslant n$, *the* 2*-norm condition number of* W_M *satisfies*

$$\kappa_2(W_M) \leqslant \tfrac{1}{2}(\eta + \sqrt{\eta^2 - 4}), \tag{3.6}$$

where

$$\eta = \left[1 + \frac{D_M^2}{(n-1)\delta^2}\right]^{(n-1)/2} \frac{n}{2}(\phi_M + \phi_M^{-1}) - n + 2. \tag{3.7}$$

Proof. Let C be an $M \times M$ companion matrix such that

$$CW_M = W_M Z.$$

It then follows that the projected companion matrix onto $\mathscr{S}_M$ matches T. In fact, as $V_s V_s^* W_M = W_M$ because $V_s V_s^*$ is the orthogonal projector onto $\mathscr{S}_M$ and the columns of W_M belong to this subspace, the above equation can be rewritten as

$$CV_s V_s^* W_M = V_s V_s^* W_M Z.$$

Using the fact that $V_s^* V_s = I$ and Theorem 1, the above equation becomes

$$V_s^* C V_s = (V_s^* W_M) Z (V_s^* W_M)^{-1} = T \tag{3.8}$$

as claimed. Bound (3.6) follows then from identifying T with $V_s^* C V_s$ in Theorem 3 by Bazan and Toint [4]. □

Note that the quality of bound (3.6) depends on the condition $D_M^2 < (n-1)\delta^2$ and that the bound is about ρ times n when this condition is fulfilled, with ρ a moderate constant. Note also that, since δ depends on the separation of the signal eigenvalues z_l but not on M, the quality of the bound ultimately depends on the behaviour of D_M^2 as a function of M. The number D_M, known as *departure from normality* of T, measures how close is T from being a normal matrix. It was studied by Bazan [2, see Lemma 7] who showed that

$$0 \leqslant D_M^2 \leqslant (n-1) + \|\hat{f}_M\|^2 + \prod_{l=1}^{n} |z_l|^2 - \sum_{l=1}^{n} |z_l|^2, \tag{3.9}$$

where $\hat{f}_M$ is the minimum 2-norm solution of the underdetermined system of equations

$$W_M^* f = Z^{*M} e, \tag{3.10}$$

where e is as before. Vector $\hat{f}_M$ features the interesting property that its norm $\|\hat{f}_M\|$ monotonically decreases to zero as M increases [5]. From this fact it follows that whenever $\|\hat{f}_M\|^2 \approx 0$ and $|z_l| \approx 1$ we should get $D_M^2 \approx 0$. Hence, unless the signal eigenvalues are extremely close to each other, the condition $D_M^2 < (n-1)\delta^2$ should be satisfied, thereby ensuring reasonably small bounds for $\kappa_2(W_M)$. Another evidence about the well-conditioning of W_M comes in terms of D_M: Ruhe [17] shows that $D_M \approx 0$ implies a well-conditioned eigenvalue problem. We therefore conclude that if the signal eigenvalues in modulus are resonably close to 1 (which happens when analysing slightly damped signals) but not extremely close to each other, then the only condition needed to ensure moderate values of $\kappa_2(W_M)$ is to keep M sufficiently large. However, as we shall illustrate later, M need not be as large as the theoretical analysis seems to require.

4. Error perturbation analysis

In this section we provide estimates for the error $|z_l - \tilde{z}_l|$, $l = 1,2,\ldots,n$, where the $\tilde{z}_l$ are computed from an approximate transition matrix $\tilde{T}$. Our goal here is to prove that signal eigenvalues are quite insensitive to small perturbations on the data. We start by noting that classical eigenvalue perturbation theory and Theorem 1 together ensure that

$$|z_l - \tilde{z}_l| \leqslant \|T - \tilde{T}\| \kappa_2(W_M), \quad 1 \leqslant l \leqslant n. \tag{4.1}$$

Since $\kappa_2(W_M)$ was already analysed before and we know it remains within moderate bounds under conditions that hold in many applications, in order to achieve our goal, we have to prove that the error $\|T - \tilde{T}\|$ does not propagate the input errors on the data matrix. We shall restrict ourselves to only analyse the error $\|T - \tilde{T}\|$ in the case where $\tilde{T}$ is estimated by using a pure least-squares technique, as described in (2.4).

Let V_s and $\tilde{V}_s$ be $N \times n$ matrices with orthonormal columns that span the exact signal subspace $\mathscr{S}_M$ and the approximate signal subspace $\tilde{\mathscr{S}}_M$, respectively. Decompose $\tilde{V}_s$ conformally with V_s as in the previous section, i.e.,

$$V_s = \begin{bmatrix} A \\ x^* \end{bmatrix} = \begin{bmatrix} y^* \\ B \end{bmatrix}, \quad \tilde{V}_s = \begin{bmatrix} \tilde{A} \\ \tilde{x}^* \end{bmatrix} = \begin{bmatrix} \tilde{y}^* \\ \tilde{B} \end{bmatrix}.$$

Note that for noise-free data the transition matrix T is the unique solution to the compatible system $AT = B$. Let $\tilde{A} = A + \Delta_A$, $\tilde{B} = B + \Delta_B$ and assume that $\tilde{A}$ is of rank n. Then the error on T is

$$\begin{aligned}\tilde{T} - T &= \tilde{A}^{\dagger}\tilde{B} - T \\ &= \tilde{A}^{\dagger}(B + \Delta_B) - T \\ &= \tilde{A}^{\dagger}(AT + \Delta_B) - T \\ &= \tilde{A}^{\dagger}(\tilde{A}T - \Delta_A T + \Delta_B) - T \\ &= \tilde{A}^{\dagger}(-\Delta_A T + \Delta_B). \end{aligned} \tag{4.2}$$

The last equality follows from the fact that $\tilde{A}^{\dagger}\tilde{A} = I$, since by assumption $\operatorname{rank}[\tilde{A}] = n$. Taking 2-norm on both sides of (4.2) yields

$$\|\tilde{T} - T\| \leqslant \|\tilde{A}^{\dagger}\|\,\|T\|(\|\Delta_A\| + \|\Delta_B\|), \tag{4.3}$$

where we used the property that $\|T\| = \|V_s^* C V_s\| \geqslant 1$ [4]. Now, note that to obtain any meaningful error estimate, V_s must be chosen as close as possible to $\tilde{V}_s$. This is always possible and can be made by choosing $V_s = VX$, where V is any $M \times n$ matrix with orthonormal columns spanning the signal space $\mathscr{S}_M$ and $X \in \mathbb{C}^{n\times n}$ a unitary matrix which solves the orthogonal Procrustes problem

$$\min \|VX - \tilde{V}_s\|_F.$$

Define $G = \tilde{V}_s^* V$ and let $G = \bar{P}\bar{\Sigma}\bar{Q}^*$ be its singular value decomposition. Then the unitary matrix solving this problem is $X = \bar{Q}\bar{P}^*$, see, for instance, [1], or [7, p. 601]. Using this X, $V_s = V\bar{Q}\bar{P}^*$. After some manipulations, we obtain that

$$\begin{aligned}\|V_s - \tilde{V}_s\|^2 &= \|(V\bar{Q}\bar{P}^* - \tilde{V}_s)^*(V\bar{Q}\bar{P}^* - \tilde{V}_s)\| \\ &= \|2I - \bar{P}\bar{Q}^* V^* \tilde{V}_s - \tilde{V}_s^* V \bar{Q}\bar{P}^*\| \\ &= 2\|(I - \bar{P}\bar{\Sigma}\bar{P}^*\|. \end{aligned}$$

We now recall that the cosines of the canonical angles between the subspaces $\mathscr{S}_M$ and $\tilde{\mathscr{S}}_M$ are the singular values of $\tilde{V}_s^* V$ (contained in $\bar{\Sigma}$); see, for instance, Stewart [20, Theorem 2.4]. Using this observation, the last relation implies that

$$\|\Delta_{V_s}\| = \|V_s - \tilde{V}_s\| = 2\sin\frac{\Theta}{2}, \tag{4.4}$$

where Θ, sometimes called *subspace angle*, denotes the largest canonical angle between $\mathscr{S}_M$ and $\tilde{\mathscr{S}}_M$. Now, since $\|\delta_A\| \leqslant \|\Delta_{V_s}\|$ and $\|\Delta_B\| \leqslant \|\Delta_{V_s}\|$, using these inequalities in (4.3) and taking (4.4) into account, we obtain the following result.

Lemma 3.

$$\|T - \tilde{T}\| \leqslant 4\|\tilde{A}^{\dagger}\|\,\|T\|\sin\frac{\Theta}{2}. \tag{4.5}$$

Note that in this estimate, both $\|\tilde{A}^{\dagger}\|$ and $\|T\|$ strongly depend on M, the number of rows of the data matrix $\tilde{Y}_{M\times N}$. But, since $T = V_s^* C V_s$, $\|V_s C V_s\| = \leqslant \sqrt{1 + \|\hat{f}_M\|^2}$ (see Theorem 2 by Bazan and Toint [4]), and $\|\hat{f}_M\|$ decreases to zero as M increases, these facts ensure that for M large enough we get $\|T\| \approx 1$. Note also that $\|\tilde{A}^{\dagger}\| = 1/(1 - \|\tilde{x}\|^2) = 1/(1 - \|\tilde{p}_M\|^2)$, where $\tilde{p}_M$ is the last column of the orthogonal projector $\tilde{\mathscr{P}}$ onto the subspace $\tilde{\mathscr{S}}_M$. This suggests that if $\|\tilde{p}_M\|$ behaves approximately as $\|p_M\|$, which decreases with M and remains near zero for M large [5], then $\|\tilde{A}^{\dagger}\|$ should not be much larger than 1. We therefore conclude that the estimate provided by Lemma 3 depends strongly on M, though the closeness of $\mathscr{S}_M$ to $\tilde{\mathscr{S}}_M$, measured by the subspace angle, is also important. This means that for large M, unless $\mathscr{S}_M$ is very far from $\tilde{\mathscr{S}}_M$, the bound (4.5) becomes close to $2\sin\Theta$. Although this result does not yet prove that input errors on the data matrix are not propagated when computing $\tilde{T}$, it ensures small errors on T provided M is large enough and the subspace angle is sufficiently small. We shall return to this discussion later.

Substituting the estimate provided by Lemma 3 into (4.1), we obtain the following theorem.

Theorem 4. *The eigenvalue error satisfies*

$$|z_l - \tilde{z}_l| \leqslant 2\,\|\tilde{A}^{\dagger}\|\,\|T\|(\eta + \sqrt{\eta^2 - 4})\sin\frac{\Theta}{2}, \quad 1 \leqslant l \leqslant n, \tag{4.6}$$

where η is as in Theorem 2.

As η becomes generally a modest constant when M is large enough (see the examples discussed

in [2,4]), the behaviour of both $\|\tilde{A}^{\dagger}\|$ and $\|T\|$ as functions of M, as discussed before, ensures that, if $\tilde{V}_s$ is sufficiently close to V_s in the sense of the Frobenius norm, then the bound for the signal eigenvalue error can be regarded as the product of $\sin\Theta$ times a moderate constant. In other words, estimate (4.6) can be rewritten as

$$|z_l - \tilde{z}_l| = \mathcal{O}(\sin\Theta), \quad 1 \leqslant l \leqslant n. \tag{4.7}$$

We now assume that $\tilde{V}_s$ is estimated by using the SVD of the observed data matrix $\tilde{Y}_{M\times N} = Y_{M\times N} + E$. The following corollary gives a bound that depends on the size of $\|E\|$ and $\sigma_n(Y_{M\times N})$, the smallest nonzero singular value of the clean data matrix $Y_{M\times N}$.

Corollary 5. *Assume a basis for the signal subspace $\tilde{\mathscr{S}}_M$ is computed from the SVD of $\tilde{Y}_{M\times N} = Y_{M\times N} + E$. Then, provided $\|E\| \ll \sigma_n(Y_{M\times N})$, the following first-order estimate for the bound on $\|T - \tilde{T}\|$ in* (4.5) *holds*:

$$\|T - \tilde{T}\| \leqslant 2\|\tilde{A}\|\,\|T\|\frac{\|E\|}{\sigma_n(Y_{M\times N})}. \tag{4.8}$$

Consequently, the bound on the reconstruction error for the signal eigenvalues in (4.6) *becomes*

$$|z_l - \tilde{z}_l| \leqslant \|\tilde{A}^{\dagger}\|\|T\|(\eta + \sqrt{\eta^2 - 4})\frac{\|E\|}{\sigma_n(Y_{M\times N})}, \quad 1 \leqslant l \leqslant n. \tag{4.9}$$

Proof. Let

$$Y_{M\times N} = [U_1\ U_2]\,\mathrm{diag}(\Sigma_1,\Sigma_2)[V_1\ V_2]^* \quad \text{and}$$
$$\tilde{Y}_{M\times N} = [\tilde{U}_1\ \tilde{U}_2]\,\mathrm{diag}(\tilde{\Sigma}_1,\tilde{\Sigma}_2)[\tilde{V}_1\ \tilde{V}_2]^*$$

be SVDs of $Y_{M\times N}$ and $\tilde{Y}_{M\times N}$, respectively, where $U_1, \tilde{U}_1, V_1$ and $\tilde{V}_1$ all have n columns. It is well-known that if $\|E\|_2 \ll \sigma_n(Y_{M\times N})$ then an orthonormal basis for the approximate signal space (generated by the columns of $\tilde{U}_1$) is formed by the columns of an $M\times n$ matrix $\hat{U}_1$, say, such that

$$\hat{U}_1 = (U_1 + U_2 P)(I + P^*P)^{-1/2}, \tag{4.10}$$

where P is a matrix of order $(M-n)\times n$ whose norm is of the order of $\|E\|$ (see, for instance [26], or [14] for a detailed explanation), and $(I + P^*P)^{-1/2}$ is the inverse of the square root of the positive-definite matrix $(I + P^*P)$. Although one is not able to find the matrix P, its norm $\|P\|$ is interestingly interpreted as the tangent of the largest canonical angle Θ, see [20]. On the other hand, since

$$U_1^*\hat{U}_1 = \hat{U}_1^*U_1 = (I + P^*P)^{-1/2}$$

and, as the singular values of this Hermitian matrix are the cosines of the canonical angles between $\mathscr{S}_M$ and $\tilde{\mathscr{S}}_M$, after some algebraic manipulations it follows that $\|U_1 - \hat{U}_1\| = 2\sin(\Theta/2)$. This ensures that U_1 is a matrix that is closest to $\hat{U}_1$ in the sense that $\|U_1 - \hat{U}_1\|_F$ is minimized. Finally, since

$$2\sin\frac{\Theta}{2} \leqslant \tan\Theta = \|P\| \leqslant \frac{\|E\|}{\sigma_n(Y_{M\times N})},$$

where the last inequality holds up to first order, see [26]. Inequalities (4.8) and (4.9) follows from subsituting this inequality in (4.5) and (4.6), respectively. □

Note that inequality (4.8) shows that reduction of input errors will happen whenever $2\|\tilde{A}^{\dagger}\|\,\|T\| < \sigma_n(Y_{M\times N})$, a fact often observed if M is large enough (see the examples analysed in [4]). On the other hand, that inequality suggests choosing the dimensions of the data matrix so that the bound be minimized to guarantee a small subspace angle. There are no theoretical results on the behaviour of $\sigma_n(Y_{M\times N})$ as a function of M, N but empirical evidence advises to choose data matrices as square as possible, since thus $\sigma_n(Y_{M\times N})$ would achieve a maximum value [4], in which case the bound would be minimized.

We conclude this section with two remarks.

1. In [6], De Groen and De Moor show that

$$\|T - \tilde{T}\| \leqslant K\left(\frac{\|E\| + \sigma_{n+1}(\tilde{Y}_{M\times N})}{\sigma_n(\tilde{Y}_{M\times N}) - \sigma_{n+1}(\tilde{Y}_{M\times N})}\right), \tag{4.11}$$

where K is a positive constant not larger than 3 provided the signal is measured for long enough time (i.e, for M large enough). They use this estimate to conclude that the eigenvalue error is of order $\sigma_{n+1}(\tilde{Y}_{M\times N})/(\sigma_n(\tilde{Y}_{M\times N}) - \sigma_{n+1}(\tilde{Y}_{M\times N}))$. The advantage of our error analysis over that by the above authors is that

Table 1
Model parameters of NMR signal, $\omega_l = 2\pi f_l$

l	r_l	d_l	f_l(Hz)	z_l	$\lvert z_l\rvert$	δ_l^2
1	$5.8921 + \imath 1.5788$	208	-1379	$0.6342 - \imath 0.7463$	0.9794	0.1787
2	$9.5627 + \imath 2.5623$	256	-6854	$0.8858 - \imath 0.4067$	0.9747	0.0643
3	$5.7956 + \imath 1.5529$	197	-271	$0.9663 - \imath 0.1661$	0.9805	0.0643
4	$2.7046 + \imath 0.7247$	117	353	$0.9642 + \imath 0.2174$	0.9884	0.0100
5	$16.4207 + \imath 4.3999$	808	478	$0.8811 + \imath 0.2729$	0.9224	0.0100

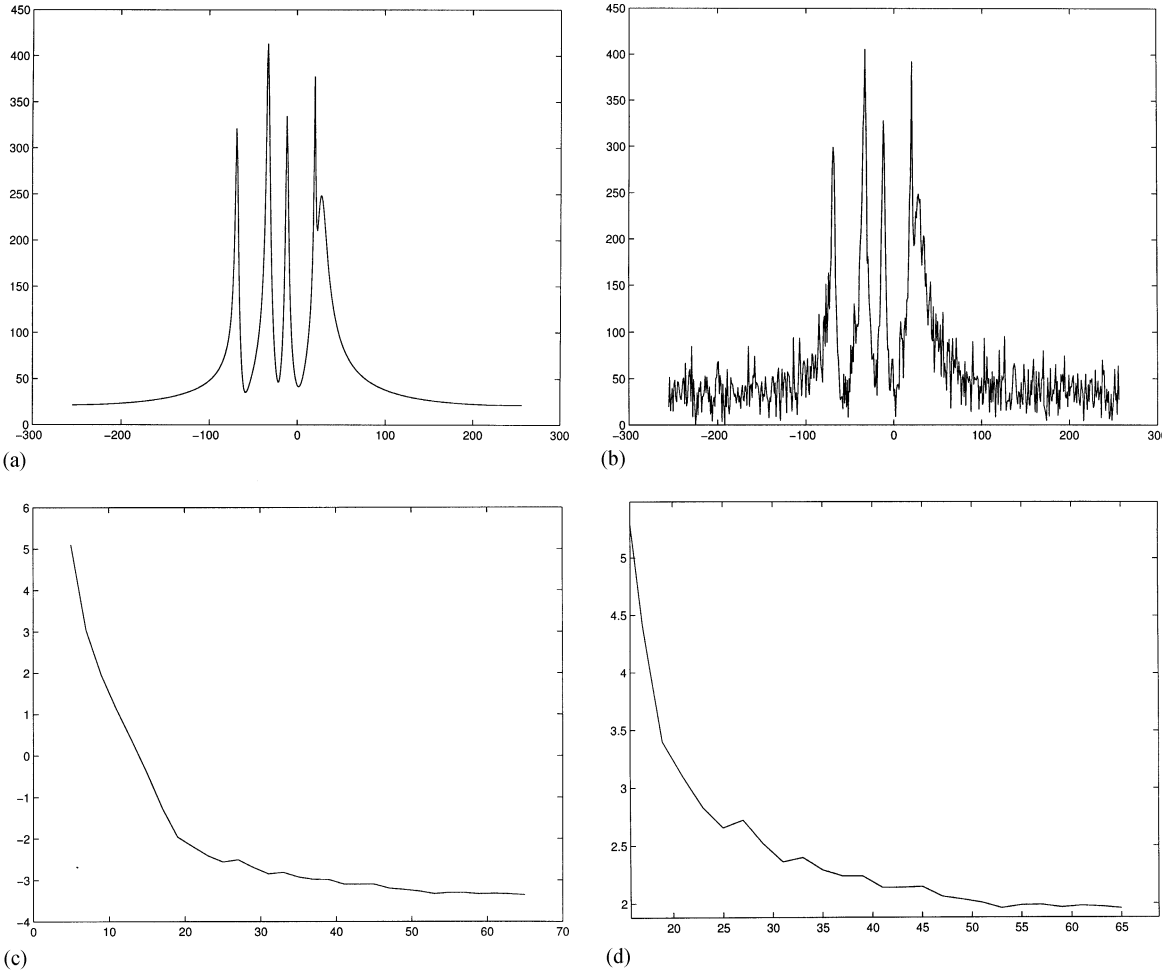


Fig. 1. (a) Fourier spectrum of pure NMR signal. (b) Fourier spectrum of noisy NMR signal, $\sigma = 1$ (the scale is arbitrary in both cases). (c) D_M^2 as a function of M. (d) Upper bound for $\kappa_2(W_M)$ as a function of M ((c) and (d) are on a logarithmic scale).

we obtain an error bound for the transition matrix (hence for the eigenvalue error, see (4.8) and(4.9)), which involves a readily computable constant of modest size. In addition to this, comparison of (4.8) with (4.11) shows that our error estimate (4.8) realizes substantial improvement.

2. The current analysis can be adapted to derive eigenvalue error estimates for subspace-based methods that exploit a multiple shift-invariance property present in the signal subspace. These methods are applied to estimating system poles from multi-input multi-output measurements, in which case, poles emerge from transition matrices satisfying a multiple shift-invariance equation like (2.3), and the "signal subspace" is identified as the range space of the extended observability matrix of the system state-space description [12,30]. It is easy to see that the error estimate (4.5) applies immediately to these methods, however the conditioning of the matrix eigenvalue problem remains to be analysed.

5. Numerical experience

In this section we report numerical results to illustrate the theory presented throughout the paper. We analyse a typical NMR signal comprising five complex exponentials [29]. Model parameters as well as the separations of the signal poles, $\delta_l = \min|z_l - z_j|$, $j \neq l$, $l,j = 1,2,\ldots,n$ shown in Table 1. As seen in that table and Fig. 1(a), the signal is relatively damped and its Fourier spectrum features two closely overlapping peaks. The number of samples is $L = 128$ and the sample rate 10 kHz (i.e. $\Delta t = 0.0001s$).

The numerical experiment consists of two parts. The goal of the first part is to show that $\kappa_2(W_M)$ becomes rather small provided M is large enough. This can be appreciated in Fig. 1(d), where we illustrate the behaviour of bound (3.6) for $\kappa_2(W_M)$ as a function of M. The condition number itself for $M \geqslant 60$ reaches values about 3.10. Explanation for this comes in terms of D_M and the separation between the z_l themselves: since $(n-1)\delta^2 = 0.0399$, the inequality $D_M^2 < (n-1)\delta^2$ in (3.7), which we have proved to ensure moderate values of the bound, is rapidly reached. Some values illustrate this fact: $D_{50}^2 = 0.0385$, $D_{60}^2 = 0.0365$, $D_{100}^2 = 0.0310$, etc. (see also Fig. 1(c)).

In the second part, we address the issue of signal eigenvalue error $|z_l - \tilde{z}_l|$: we compute bound (4.1) for the eigenvalue error, the error itself, and related quantities such as $\sin\Theta$, $2\sin(\Theta/2)$, $\|E\|/\sigma_n(Y_{M\times N})$ and $\|T - \tilde{T}\|$. The $\tilde{z}_l$ were extracted from matrices $\tilde{T}$, computed according (2.4), where $\tilde{A}$ and $\tilde{B}$ in that formula are estimated from the SVD of a Hankel matrix $\tilde{Y}_{M\times N} = Y_{M\times N} + E$ containing samples of the perturbed signal $\tilde{h}_k = h_k + \varepsilon_k$, where we use zero-mean Gaussian random numbers with standard deviation σ as noise. In order to assess the effect that the dimensions M, N could yield on the eigenvalue error, the experiment was performed for M, N constrained to $M + N = 129$, for $N = 5,6,\ldots,64$.

We report mean values of computed quantities corresponding to 100 noise realizations for $\sigma = 0.1$ (low noise-to-signal ratio) and $\sigma = 1$ (matrix noise to signal ratio $\|E\|/\|Y_{M\times N}\| \approx 13\%$, see Fig. 2). The behaviour of $\sin\Theta$, $2\sin(\Theta/2)$, and $\|E\|/\sigma_n(Y_{M\times N})$, displayed in Fig. 3(a), shows that at low noise-to-signal ratios ($\sigma = 0.1$) those quantities are very comparables when the data matrix progressively becomes less overdetermined and that the sine measuring the closeness between the exact and approximate signal subspace seems to monotonically

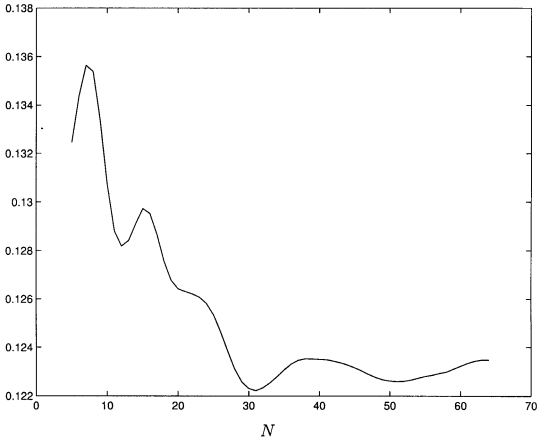
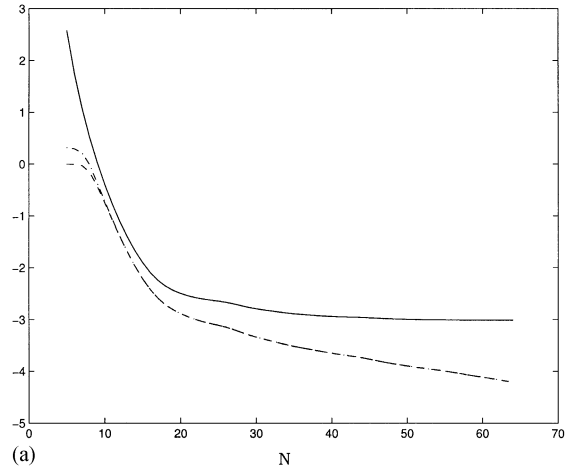


Fig. 2. Matrix noise-to-signal ratio $\|E\|/\|Y_{M\times N}\|$, $M + N = 129$, $5 \leqslant N \leqslant 64$.



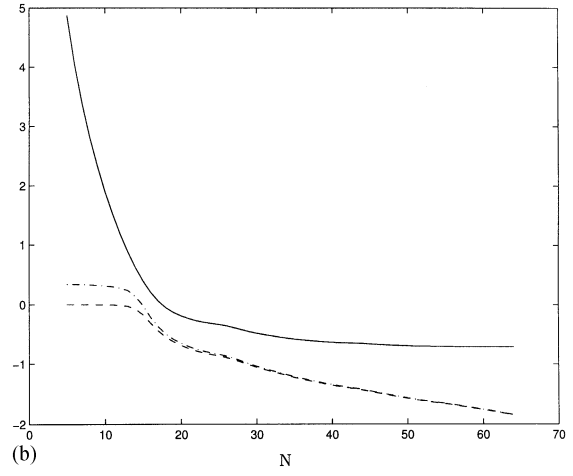
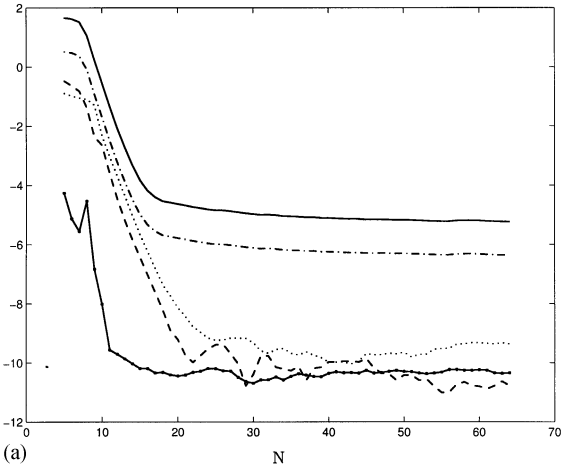


Fig. 3. $\|E\|/\sigma_n(Y_{M\times N})$ (solid line), $\sin\Theta$ (dashed line) and $2\sin(\Theta/2)$ (dash-dotted line), all as functions of M, $M+N=129$, $5\leqslant N\leqslant 64$ ((a): $\sigma=0.1$, (b): $\sigma=1$, the scale is logarithmic in both cases).



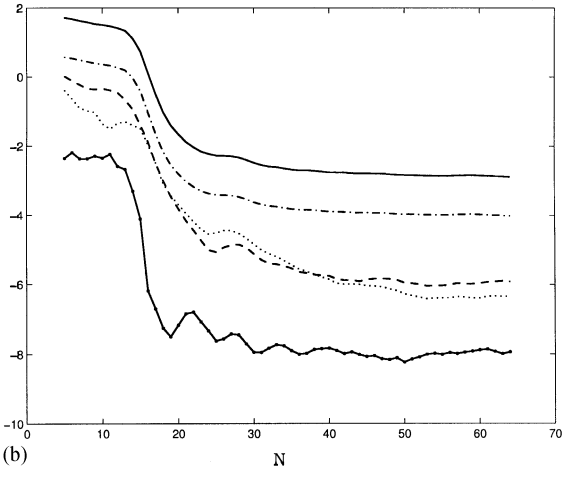


Fig. 4. Bound (4.1) for $\kappa_2(W_M)$ (solid line), $\|T-\tilde{T}\|$ (dash-dotted line), and $|z_l-\tilde{z}_l|$ for $l=3,4,5$ (solid-dotted line, dotted line, and dashed line, in the same order). (a): $\sigma=0.1$, (b): $\sigma=1$. The dimensions M,N and the scale are as in Fig. 2.

decrease as $\tilde{Y}_{M\times N}$ becomes a square one. The same comment applies for $\sigma=1$, though the distance of $\|E\|/\sigma_n(Y_{M\times N})$ to $\sin\Theta$ becomes more pronounced (see Fig. 3(b)).

Bound (4.1) for the eigenvalue error, the error $\|T-\tilde{T}\|$, and the eigenvalue error itself for z_3, z_4 and z_5, all are displayed in Fig. 4(a) and (b). Here again observe that the eigenvalue error strongly decreases when the data matrix becomes less over-determined and that these errors are somewhat minimized when the dimensions of the data matrix satisfy $M\approx N$. These figures also confirm that the eigenvalue error is of the same order of magnitude as $\sin\Theta$, as predicted in theory, and that the third pole z_3 (marked by solid-doted lines) is notoriously less sensitive than the closest poles z_4 and z_5, as expected. Extensive numerical experiments which illustrate the sensitivity of the model parameters themselves related to this NMR signal, for several noise levels, can be encountered in [28,29].

6. Conclusions

We have performed an eigenvalue perturbation analysis for subspace-based methods that exploit the shift-invariance property present in the signal subspace. We concluded that signal eigenvalues become quite insensitive to small perturbations on the data provided the dimension of the data matrix is large enough and the eigenvalues themselves in modulus are not much smaller than 1. The analysis also allowed us to derive estimates for the signal eigenvalue error. These estimates suggest that the error itself could be approximately of the same order of magnitude as the sine of the largest canonical angle between the exact and approximate signal subspace. This was numerically verified, even for moderate levels of noise, and illustrated by simulations using a signal often mentioned in the literature. The numerical experience also suggested that the sine measuring the closeness between the exact and the approximate signal subspace is minimized when the data matrix is square, however this needs to be analytically demonstrated.

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References

[1] K.S. Arun, A unitarily constrained total least squares problem in signal processing, SIAM J. Matrix Anal. Appl. 13 (3) (July 1992) 729–745.

[2] F.S.V. Bazán, Conditioning of rectangular vandermonde matrices with nodes in the unit disk, SIAM Matrix Anal. Appl., to appear.

[3] F.S.V. Bazán, C. Bavastri, An optimized pseudo-inverse algorithm (OPIA) for multi-input multi-output modal parameter identification, Mech. Systems Signal Process. 10 (1996) 365–380.

[4] F.S.V. Bazán, Ph.L. Toint, On the 2-condition number of infinite Hankel matrices of finite rank, Technical Report 10, Department of Mathematics, FUNDP, Namur, Belgium, June 1998.

[5] F.S.V. Bazán, Ph.L. Toint, Singular values of predictor matrices and signal eigenvalue bounds, Technical Report 5, Department of Mathematics, FUNDP, Namur, Belgium, February 1998.

[6] P. De Groen, B. De Moor, The fit of a sum of exponentials to noisy data, Comput. Appl. Math. 20 (1987) 175–187.

[7] G.H. Golub, C.F. Van Loan, Matrix Computations, 3rd Edition, Johns Hopkins University Press, Baltimore, 1996.

[8] T. Henderson, Geometric methods for determining system poles from transient response, IEEE Trans. On Acoust. Speech Signal Process. ASSP-29 (1981) 982–988.

[9] Y. Hua, T.P. Sarkar, Perturbation analysis of TK method for harmonic retrieval problems, IEEE Trans. On Acoust. Speech Signal Process. ASSP-36 (1988) 228–240.

[10] Y. Hua, T.P. Sarkar, Matrix pencil method for estimating parameters of exponentially damped/undamped sinusoids in noise, IEEE Trans. On Acoust. Speech Signal Process. ASSP-38 (5) (May 1990) 814–824.

[11] Y. Hua, T.P. Sarkar, On SVD for estimating generalized eigenvalues of singular matrix pencil in noise, IEEE Trans. Signal Process. 39 (April 1991) 892–900.

[12] S. Kung, A new identification and model reduction algorithm via singular value decompositions, in: Proceedings of the 12th Asilomar Conference on Circuits Systems and Computers, Pacific Grove California, 1978, pp. 705–714.

[13] S. Kung, K.S. Arun, D.V. Bhaskar Rao, State-space and singular value decomposition-based approximation methods for the harmonic retrieval problem, Opt. Soc. Amer. 73 (December 1983) 1799–1811.

[14] F. Li, Y. Lu, Bias analysis for esprit-type algorithms, IEEE Trans. Antenn. Propag. 42 (3) (March 1994) 418–423.

[15] R. Roy, T. Kailath, Total least squares esprit, in: Proceedings 21st Ann. Asilomar Conference on Signals, Systems and Computers, Pacific Grove, CA, 1987, pp. 297–301.

[16] R. Roy, T. Kailath, Esprit-estimation of signal parameters via rotational invariance technique, IEEE Trans. Acoust. Speech Signal Process. ASSP-37 (1989) 984–985.

[17] A. Ruhe, On the closeness of eigenvalues and singular values for almost normal matrices, Linear Algebra Appl. 11 (1975) 87–94.

[18] R.O. Schmidt, Multiple emitter location and signal parameter estimation, IEEE Trans. Antenn. Propag. AP-34 (1986) 276–281.

[19] W.M. Steedly, R.L. Moses, The Cramer–Rao bound for pole and amplitude estimates of damped exponential signals in noise, in: Proceedings of the International Conference on Acoustics, Speech, and Signal Processing, Toronto, Ontario, 1991, pp. 3569–3572.

[20] G.W. Stewart, Error and perturbation bounds for subspaces associated with certain eigenvalue problems, SIAM Rev. 32 (1973) 727–764.

[21] P. Stoica, A. Nehorai, Music, maximum likelihood, and Cramér – Rao bound, IEEE Trans. Acoust. Speech Signal Process. ASSP-37 (1989) 720–741.
[22] P. Stoica, T. Soderstrom, Statistical analysis of music and subspace rotation estimates of sinusoidal frequencies, IEEE Trans. Acoust. Speech Signal Process. ASSP-39 (August 1991) 1836–1847.
[23] P. Stoica, T. Soderstrom, F. Ti, Asymptotic properties of the high-order Yule–Walker estimates of sinusoidal frequencies, IEEE Trans. Acoust. Speech Signal Process. ASSP-37 (11) (1989) 1721–1734.
[24] A. Swindlehurst, B. Ottersten, R. Roy, T. Kailath, Multiple invariance ESPRIT, IEEE Trans. Signal Process. 40 (April 1992) 867–881.
[25] A. Swindlehurst, R. Roy, B. Ottersten, T. Kailath, A subspace fitting method for identification of linear state space models, IEEE Trans. Autom. Control 40 (2) (February 1995) 331–336.
[26] R.J. Vaccaro, A second perturbation expansion for the SVD, SIAM J. Matrix Anal. Appl. 15 (2) (April 1994) 661–671.
[27] A. Van Der Veen, E.F. Deprettere, A. Lee Swindlehurst, Subspace-based signal analysis using singular value decomposition, Proc. IEEE 81 (9) (September 1993) 1277–1309.
[28] S. Van Huffel, Enhanced resolution based on minimum variance estimation and exponential data modeling, Signal Processing 33 (1993) 333–355.
[29] S. Van Huffel, H. Chen, C. Decanniere, P. Van Hecke, Algorithm for time-domain NMR data fitting based on total least squares, J. Magn. Reson. A 110 (1994) 228–237.
[30] M. Viberg, B. Ottersten, B. Walhberg, L. Ljung, Performance of subspace based state space system identification methods, in: Proc. of the 12th IFAC World Congress, Vol. 7, Sidney, Australia, July 1993, pp. 369–372.
[31] M.D. Zoltowski, D. Stavrinides, Sensor array signal processing via procustes rotations based eigenanalysis of the esprit data pencil, IEEE Acoust. Speech Signal Process. ASSP-37 (1989) 832–861.