# Another proof and a generalization of a theorem of H. H. Bauschke on monotone operators

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*Dedicated to Alfredo N. Iusem on the occasion of his 70th birthday*

**Abstract**

Using the concept of *partial-inverse* of monotone operators due to Spingarn, we present a new and simple proof of a result − Theorem 2 in [4] − of Heinz H. Bauschke. Our proof is based on the maximal monotonicity of the partial-inverse and on the (asymptotic) closedness principle on the graph of maximal monotone operators in the weak $\times$ strong topology. We also present a generalization of Bauschke's theorem to the more general setting of $\varepsilon$–enlargements of monotone maps.

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## 1 Introduction

In [4, Theorem 2], Heinz H. Bauschke first presented and proved the following result:

**Theorem 1.1** (Bauschke)**.** *Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximal monotone, and let $C$ be a closed linear subspace of $\mathcal{H}$. Let $(x_n, u_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{gra} A$ such that $(x_n, u_n) \rightharpoonup (x, u) \in \mathcal{H} \times \mathcal{H}$. Suppose that $x_n - P_C x_n \to 0$ and $P_C u_n \to 0$, where $P_C$ denotes the projector onto $C$. Then $(x, u) \in (\operatorname{gra} A) \cap (C \times C^{\perp})$ and $\langle x_n, u_n\rangle \to \langle x, u\rangle = 0$.*

In addition to being interesting by its own right, as a simple and elegant asymptotic closedness principle in monotone operator theory, Theorem 1.1 has been shown to be an important tool for proving the weak convergence of many modern splitting algorithms for solving monotone inclusion and convex optimization problems (see, e.g., [1, 2, 4, 5, 9, 10, 11]). The proof in [4, Theorem 2] makes use of an asymptotic principle for *firmly nonexpansive operators*, while a different proof in [5, Proposition 25.3] relies, in particular, on the use of *Fitzpatrick (convex) functions*. A generalization of Theorem 1.1 (including compositions with linear operators) is also given in [2, Proposition 2.4].

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The proof of the latter result is obtained after a product space embedding and a characterization of the projector onto the graph of linear operators (see [1, Lemma 3.1]), followed by a direct application of Theorem 1.1.

On the other hand, it is well-known that the concepts of *maximal monotonicity*, *nonexpansiveness* and *convexity* are strongly connected through many important results in convex analysis and monotone operator theory. In this sense, and with the above discussion in mind, it is natural to ask whether or not there is a proof of Theorem 1.1 relying solely on the concept of maximal monotonicity.

In this short note, we give a positive answer to this question by employing the concept of Spingarn's *partial–inverse* of a monotone operator with respect to a closed linear subspace (see Definition 2.1 below), originally introduced and studied by J. E. Spingarn in the seminal paper [14]. In addition to being new, the proof of Theorem 1.1 given here is structurally simple and emphasizes the importance and the role played in monotone operator theory by the notion of the *partial-inverse* of a monotone map . We also present a generalization of Theorem 1.1 (see Theorem 4.3 below) – which we believe may be useful for future developments in modern inexact operator-splitting algorithms – to the more general setting of $\varepsilon$–enlargements of maximal monotone operators.

The rest of the material is organized as follows. In Section 2, we review the concept of Spingarn's partial-inverse. In Section 3, we present our proof of Bauschke's theorem and in Section 4 we present a generalization of Theorem 1.1 to $\varepsilon$–enlargements of monotone operators.

**Basic notation:** Throughout this note, $\mathcal{H}$ denotes a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$. $\mathcal{H}\times\mathcal{H}$ denotes the Cartesian product (always endowed with product topologies) of $\mathcal{H}$ by $\mathcal{H}$. By $\to$ and $\rightharpoonup$ we also denote *strong* and *weak* convergence, respectively. The projector onto a closed linear subspace $C$ of $\mathcal{H}$ will be denoted by $P_C$ and the orthogonal complement of $C$ will be denoted by $C^{\perp}$. By $I$ we denote the identity map in $\mathcal{H}$. The *graph* of a set-valued map $A:\mathcal{H}\rightrightarrows\mathcal{H}$ is defined by $\operatorname{gra} A := \{(z,v)\in\mathcal{H}\times\mathcal{H} \mid v\in A(z)\}$. A set-valued map $A:\mathcal{H}\rightrightarrows\mathcal{H}$ is said to be a *monotone operator* in $\mathcal{H}$ whenever $\langle z-z', v-v'\rangle \geq 0$ for all $(z,v),(z',v')\in\operatorname{gra} A$ and a *maximal monotone operator* if $A$ is monotone and its *graph* is not properly contained in the graph of any other monotone operator in $\mathcal{H}$.

## 2 The partial-inverse of a monotone operator

Since its first appearance in [14], the *partial-inverse* operator has found numerous applications in the design and analysis of different proximal and proximal operator-splitting algorithms for monotone inclusion and convex optimization problems (see, e.g., [1, 3, 6, 12, 13, 15] and references therein).

As we mentioned earlier, the partial-inverse will be an important ingredient in our new proof of Bauschke's theorem (see Section 3 below).

**Definition 2.1** (Spingarn)**.** *The* partial-inverse *of a set-valued map $A:\mathcal{H}\rightrightarrows\mathcal{H}$ with respect to a closed linear subspace $C$ of $\mathcal{H}$ is (the set-valued map) $A_C:\mathcal{H}\rightrightarrows\mathcal{H}$ whose graph is*

$$\operatorname{gra} A_C := \{(P_C x + P_{C^\perp} u, P_C u + P_{C^\perp} x) \mid (x,u)\in\operatorname{gra} A\}.$$

Theorem 2.2 below is originally due to J. E. Spingarn [14, Proposition 2.1]. For the convenience of the reader and sake of completeness, we present a detailed proof here.

**Theorem 2.2** (Spingarn)**.** *If $A:\mathcal{H}\rightrightarrows\mathcal{H}$ is maximal monotone, then $A_C$ is also maximal monotone.*

*Proof.* Suppose $A$ is maximal monotone in $\mathcal{H}$ and let $(z,v) := (P_C x + P_{C^\perp} u, P_C u + P_{C^\perp} x)$ and $(z',v') := (P_C x' + P_{C^\perp} u', P_C u' + P_{C^\perp} x')$ be elements in $\operatorname{gra} A_C$. Using the latter definitions, the fact that $C \perp C^\perp$ and the identity $P_C + P_{C^\perp} = I$, we find

$$\begin{aligned}
\langle z - z', v - v' \rangle &= \langle P_C(x - x') + P_{C^\perp}(u - u'), P_C(u - u') + P_{C^\perp}(x - x') \rangle \\
&= \langle P_C(x - x'), P_C(u - u') \rangle + \langle P_{C^\perp}(u - u'), P_{C^\perp}(x - x') \rangle \\
&= \langle P_C(x - x') + P_{C^\perp}(x - x'), P_C(u - u') + P_{C^\perp}(u - u') \rangle \\
&= \langle x - x', u - u' \rangle \\
&\geq 0, \qquad (1)
\end{aligned}$$

which, in turn, proves the monotonicity of $A_C$, where in (1) we used the monotonicity of $A$ combined with the inclusions $(x,u), (x',u') \in \operatorname{gra} A$.

To prove the maximal monotonicity of $A_C$, assume first that $(z,v) \in \mathcal{H} \times \mathcal{H}$ satisfies

$$\langle z - z', v - v' \rangle \geq 0 \qquad \forall (z', v') \in \operatorname{gra} A_C. \qquad (2)$$

Let $(x',u') \in \operatorname{gra} A$ and let, by Definition 2.1, $(z',v') := (P_C x' + P_{C^\perp} u', P_C u' + P_{C^\perp} x') \in \operatorname{gra} A_C$. Using this, (again) the identity $P_C + P_{C^\perp} = I$ and the fact that $C \perp C^\perp$, we now find

$$\begin{aligned}
\langle z - z', v - v' \rangle &= \langle z - (P_C x' + P_{C^\perp} u'), v - (P_C u' + P_{C^\perp} x') \rangle \\
&= \langle P_C(z - x') + P_{C^\perp}(z - u'), P_C(v - u') + P_{C^\perp}(v - x') \rangle \\
&= \langle P_C(z - x'), P_C(v - u') \rangle + \langle P_{C^\perp}(z - u'), P_{C^\perp}(v - x') \rangle \\
&= \langle P_C(z - x') + P_{C^\perp}(v - x'), P_C(v - u') + P_{C^\perp}(z - u') \rangle \\
&= \langle P_C z + P_{C^\perp} v - x', P_C v + P_{C^\perp} z - u' \rangle. \qquad (3)
\end{aligned}$$

From (2) and (3) we obtain

$$\langle P_C z + P_{C^\perp} v - x', P_C v + P_{C^\perp} z - u' \rangle \geq 0 \qquad \forall (x', u') \in \operatorname{gra} A$$

which, in turn, in view of the maximality of $A$, gives $(x,u) := (P_C z + P_{C^\perp} v, P_C v + P_{C^\perp} z) \in \operatorname{gra} A$. Simple computations show that $(z,v) = (P_C x + P_{C^\perp} u, P_C u + P_{C^\perp} x)$, which, by Definition 2.1, gives $(z,v) \in \operatorname{gra} A_C$, proving the maximal monotonicity of $A_C$. $\square$

## 3 A new proof of Theorem 1.1

*Proof.* Using the assumption $(x_n, u_n) \in \operatorname{gra} A$ and Definition 2.1, we obtain $(z_n, v_n) \in \operatorname{gra} A_C$, where

$$z_n := P_C x_n + P_{C^\perp} u_n, \qquad v_n := P_C u_n + P_{C^\perp} x_n. \qquad (4)$$

From the assumptions $x_n - P_C x_n \to 0$ and $P_C u_n \to 0$, the identity $x_n = P_C x_n + P_{C^\perp} x_n$ and the definition of $v_n$ above, we also obtain $v_n \to 0$. Moreover, since $P_C$ and $P_{C^\perp}$ are weakly continuous and by assumption $(x_n, u_n) \rightharpoonup (x, u)$, we also have $z_n \rightharpoonup P_C x + P_{C^\perp} u$. Altogether, we proved that

$$(z_n, v_n) \in \operatorname{gra} A_C, \quad v_n \to 0 \;\text{ and }\; z_n \rightharpoonup P_C x + P_{C^\perp} u,$$

which, in turn, combined with Theorem 2.2 and the closedness property of the graph of maximal monotone operators in the weak $\times$ strong topology of $\mathcal{H} \times \mathcal{H}$ (see, e.g., Proposition 20.38(ii) in [5]) gives the inclusion

$$(P_C x + P_{C^\perp} u, 0) \in \operatorname{gra} A_C, \quad \text{i.e.,} \quad 0 \in A_C(P_C x + P_{C^\perp} u). \tag{5}$$

In view of (5) and Definition 2.1, we have that there exists $(x', u') \in \operatorname{gra} A$ such that

$$P_C x + P_{C^\perp} u = P_C x' + P_{C^\perp} u', \qquad 0 = P_C u' + P_{C^\perp} x',$$

which in turn (using the fact that $C \cap C^\perp = \{0\}$) yields $P_C x = P_C x'$, $P_{C^\perp} u = P_{C^\perp} u'$ and $P_C u' = P_{C^\perp} x' = 0$. As a consequence of the three latter identities and the facts that $x' = P_C x' + P_{C^\perp} x'$ and $u' = P_C u' + P_{C^\perp} u'$ we obtain $x' = P_C x$ and $u' = P_{C^\perp} u$ and so, since $(x', u') \in \operatorname{gra} A$, the inclusion $(P_C x, P_{C^\perp} u) \in \operatorname{gra} A$.

Consequently, to prove the desired inclusion in Theorem 1.1, it suffices to show that $x = P_C x$ and $u = P_{C^\perp} u$. To this end, and to finish the proof of the theorem, we now follow the same arguments as in [4, Theorem 2]. Since $P_C$ is weakly continuous, we have

$$x \leftharpoonup x_n = P_C x_n + P_{C^\perp} x_n \rightharpoonup P_C x + 0 = P_C x$$

and so $x = P_C x \in C$. A similar reasoning also yields $u = P_{C^\perp} u \in C^\perp$. To finish the proof, note that

$$\langle x_n, u_n \rangle = \langle P_C x_n, P_C u_n \rangle + \langle P_{C^\perp} x_n, P_{C^\perp} u_n \rangle \to \langle P_C x, 0 \rangle + \langle 0, P_{C^\perp} u \rangle = 0 = \langle P_C x, P_{C^\perp} u \rangle = \langle x, u \rangle.$$

□

# 4 A generalization of Bauschke's theorem

In this last section, we propose and prove a generalization of Theorem 1.1 to the more general setting of $\varepsilon$–enlargements of maximal monotone operators. The main result is Theorem 4.3 below, which reduces to Bauschke's theorem whenever $\varepsilon_n \equiv 0$. Theorem 4.3 is potentially useful for future investigations concerning the weak convergence of inexact variants of projective-splitting type algorithms (see, e.g., [9, 10, 11]).

Next is the definition of $\varepsilon$–enlargements (see, e.g., [7]).

**Definition 4.1** (Burachik–Iusem–Svaiter)**.** *The* $\varepsilon$–enlargement *of a set-valued map* $A : \mathcal{H} \rightrightarrows \mathcal{H}$ *is (the set-valued map)* $A^{[\varepsilon]} : \mathcal{H} \rightrightarrows \mathcal{H}$ *whose graph is defined as*

$$\operatorname{gra} A^{[\varepsilon]} := \{(z, v) \in \mathcal{H} \times \mathcal{H} \mid \langle z - z', v - v' \rangle \geq -\varepsilon \quad \forall (z', v') \in \operatorname{gra} A\}.$$

We shall also need the following result from [8].

**Proposition 4.2** (Burachik–Sagastizábal–Scheimberg)**.** *Let* $A : \mathcal{H} \rightrightarrows \mathcal{H}$ *be maximal monotone and let* $C$ *be a closed linear subspace of* $\mathcal{H}$*. Then, for all* $\varepsilon \geq 0$,

$$(A^{[\varepsilon]})_C = (A_C)^{[\varepsilon]}. \tag{6}$$

Next is the main result of this section, namely a generalization of Bauschke's theorem to the more general setting of $\varepsilon$–enlargements.

**Theorem 4.3.** *Let $A : \mathcal{H} \rightrightarrows \mathcal{H}$ be maximal monotone and let $C$ be a closed linear subspace of $\mathcal{H}$. Let $(x_n, u_n)_{n\in\mathbb{N}}$ be a sequence in $\operatorname{gra} A^{[\varepsilon_n]}$ such that $(x_n, u_n) \rightharpoonup (x, u) \in \mathcal{H} \times \mathcal{H}$ and $\varepsilon_n \to 0$. Suppose that $x_n - P_C x_n \to 0$ and $P_C u_n \to 0$, where $P_C$ denotes the projector onto $C$. Then $(x, u) \in (\operatorname{gra} A) \cap (C \times C^\perp)$ and $\langle x_n, u_n\rangle \to \langle x, u\rangle = 0$.*

*Proof.* The proof follows the same outline of the proof of Theorem 1.1 in Section 3. Using the assumption $(x_n, u_n) \in \operatorname{gra} A^{[\varepsilon_n]}$, Definition 2.1 for $A^{[\varepsilon_n]}$ and Proposition 4.2, we obtain

$$(z_n, v_n) \in \operatorname{gra}(A_C)^{[\varepsilon_n]}, \text{ i.e., } v_n \in \operatorname{gra}(A_C)^{[\varepsilon_n]}(z_n),$$

where $z_n$ and $v_n$ are as in (4). Using the same reasoning as in the proof of Section 3 we also conclude that $v_n \to 0$ and $z_n \rightharpoonup P_C x + P_{C^\perp} u$. Altogether, and from the assumption $\varepsilon_n \to 0$, we then obtain

$$(z_n, v_n) \in \operatorname{gra}(A_C)^{[\varepsilon_n]}, \quad v_n \to 0, \ \varepsilon_n \to 0 \ \text{ and } \ z_n \rightharpoonup P_C x + P_{C^\perp} u,$$

which combined with Theorem 2.2 and the closedness property of the graph of $\varepsilon$–enlargement of maximal monotone operators in the weak $\times$ strong topology of $\mathcal{H} \times \mathcal{H}$ (see, e.g., Proposition 4.3(b) in [16], for the maximal monotone operator $A_C$) gives the inclusion in (5). The rest of the proof is analogous to the corresponding one in Section 3. □

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