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Maximal Monotonicity, Conjugation and the Duality Product in Non-Reflexive Banach Spaces

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In this work we study some conditions which guarantee that a convex function represents a maximal monotone operator in non-reflexive Banach spaces.

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1. Introduction

Let X be a real Banach space and X^* its topological dual, both with norms denoted by $\|\cdot\|$. The duality product in $X \times X^*$ will be denoted by:

$$\pi : X \times X^* \to \mathbb{R}, \quad \pi(x, x^*) := \langle x, x^* \rangle = x^*(x). \tag{1}$$

A point to set operator $T : X \rightrightarrows X^*$ is a relation on $X \times X^*$:

$$T \subset X \times X^*$$

and $T(x) = \{x^* \in X^* \mid (x, x^*) \in T\}$. An operator $T : X \rightrightarrows X^*$ is *monotone* if

$$\langle x - y, x^* - y^* \rangle \geq 0, \quad \forall (x, x^*), (y, y^*) \in T$$

and it is *maximal monotone* if it is monotone and maximal (with respect to the inclusion) in the family of monotone operators of X into X^*. The domain of $T : X \rightrightarrows X^*$ is defined by $D(T) := \{x \in X \mid T(x) \neq \emptyset\}$.

Fitzpatrick proved constructively that maximal monotone operators are representable by convex functions. Before discussing his findings, let us establish some notation. We

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denote the set of extended-real valued functions on X by $\overline{\mathbb{R}}^X$. The *epigraph* of $f \in \overline{\mathbb{R}}^X$ is defined by

$$\mathrm{E}(f) := \{(x, \mu) \in X \times \mathbb{R} \mid f(x) \leq \mu\}.$$

We say that $f \in \overline{\mathbb{R}}^X$ is lower semicontinuous (l.s.c. from now on) if $\mathrm{E}(f)$ is closed in the strong topology of $X \times \mathbb{R}$.

Let $T : X \rightrightarrows X^*$ be maximal monotone. The *Fitzpatrick function of* T is [4]

$$\varphi_T \in \overline{\mathbb{R}}^{X \times X^*}, \quad \varphi_T(x, x^*) := \sup_{(y,y^*) \in T} \langle x - y, y^* - x^* \rangle + \langle x, x^* \rangle \tag{2}$$

and the *Fitzpatrick family* associated with T is

$$\mathcal{F}_T := \left\{ h \in \overline{\mathbb{R}}^{X \times X^*} \;\middle|\; \begin{array}{l} h \text{ is convex and l.s.c.} \\ h(x, x^*) \geq \langle x, x^* \rangle, \ \forall (x, x^*) \in X \times X^* \\ (x, x^*) \in T \Rightarrow h(x, x^*) = \langle x, x^* \rangle \end{array} \right\}.$$

In the next theorem we summarize the Fitzpatrick's results:

Theorem 1.1 ([4, Theorem 3.10]). *Let X be a real Banach space and $T : X \rightrightarrows X^*$ be maximal monotone. Then for any $h \in \mathcal{F}_T$*

$$(x, x^*) \in T \iff h(x, x^*) = \langle x, x^* \rangle$$

and φ_T is the smallest element of the family $\mathcal{F}_T$.

Fitzpatrick's results described above were rediscovered by Martínez-Legaz and Théra [9], and Burachik and Svaiter [2].

It seems interesting to study conditions under which a convex function $h \in \overline{\mathbb{R}}^X$ represents a maximal monotone operator, that is, $h \in \mathcal{F}_T$ for some maximal monotone operator T. Our aim is to extend previous results on this direction. We will need some auxiliary results and additional notation for this aim.

The Fenchel-Legendre conjugate of $f \in \overline{\mathbb{R}}^X$ is

$$f^* \in \overline{\mathbb{R}}^{X^*}, \quad f^*(x^*) := \sup_{x \in X} \langle x, x^* \rangle - f(x).$$

Whenever necessary, we will identify X with its image under the canonical injection of X into X^{**}. Burachik and Svaiter proved that the family $\mathcal{F}_T$ is invariant under the mapping

$$\mathcal{J} : \overline{\mathbb{R}}^{X \times X^*} \to \overline{\mathbb{R}}^{X \times X^*}, \quad \mathcal{J}\, h(x, x^*) := h^*(x^*, x). \tag{3}$$

This means that if $T : X \rightrightarrows X^*$ is maximal monotone, then [2]

$$\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T. \tag{4}$$

In particular, for any $h \in \mathcal{F}_T$ it holds that $h \geq \pi$, $\mathcal{J}h \geq \pi$, that is,

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \ \ \forall (x, x^*) \in X \times X^*.$$

So, the above conditions are *necessary* for a convex function h on $X \times X^*$ to represent a maximal monotone operator. Burachik and Svaiter proved that these conditions are also *sufficient*, in a reflexive Banach space, for h to represent a maximal monotone operator [3]:

Theorem 1.2 ([3, Theorem 3.1]). *Let $h \in \overline{\mathbb{R}}^{X \times X^*}$ be proper, convex, l.s.c. and*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*. \tag{5}$$

If X is reflexive, then

$$T := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone and $h, \mathcal{J}h \in \mathcal{F}_T$.

Marques Alves and Svaiter generalized Theorem 1.2 to non-reflexive Banach spaces as follows:

Theorem 1.3 ([5, Corollary 4.4]). *If $h \in \overline{\mathbb{R}}^{X \times X^*}$ is convex and*

$$\begin{aligned} h(x, x^*) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*, \\ h^*(x^*, x^{**}) \geq \langle x^*, x^{**} \rangle, \quad \forall (x^*, x^{**}) \in X^* \times X^{**} \end{aligned} \tag{6}$$

then

$$T := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$$

is maximal monotone and $\mathcal{J}h \in \mathcal{F}_T$. Moreover, if h is l.s.c. then $h \in \mathcal{F}_T$.

Condition (6) of Theorem 1.3 enforces the operator T to be of type (NI) [6] and is not necessary for maximal monotonicity of T in a non-reflexive Banach space. Note that the weaker condition (5) of Theorem 1.2 is still *necessary* in non-reflexive Banach spaces for the inclusion $h \in \mathcal{F}_T$, where T is a maximal monotone operator. The main result of this paper is another generalization of Theorem 1.2 to non-reflexive Banach spaces which uses condition (5) instead of (6). To obtain this generalization, we add a regularity assumption on the domain of h.

If $T : X \rightrightarrows X^*$ is maximal monotone, it is easy to prove that φ_T is minimal in the family of all convex functions in $X \times X^*$ which majorizes the duality product. So, it is natural to ask whether the converse also holds, that is:

> Is any minimal element of this family (convex functions which majorizes the duality product) a Fitzpatrick function of some maximal monotone operator?

To give a partial answer to this question, Martínez-Legaz and Svaiter proved the following results, which we will use latter on:

Theorem 1.4 ([8, Theorem 5]). *Let $\mathcal{H}$ be the family of convex functions in $X \times X^*$ which majorizes the duality product:*

$$\mathcal{H} := \left\{ h \in \overline{\mathbb{R}}^{X \times X^*} \mid h \text{ is proper, convex and } h \geq \pi \right\}. \tag{7}$$

The following statements hold true:

1. *The family $\mathcal{H}$ is (downward) inductively ordered;*
2. *For any $h \in \mathcal{H}$ there exists a minimal $h_0 \in \mathcal{H}$ such that $h \geq h_0$;*
3. *Any minimal element g of $\mathcal{H}$ is l.s.c. and satisfies $\mathcal{J}g \geq g$.*

Note that item *2.* is a direct consequence of item *1.*. Combining item *3.* with Theorem 1.2, Martínez-Legaz and Svaiter concluded that *in a reflexive* Banach space, any minimal element of $\mathcal{H}$ is the Fitzpatrick function of some maximal monotone operator [8, Theorem 5]. We will also present a partial extension of this result for non-reflexive Banach spaces.

2. Basic results and notation

The weak-star topology of X^* will be denoted by ω^* and by s we denote the strong topology of X. A function $h \in \overline{\mathbb{R}}^{X\times X^*}$ is lower semicontinuous in the strong $\times$ weak-star topology if $\mathrm{E}(h)$ is a closed subset of $X \times X^* \times \mathbb{R}$ in the $s \times \omega^* \times |\cdot|$ topology.

The *indicator function* of $V \subset X$ is δ_V, $\delta_V(x) := 0$, $x \in V$ and $\delta_V(x) := \infty$, otherwise. The closed convex closure of $f \in \overline{\mathbb{R}}^X$ is defined by

$$\mathrm{cl\,conv} f \in \overline{\mathbb{R}}^X, \quad \mathrm{cl\,conv} f(x) := \inf\{\mu \in \mathbb{R} \,|\, (x,\mu) \in \mathrm{cl\,conv}\,\mathrm{E}(f)\}$$

where for $U \subset X$, $\mathrm{cl\,conv}\, U$ is the closed convex hull (in the s topology) of U. The *effective domain* of a function $f \in \overline{\mathbb{R}}^X$ is

$$\mathrm{D}(f) := \{x \in X \,|\, f(x) < \infty\},$$

and f is *proper* if $\mathrm{D}(f) \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$. If f is proper, convex and l.s.c., then f^* is proper. For $h \in \overline{\mathbb{R}}^{X\times X^*}$, we also define

$$\mathrm{Pr}_X\,\mathrm{D}(h) := \{x \in X \,|\, \exists\, x^* \in X^* \,|\, (x,x^*) \in \mathrm{D}(h)\}.$$

Let $T : X \rightrightarrows X^*$ be maximal monotone. In [2] Burachik and Svaiter defined and studied the biggest element of $\mathcal{F}_T$, namely, the $\mathcal{S}$-function, $\mathcal{S}_T \in \mathcal{F}_T$ defined by

$$\mathcal{S}_T \in \overline{\mathbb{R}}^{X\times X^*}, \quad \mathcal{S}_T := \sup_{h\in\mathcal{F}_T} \{h\},$$

or, equivalently

$$\mathcal{S}_T = \mathrm{cl\,conv}(\pi + \delta_T).$$

Recall that $\mathcal{J}(\mathcal{F}_T) \subset \mathcal{F}_T$. Additionally [2]

$$\mathcal{J}\,\mathcal{S}_T = \varphi_T \tag{8}$$

and, in a reflexive Banach space, $\mathcal{J}\varphi_T = \mathcal{S}_T$.

In what follows we present the Attouch-Brezis's version of the Fenchel-Rockafellar duality theorem:

Theorem 2.1 ([1, Theorem 1.1]). *Let Z be a Banach space and $\varphi, \psi \in \overline{\mathbb{R}}^{Z}$ be proper, convex and l.s.c. functions. If*

$$\bigcup_{\lambda>0} \lambda\,[\mathrm{D}\,(\varphi) - \mathrm{D}\,(\psi)], \tag{9}$$

is a closed subspace of Z, then

$$\inf_{z\in Z} \varphi(z) + \psi(z) = \max_{z^*\in Z^*} -\varphi^*(z^*) - \psi^*(-z^*). \tag{10}$$

Given X, Y Banach spaces, $\mathcal{L}(Y, X)$ denotes the set of continuous linear operators of Y into X. The range of $A \in \mathcal{L}(Y, X)$ is denoted by $\mathrm{R}(A)$ and the adjoint by $A^* \in \mathcal{L}(X^*, Y^*)$:

$$\langle Ay, x^*\rangle = \langle y, A^*x^*\rangle \quad \forall y \in Y,\ x^* \in X^*,$$

where X^*, Y^* are the topological duals of X and Y, respectively. The next proposition is a particular case of Theorem 3 of [10]. For the sake of completeness, we give the proof in the Appendix A.

Proposition 2.2. *Let X, Y Banach spaces and $A \in \mathcal{L}(Y, X)$. For $h \in \overline{\mathbb{R}}^{X\times X^*}$, proper convex and l.s.c., define $f \in \overline{\mathbb{R}}^{Y\times Y^*}$*

$$f(y, y^*) := \inf_{x*\in X^*} h(Ay, x^*) + \delta_{\{0\}}(y^* - A^*x^*).$$

If

$$\bigcup_{\lambda>0} \lambda\,[\mathrm{Pr}_X\,\mathrm{D}(h) - \mathrm{R}(A)], \tag{11}$$

is a closed subspace of X, then

$$f^*(z^*, z) = \min_{u^*\in X^*} h^*(u^*, Az) + \delta_{\{0\}}(z^* - A^*u^*).$$

Martínez-Legaz and Svaiter [7] defined, for $h \in \overline{\mathbb{R}}^{X\times X^*}$ and $(x_0, x_0^*) \in X\times X^*$, $h_{(x_0,x_0^*)} \in \overline{\mathbb{R}}^{X\times X^*}$

$$\begin{aligned} h_{(x_0,x_0^*)}(x, x^*) &:= h(x + x_0, x^* + x_0^*) - [\langle x, x_0^*\rangle + \langle x_0, x^*\rangle + \langle x_0, x_0^*\rangle] \\ &= h(x + x_0, x^* + x_0^*) - \langle x + x_0, x^* + x_0^*\rangle + \langle x, x^*\rangle. \end{aligned} \tag{12}$$

The operation $h \mapsto h_{(x_0,x_0^*)}$ preserves many properties of h, as convexity and lower semicontinuity. Moreover, one can easily prove the following Proposition:

Proposition 2.3. *Let $h \in \overline{\mathbb{R}}^{X\times X^*}$. Then it holds that*

1. $h \geq \pi \iff h_{(x_0,x_0^*)} \geq \pi,\ \forall (x_0, x_0^*) \in X \times X^*$;
2. $\mathcal{J}h_{(x_0,x_0^*)} = (\mathcal{J}h)_{(x_0,x_0^*)},\ \forall (x_0, x_0^*) \in X \times X^*$.

3. Main results

In the next theorem we generalize Theorem 1.2 to non-reflexive Banach spaces under condition (5) instead of condition (6) used in Theorem 1.3. To obtain this generalization, we add a regularity assumption (14) on the domain of h.

Theorem 3.1. *Let* $h \in \overline{\mathbb{R}}^{X\times X^*}$ *be proper, convex and*

$$h(x,x^*) \geq \langle x,x^*\rangle, \quad h^*(x^*,x) \geq \langle x,x^*\rangle, \ \ \forall (x,x^*) \in X\times X^*. \tag{13}$$

If

$$\bigcup_{\lambda>0} \lambda \operatorname{Pr}_X \operatorname{D}(h), \tag{14}$$

is a closed subspace of X*, then*

$$T := \{(x,x^*) \in X\times X^* \mid h^*(x^*,x) = \langle x,x^*\rangle\}$$

is maximal monotone and $\mathcal{J}h \in \mathcal{F}_T$*.*

Proof. First, define $\bar h := \operatorname{cl} h$ and note that $\bar h$ is proper, convex, l.s.c., satisfies (13), (14) and $\mathcal{J}\bar h = \mathcal{J}h$. So, it suffices to prove the theorem for the case where h is l.s.c., and we assume it from now on in this proof. Monotonicity of T follows from Theorem 5 of [7]. Note that for any $x \in X$

$$T(x) = \{x^* \in X^* \mid h^*(x^*,x) - \langle x,x^*\rangle \leq 0\}.$$

Therefore, $T(x)$ is convex and ω^*-closed.

To prove maximality of T, take $(x_0,x_0^*) \in X\times X^*$ such that

$$\langle x-x_0, x^*-x_0^*\rangle \geq 0, \ \ \forall (x,x^*) \in T \tag{15}$$

and suppose $x_0^* \notin T(x_0)$. As $T(x_0)$ is convex and ω^*-closed, using the geometric version of the Hahn-Banach theorem in X^* endowed with the ω^* topology we conclude that (even if $T(x_0)$ is empty) there exists $z_0 \in X$ such that

$$\langle z_0, x_0^*\rangle < \langle z_0, x^*\rangle, \ \ \forall\, x^* \in T(x_0). \tag{16}$$

Let $Y := \operatorname{span}\{x_0, z_0\}$. Define $A \in \mathcal{L}(Y,X)$, $A\,y := y$, $\forall\, y \in Y$ and the convex function $f \in \overline{\mathbb{R}}^{Y\times Y^*}$,

$$f(y,y^*) := \inf_{x*\in X^*} h(Ay,x^*) + \delta_{\{0\}}(y^* - A^*x^*). \tag{17}$$

Using Proposition 2.2 we obtain

$$f^*(y^*,y) = \min_{x*\in X^*} h^*(x^*,Ay) + \delta_{\{0\}}(y^* - A^*x^*). \tag{18}$$

Using (13), (17) and (18) it is easy to see that

$$f(y,y^*) \geq \langle y,y^*\rangle, \quad f^*(y^*,y) \geq \langle y,y^*\rangle, \ \ \forall (y,y^*) \in Y\times Y^*. \tag{19}$$

Define $g := \mathcal{J}f$. As Y is reflexive we have $\mathcal{J}g = \operatorname{cl} f$. Therefore, using (19) we also have

$$g(y,y^*) \geq \langle y,y^*\rangle, \quad g^*(y^*,y) \geq \langle y,y^*\rangle, \quad \forall (y,y^*) \in Y \times Y^*. \tag{20}$$

Now, using (20) and item *1.* of Proposition 2.3 we obtain

$$\begin{aligned} & g_{(x_0,A^*x_0^*)}(y,y^*) + \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|^2 \\ & \geq \langle y,y^*\rangle + \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|^2 \geq 0, \quad \forall (y,y^*) \in Y \times Y^* \end{aligned} \tag{21}$$

and

$$\begin{aligned} & (\mathcal{J}g)_{(x_0,A^*x_0^*)}(y,y^*) + \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|^2 \\ & \geq \langle y,y^*\rangle + \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|^2 \geq 0, \quad \forall (y,y^*) \in Y \times Y^*. \end{aligned} \tag{22}$$

Using Theorem 2.1 and item *2.* of Proposition 2.3 we conclude that there exists $(\tilde{z},\tilde{z}^*) \in Y \times Y^*$ such that

$$\inf g_{(x_0,A^*x_0^*)}(y,y^*) + \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|^2 + (\mathcal{J}g)_{(x_0,A^*x_0^*)}(\tilde{z},\tilde{z}^*) + \frac{1}{2}\|\tilde{z}\|^2 + \frac{1}{2}\|\tilde{z}^*\|^2 = 0. \tag{23}$$

From (21), (22) and (23) we have

$$\inf_{(y,y^*)\in Y \times Y^*} g_{(x_0,A^*x_0^*)}(y,y^*) + \frac{1}{2}\|y\|^2 + \frac{1}{2}\|y^*\|^2 = 0. \tag{24}$$

As Y is reflexive, from (12), (24) we conclude that there exists $(\hat{y},\hat{y}^*) \in Y \times Y^*$ such that

$$g\left(\hat{y}+x_0, \hat{y}^* + A^*x_0^*\right) - \langle \hat{y}+x_0, \hat{y}^*+A^*x_0^*\rangle + \langle \hat{y},\hat{y}^*\rangle + \frac{1}{2}\|\hat{y}\|^2 + \frac{1}{2}\|\hat{y}^*\|^2 = 0. \tag{25}$$

Using (25) and the first inequality of (20) (and the definition of g) we have

$$f^*(\hat{y}^* + A^*x_0^*, \hat{y}+x_0) = \langle \hat{y}+x_0, \hat{y}^*+A^*x_0^*\rangle \tag{26}$$

and

$$\langle \hat{y},\hat{y}^*\rangle + \frac{1}{2}\|\hat{y}\|^2 + \frac{1}{2}\|\hat{y}^*\|^2 = 0. \tag{27}$$

Using (18) we have that there exists $w_0^* \in X^*$ such that

$$f^*(\hat{y}^* + A^*x_0^*, \hat{y}+x_0) = h^*(w_0^*, A\left(\hat{y}+x_0\right)), \quad \hat{y}^* + A^*x_0^* = A^*w_0^*. \tag{28}$$

So, combining (26) and (28) we have

$$h^*(w_0^*, A\left(\hat{y}+x_0\right)) = \langle \hat{y}+x_0, A^*w_0^*\rangle = \langle A(\hat{y}+x_0), w_0^*\rangle.$$

In particular, $w_0^* \in T(A(\hat{y}+x_0))$. As $x_0 \in Y$, we can use (15) and the second equality of (28) to conclude that

$$\langle A(\hat{y}+x_0) - x_0, w_0^* - x_0^*\rangle = \langle \hat{y}, A^*(w_0^*-x_0^*)\rangle = \langle \hat{y},\hat{y}^*\rangle \geq 0. \tag{29}$$

Using (27) and (29) we conclude that $\hat{y} = 0$ and $\hat{y}^* = 0$. Therefore,

$$w_0^* \in T(x_0), \quad A^* x_0^* = A^* w_0^*.$$

As $z_0 \in Y$, we have $z_0 = A\, z_0$ and so

$$\langle z_0, x_0^* \rangle = \langle A\, z_0, x_0^* \rangle = \langle z_0, A^* x_0^* \rangle = \langle z_0, A^* w_0^* \rangle = \langle A\, z_0, w_0^* \rangle = \langle z_0, w_0^* \rangle,$$

that is,

$$\langle z_0, x_0^* \rangle = \langle z_0, w_0^* \rangle, \quad w_0^* \in T(x_0)$$

which contradicts (16). Therefore, $(x_0, x_0^*) \in T$ and so T is maximal monotone and $\mathcal{J}h \in \mathcal{F}_T$. □

Observe that if h is convex, proper and l.s.c. in the strong $\times$ weak-star topology, then $\mathcal{J}^2 h = h$. Therefore, using this observation we have the following corollary of Theorem 3.1:

Corollary 3.2. *Let $h \in \overline{\mathbb{R}}^{X \times X^*}$ be proper, convex, l.s.c. in the strong $\times$ weak-star topology and*

$$h(x, x^*) \geq \langle x, x^* \rangle, \quad h^*(x^*, x) \geq \langle x, x^* \rangle, \quad \forall (x, x^*) \in X \times X^*.$$

If

$$\bigcup_{\lambda > 0} \lambda \operatorname{Pr}_X \operatorname{D}(h),$$

is a closed subspace of X, then

$$T := \{(x, x^*) \in X \times X^* \mid h(x, x^*) = \langle x, x^* \rangle\}$$

is maximal monotone and $h, \mathcal{J}h \in \mathcal{F}_T$.

Proof. Using Theorem 3.1 we conclude that the set

$$S := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^* \rangle\}$$

is maximal monotone. Take $(x, x^*) \in S$. As π is Gateaux differentiable, $h \geq \pi$ and $\pi(x, x^*) = h(x, x^*)$, we have (see Lemma 4.1 of [5])

$$D\pi(x, x^*) \in \partial \mathcal{J}h(x, x^*),$$

where $D\pi$ stands for the Gateaux derivative of π. As $D\pi(x, x^*) = (x^*, x)$, we conclude that

$$\mathcal{J}h(x, x^*) + \mathcal{J}^2 h(x, x^*) = \langle (x, x^*), (x^*, x) \rangle.$$

Substituting $\mathcal{J}h(x, x^*)$ by $\langle x, x^* \rangle$ in the above equation we conclude that $\mathcal{J}^2 h(x, x^*) = \langle x, x^* \rangle$. Therefore, as $\mathcal{J}^2 h(x, x^*) = h(x, x^*)$,

$$S \subset T.$$

To end the proof use the maximal monotonicity of S (Theorem 3.1) and the monotonicity of T (see Theorem 5 of [7]) to conclude that $S = T$. □

It is natural to ask whether we can drop lower semicontinuity assumptions. In the context of non-reflexive Banach spaces, we should use the l.s.c. closure in the strong × weak-star topology. Unfortunately, as the duality product is not continuous in this topology, it is not clear whether the below implication holds:

$$h \geq \pi \overset{?}{\Rightarrow} \mathrm{cl}_{s\times\omega^*} h \geq \pi.$$

Corollary 3.3. *Let* $h \in \overline{\mathbb{R}}^{X\times X^*}$ *be proper, convex and*

$$h(x, x^*) \geq \langle x, x^*\rangle, \quad h^*(x^*, x) \geq \langle x, x^*\rangle, \quad \forall (x, x^*) \in X \times X^*.$$

If

$$\bigcup_{\lambda>0} \lambda \, \mathrm{Pr}_X \mathrm{D}(h)$$

is a closed subspace of X*, then*

$$\mathrm{cl}_{s\times\omega^*} h \in \mathcal{F}_T,$$

where $\mathrm{cl}_{s\times\omega^*}$ *denotes the l.s.c. closure in the strong × weak-star topology and* T *is the maximal monotone operator defined as in Theorem* 3.1*:*

$$T := \{(x, x^*) \in X \times X^* \mid h^*(x^*, x) = \langle x, x^*\rangle\}.$$

In particular, $\mathrm{cl}_{s\times\omega^*} h \geq \pi$*.*

Proof. First use Theorem 3.1 to conclude that T is maximal monotone and $\mathcal{J}h \in \mathcal{F}_T$. In particular,

$$\mathcal{S}_T \geq \mathcal{J}h \geq \varphi_T.$$

Therefore,

$$\mathcal{J}\varphi_T \geq \mathcal{J}^2 h \geq \mathcal{J}\mathcal{S}_T.$$

As $\mathcal{J}\mathcal{S}_T = \varphi_T \in \mathcal{F}_T$ and $\mathcal{J}\varphi_T \in \mathcal{F}_T$, we conclude that $\mathrm{cl}_{s\times\omega^*} h = \mathcal{J}^2 h \in \mathcal{F}_T$. □

In the next corollary we give a partial answer for an open question proposed by Martínez-Legaz and Svaiter in [8], in the context of non-reflexive Banach spaces.

Corollary 3.4. *Let* $\mathcal{H}$ *be the family of convex functions on* $X \times X^*$ *bounded below by the duality product, as defined in* (7)*. If* g *is a minimal element of* $\mathcal{H}$ *and*

$$\bigcup_{\lambda>0} \lambda \, \mathrm{Pr}_X \, \mathrm{D}(g)$$

is a closed subspace of X*, then there exists a maximal monotone operator* T *such that* $g = \varphi_T$*, where* φ_T *is the Fitzpatrick function of* T*.*

Proof. Using item *3.* of Theorem 1.4 and Theorem 3.1 we have that

$$T := \{(x, x^*) \in X \times X^* \,|\, g^*(x^*, x) = \langle x, x^*\rangle\}$$

is maximal monotone, $\mathcal{J}g \in \mathcal{F}_T$ and

$$T \subset \{(x, x^*) \in X \times X^* \mid g(x, x^*) = \langle x, x^*\rangle\}.$$

As g is convex and bounded below by the duality product, using Theorem 5 of [7], we conclude that the rightmost set on the above inclusion is monotone. Since T is maximal monotone, the above inclusion holds as an equality and, being l.s.c., $g \in \mathcal{F}_T$. To end the proof, note that $g \geq \varphi_T \in \mathcal{H}$. □

A. Proof of Proposition 2.2

Proof of Proposition 2.2. Using the Fenchel-Young inequality we have, for any $(y, y^*), (z, z^*) \in Y \times Y^*$ and $x^*, u^* \in X^*$,

$$h(Ay, x^*) + \delta_{\{0\}}(y^* - A^*x^*) + h^*(u^*, Az) + \delta_{\{0\}}(z^* - A^*u^*) \geq \langle Ay, u^*\rangle + \langle Az, x^*\rangle.$$

Taking the infimum over $x^*, u^* \in X^*$ on the above inequality we get

$$\begin{aligned} &f(y, y^*) + \inf_{u^* \in X^*} h^*(u^*, Az) + \delta_{\{0\}}(z^* - A^*u^*) \\ &\geq \langle y, z^*\rangle + \langle z, y^*\rangle = \langle (z^*, z), (y, y^*)\rangle, \end{aligned}$$

that is,

$$\langle (z^*, z), (y, y^*)\rangle - f(y, y^*) \leq \inf_{u^* \in X^*} h^*(u^*, Az) + \delta_{\{0\}}(z^* - A^*u^*).$$

Now, taking the supremum over $(y, y^*) \in Y \times Y^*$ on the left hand side of the above inequality we obtain

$$f^*(z^*, z) \leq \inf_{u^* \in X^*} h^*(u^*, Az) + \delta_{\{0\}}(z^* - A^*u^*). \tag{30}$$

For a fixed $(z, z^*) \in Y \times Y^*$ such that $f^*(z^*, z) < \infty$, define $\varphi, \psi \in \overline{\mathbb{R}}^{Y \times X \times Y^* \times X^*}$,

$$\begin{aligned} &\varphi(y, x, y^*, x^*) := f^*(z^*, z) - \langle y, z^*\rangle - \langle z, y^* + A^*x^*\rangle + \delta_{\{0\}}(y^*) + h(x, x^*), \\ &\psi(y, x, y^*, x^*) := \delta_{\{0\}}(x - Ay). \end{aligned}$$

Direct calculations yields

$$\bigcup_{\lambda > 0} \lambda[\mathrm{D}(\varphi) - \mathrm{D}(\psi)] = Y \times \bigcup_{\lambda > 0} \lambda[\mathrm{Pr}_X \mathrm{D}(h)] - \mathrm{R}(A)] \times Y^* \times X^*. \tag{31}$$

Using (11), (31) and Theorem 2.1 for φ and ψ, we conclude that there exists $(y^*, x^*, y^{**}, x^{**}) \in Y^* \times X^* \times Y^{**} \times X^{**}$ such that

$$\inf \varphi + \psi = -\varphi^*(y^*, x^*, y^{**}, x^{**}) - \psi^*(-y^*, -x^*, -y^{**}, -x^{**}). \tag{32}$$

Now, notice that

$$(\varphi + \psi)(y, x, y^*, x^*) \geq f^*(z^*, z) + f(y, A^*x^*) - \langle (z^*, z), (y, A^*x^*)\rangle \geq 0. \tag{33}$$

Using (32) and (33) we get

$$\varphi^*(y^*, x^*, y^{**}, x^{**}) + \psi^*(-y^*, -x^*, -y^{**}, -x^{**}) \leq 0. \tag{34}$$

Direct calculations yields

$$\begin{aligned} \psi^*(-y^*, -x^*, -y^{**}, -x^{**}) &= \sup_{(y, z^*, w^*)} \langle y, -y^* - A^*x^*\rangle + \langle z^*, -y^{**}\rangle + \langle w^*, -x^{**}\rangle \\ &= \delta_{\{0\}}(y^* + A^*x^*) + \delta_{\{0\}}(y^{**}) + \delta_{\{0\}}(x^{**}). \end{aligned} \tag{35}$$

Now, using (34) and (35) we conclude that

$$y^{**} = 0,\ x^{**} = 0 \quad \text{and} \quad y^* = -A^* x^*.$$

Therefore, from (34) we have

$$\begin{aligned} &\varphi^*(-A^*x^*, x^*, 0, 0) \\ &= \sup_{(y,x,w^*)} \big(\langle y, z^* - A^*x^* \rangle + \langle x, x^* \rangle + \langle Az, w^* \rangle - h(x, w^*) \big) - f^*(z^*, z) \\ &= h^*(x^*, Az) + \delta_{\{0\}}(z^* - A^*x^*) - f^*(z^*, z) \le 0, \end{aligned}$$

that is, there exists $x^* \in X^*$ such that

$$f^*(z^*, z) \ge h^*(x^*, Az) + \delta_{\{0\}}(z^* - A^*x^*).$$

Finally, using (30) we conclude the proof. □

References

[1] H. Attouch, H. Brezis: Duality for the sum of convex functions in general Banach spaces, in: Aspects of Mathematics and its Applications, J. A. Barroso (ed.), North-Holland Math. Library 34, North-Holland, Amsterdam (1986) 125–133.

[2] R. S. Burachik, B. F. Svaiter: Maximal monotone operators, convex functions and a special family of enlargements, Set-Valued Anal. 10(4) (2002) 297–316.

[3] R. S. Burachik, B. F. Svaiter: Maximal monotonicity, conjugation and the duality product, Proc. Amer. Math. Soc. 131(8) (2003) 2379–2383.

[4] S. Fitzpatrick: Representing monotone operators by convex functions, in: Functional Analysis and Optimization, Workshop / Miniconference (Canberra, 1988), Proc. Cent. Math. Anal. Aust. Natl. Univ. 20, Australian National University, Canberra (1988) 59–65.

[5] M. Marques Alves, B. F. Svaiter: Brøndsted-Rockafellar property and maximality of monotone operators representable by convex functions in non-reflexive Banach spaces, J. Convex Analysis 15(4) (2008) 693–706.

[6] M. Marques Alves, B. F. Svaiter: A new old class of maximal monotone operators, J. Convex Analysis 16(3&4) (2009) 881–890.

[7] J.-E. Martínez-Legaz, B. F. Svaiter: Monotone operators representable by l.s.c. convex functions, Set-Valued Anal. 13(1) (2005) 21–46.

[8] J.-E. Martínez-Legaz, B. F. Svaiter: Minimal convex functions bounded below by the duality product, Proc. Amer. Math. Soc. 136(3) (2008) 873–878.

[9] J.-E. Martínez-Legaz, M. Théra: A convex representation of maximal monotone operators, J. Nonlinear Convex Anal. 2(2) (2001) 243–247.

[10] S. Simons: Quadrivariate versions of the Attouch-Brezis theorem and strong representability, arXiv:0809.0325 (2008).